Randomized Contractions

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Update meeting on graph separation problems,
Warsaw, 9^{th} April 2013

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- CCHPP, *Designing FPT algorithms for cut problems using randomized contractions*, FOCS 2012

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- A Σ-*labeled graph* consists of:
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  - a set of constraints E (called *edges*) of form $((v, w), \varphi_{(v,w)})$ such that $\varphi_{(v,w)}$ is a permutation of Σ.
- A labeling $\Lambda : V \to \Sigma$ is *consistent* if $(\Lambda(v), \Lambda(w)) \in \varphi_{(v,w)}$ for each constraint $((v, w), \varphi_{(v,w)}) \in E$.

Example

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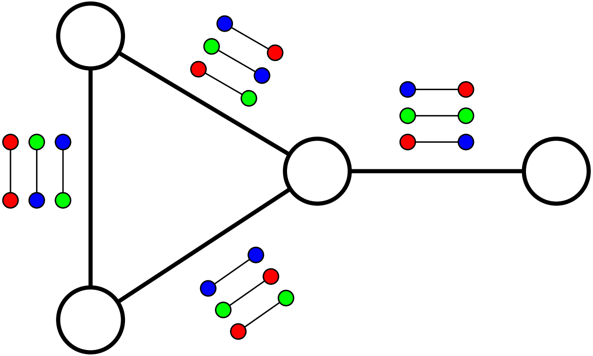

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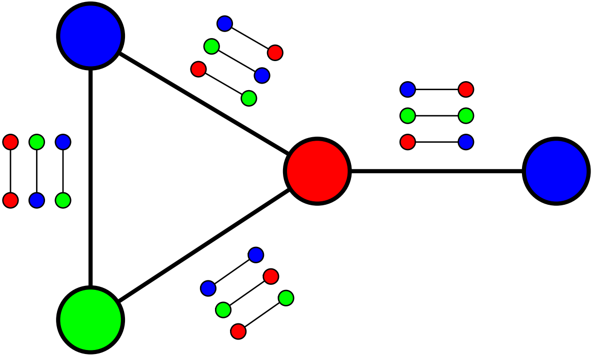

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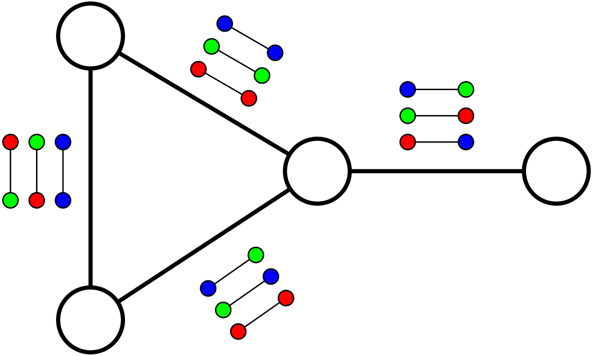

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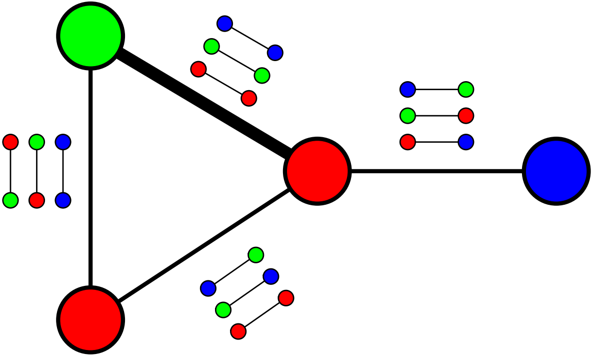

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- We show an algorithm working in time $O^*(2^{O(k^2 \log |\Sigma|)})$.

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Why ULC?

- Generalizes many graph separation problems:
    - Edge Bipartization;
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- Hardness of robust approximation for ULC is the base of the Unique Games Conjecture.

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- **Strategy**:
  - If there is a *nice* separator, perform divide-and-conquer on it,
  - otherwise, exploit the high-connectivity structure of the graph to solve the problem directly.

