# Fixed Parameter Tractability: an alternative approach to NP-hard problems

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- but maybe some instances are 'easier' than other?
- For example, seeking a clique of size 10 seems easier than of size $10^6$.

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- We are looking for an algorithm running in time $f(k)n^{O(1)}$, where $f$ is some (usually exponential) function.
- **Intuition:** The whole 'exponentiality' of the algorithm has to be encapsulated in the constant factor depending only on the parameter.

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- If none of the branches returned 'YES', return 'NO'.
- Running time: $2^k n^{O(1)}$.

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- Examples:
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  - Set Cover (W[2]-complete, parameterized by solution size).

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- **Intuition:** Question about a vertex cover of size 30 in a graph with $10^6$ vertices is a nonsense.
- Maybe in such a situation we could preprocess the graph to shrink its size?

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- Abstract nonsens: decidable problem is FPT iff it has a kernel.
- What is really interesting is whether a problem admits a **polynomial** kernel, i.e., such that $f$ is a polynomial.

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- Reduction 1: if there is an isolated vertex, remove it.
- Reduction 2: if there is a vertex of degree more than $k$, delete it and decrease $k$ by 1.
- Reduction 3: if there are more than $k^2$ edges in the graph, answer 'NO'.

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- It can be done better — a kernel with $2k$ vertices,
  - One approach: Hall's theorem, Expansion Lemma.
  - Second approach: Linear Programming.
  - Unless polynomial hierarchy collapses, the number of edges in the kernel cannot be $O(k^{2-\epsilon})$. [Dell, Melkebeek, STOC 2010]

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  - mathing theory: Hall's theorem, Tutte's theorem;
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  - set theory: $\Delta$–lemma,
  - topological properties: grid theorem, separator theorems.
- What is really interesting, is that we can show **negative** results about existence of polynomial kernels!

# Compositionality

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  - $(x', k')$ is a YES-instance iff at least one $(x_i, k)$ is a YES-instance.

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**Theorem [Fortnow, Santhanam, STOC 2008; Bodlaender et al.]**

If a problem $L$ is in $NP$