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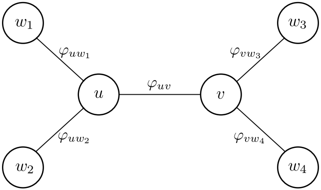

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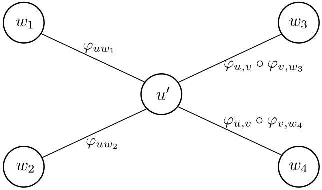

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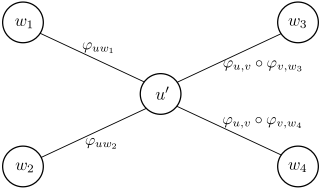

Observation

Labelings of G that respect constraint $((u, v), \varphi_{uv})$ correspond one-to-one to labelings of G/uv, where the correspondence retains the set of disrespected constraints.

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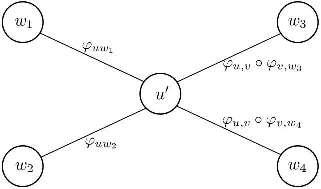

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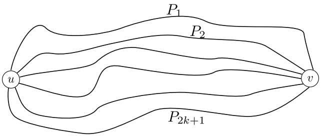
Corollary

If we infer that uv is not contained in some optimum solution, then it is safe to contract uv.

High connectivity lemma



- Assume that there are $2k+1$ edge-disjoint paths from u to v.

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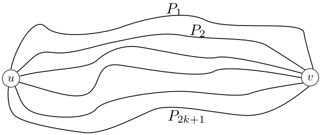

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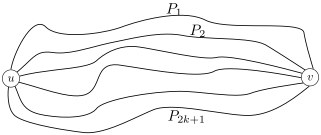

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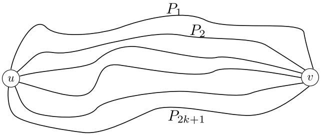

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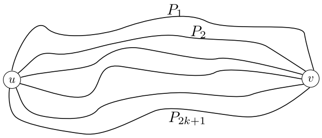

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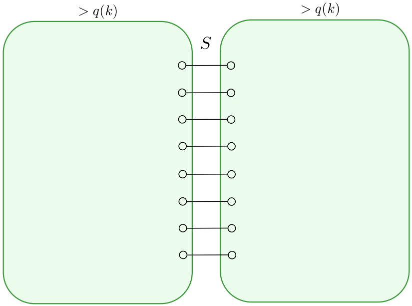
- Majority of paths are for sure not hit...
- so a majority candidate is always correct:

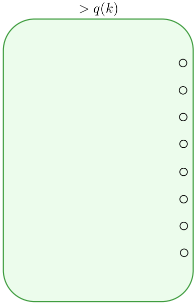
We can infer $\Lambda(v)$ even if we do not know which constraints are not respected!

Divide and conquer



Assume that we have a $2k$-edge separator S, and suppose both sides are connected and of size larger than $q(k)$.

Divide and conquer



Look at one side, iterate through all labelings of endpoints of S.

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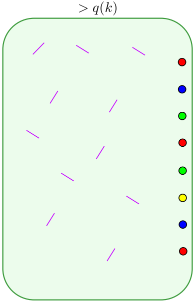

For each labeling, mark some optimum solution of size $\leqslant k$ (or nothing if there is no such).

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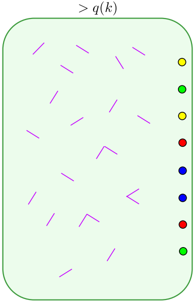

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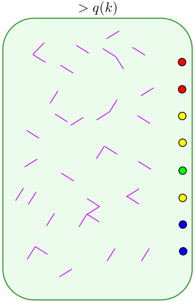

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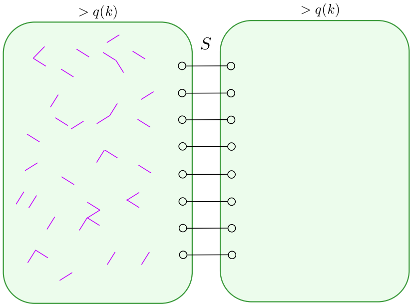
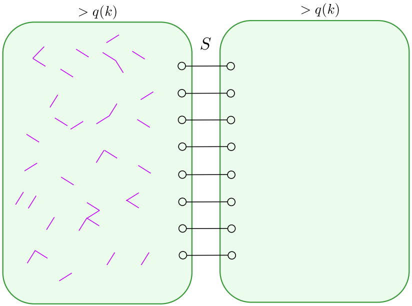


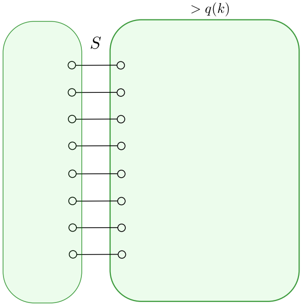
Recall that we had both of the sides.

Divide and conquer



Claim. Each unmarked edge
is not contained in some optimum solution.

Divide and conquer



Contract all the unmarked edges;
if $q(k) \geqslant k \cdot |\Sigma|^{2k} + 1$, something gets contracted.

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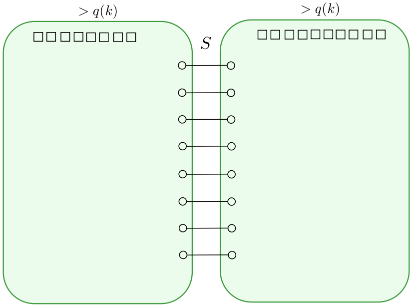
BORDER ULC

Input: a Σ-labeled graph G, an integer k,
and a set T of at most $4k$ terminals

Output: for every labeling Λ_0 of T, an optimum set of edges F after removing which Λ_0 can be extended on G,
or $\perp$ if no such F of cardinality $\leqslant k$ exists.

Recursive understanding



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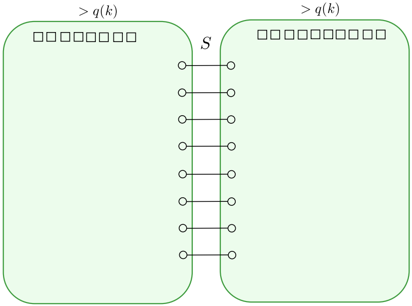
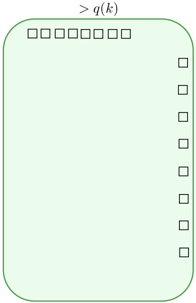


One of the sides has at most $2k$ terminals.
(assume it is the left one)

Recursive understanding



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The border becomes also terminals.

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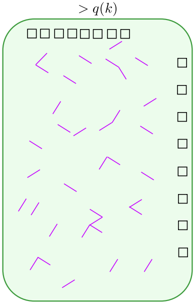

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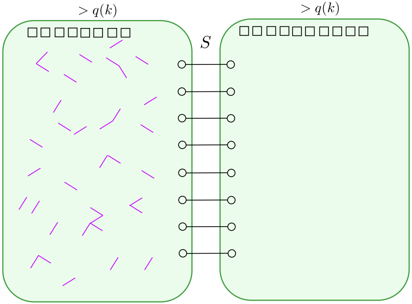

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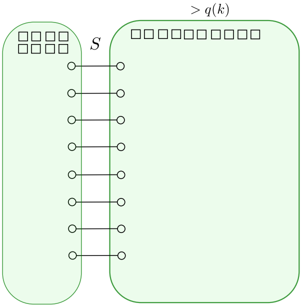

Contract all the unmarked edges.

The whole algorithm

Is there a good separation?

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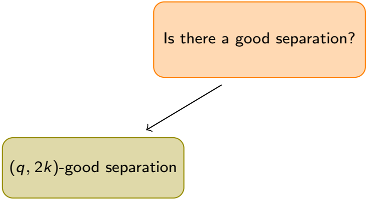

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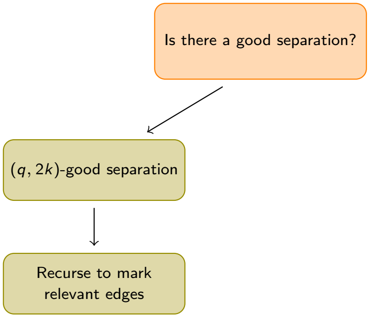

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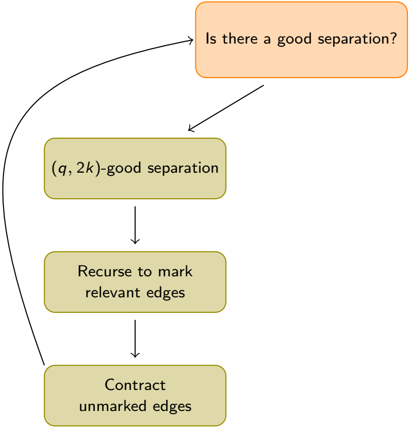

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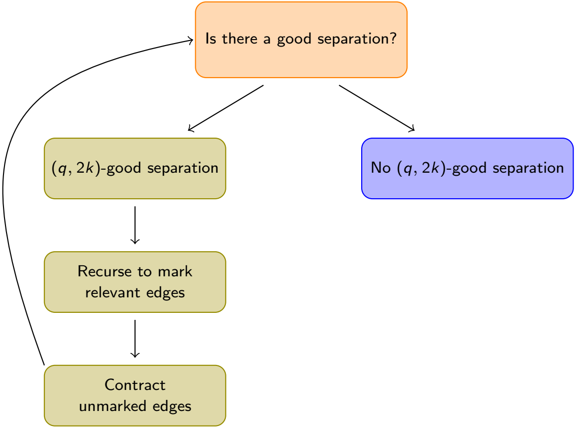

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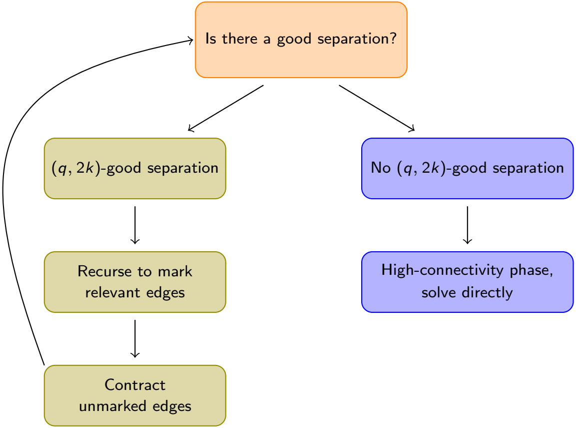

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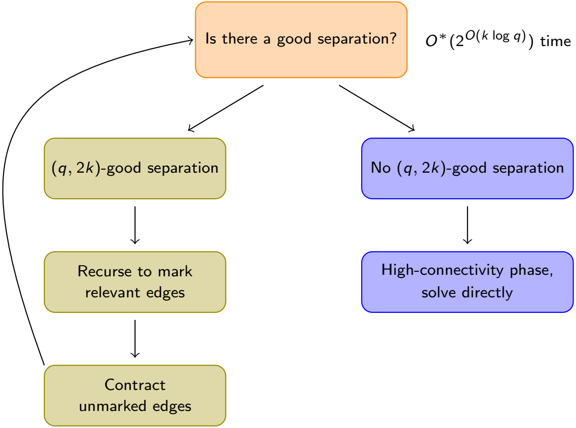

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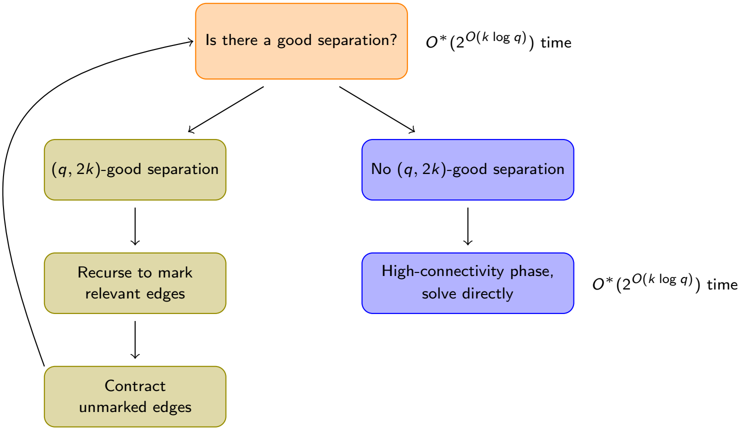

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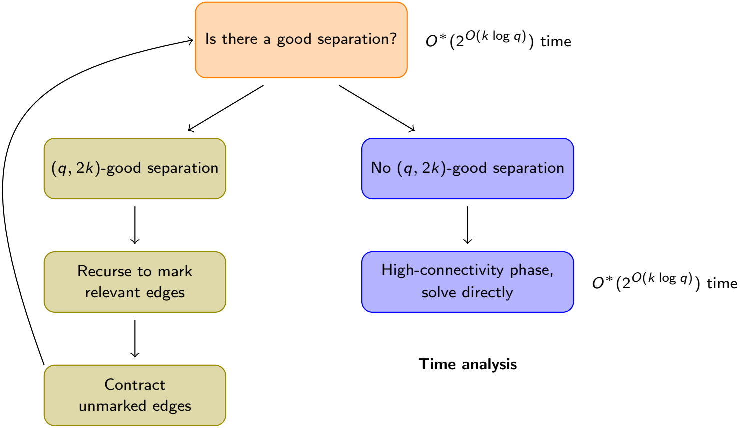

Time analysis

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Is there a good separation? $O^*(2^{O(k \log q)})$ time

$(q, 2k)$-good separation

Recurse to mark relevant edges

Contract unmarked edges

No $(q, 2k)$-good separation

High-connectivity phase, solve directly $O^*(2^{O(k \log q)})$ time

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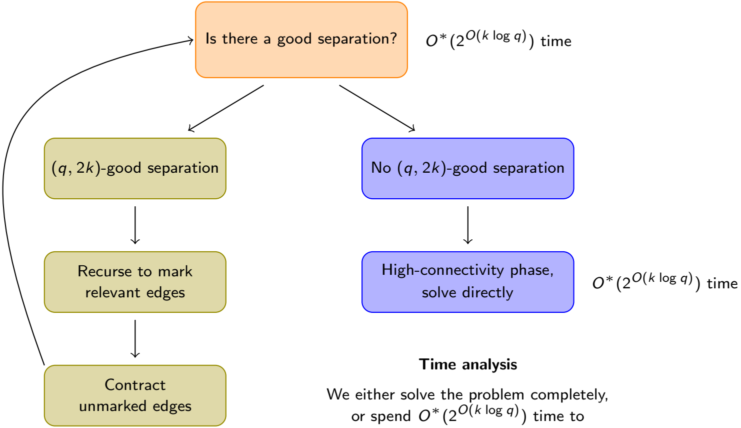

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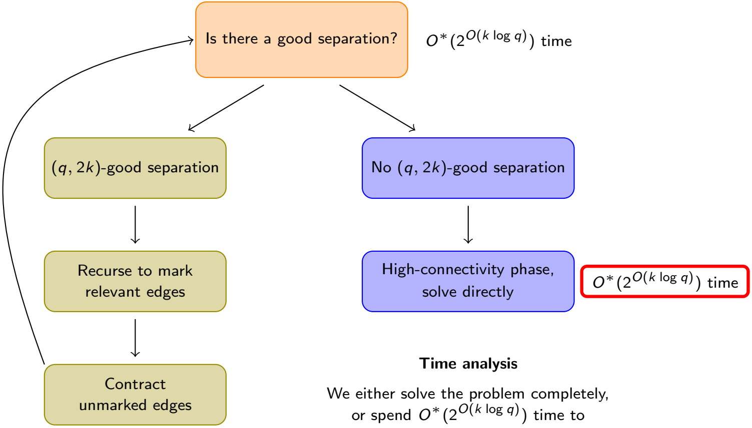

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- It can contain at most k small connected components of size at most q, and at most one big of arbitrarily large size.

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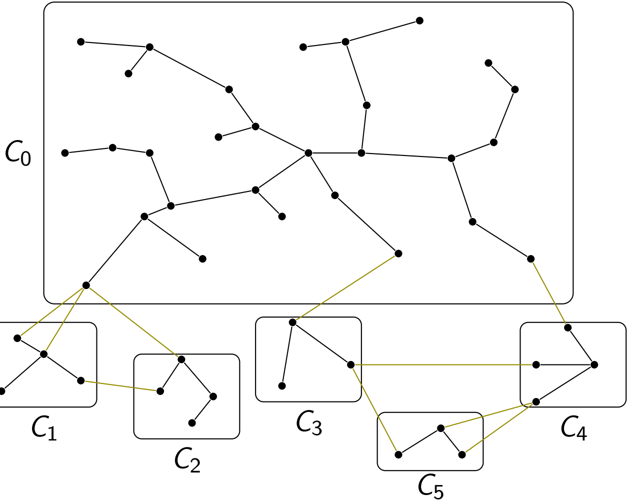
- With probability $\frac{1}{2}$ it becomes **red**, and with probability $\frac{1}{2}$ **blue**.
- We aim at the event that:
  - the whole F becomes **red**;
  - for every small component, some its spanning tree becomes **blue**;
  - for every endpoint v of an edge from F in the big component, we have a **blue** tree on $q+1$ vertices adjacent to v (**anchor**).

Colour coding



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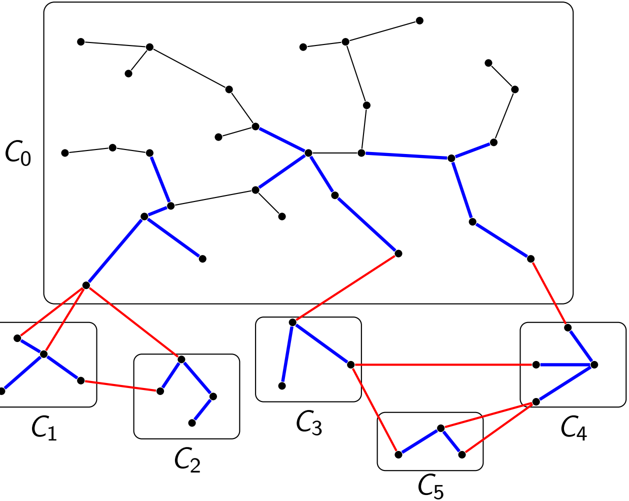

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- Such components are called **stains**:
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  - All small components become small stains, while anchors are contained in large stains.

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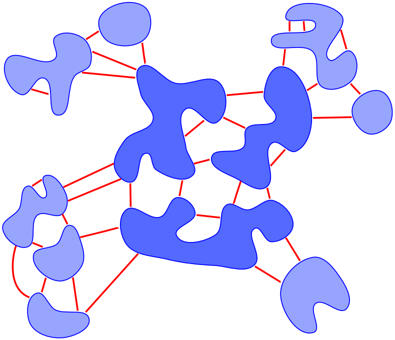
- At a cost of $|\Sigma|$ overhead, we have all the large stains labeled!

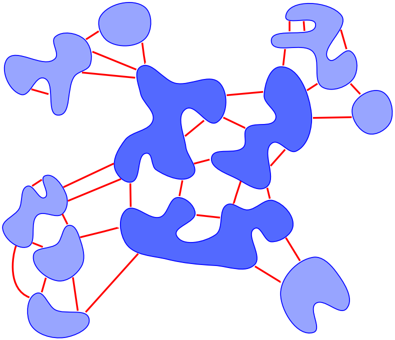
Small stains



Goal:

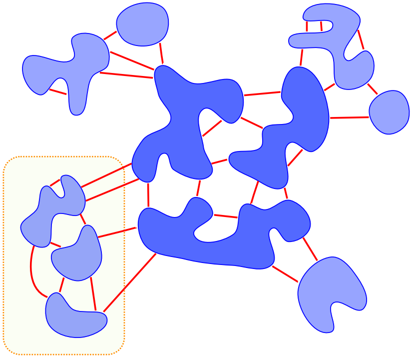
determine which red edges are in the solution, and which not.

Small stains



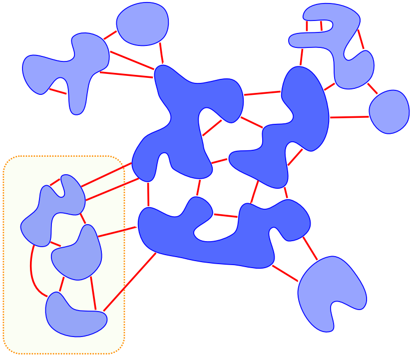
Edges with both endpoints in large stains:
easy

Small stains



Group of stains:
connected component of $G \setminus \mathrm{large\ stains}$

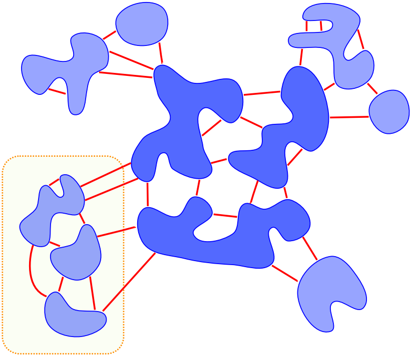
Small stains



Anchors:

every group either goes fully into F, or fully out of F.

Small stains



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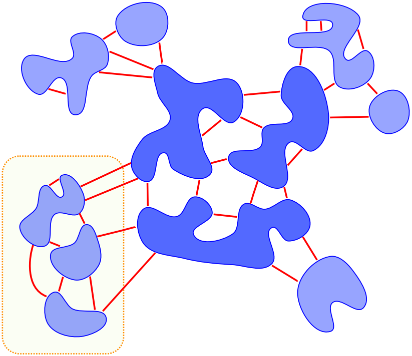
Small stains



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- **Thank you for attention!**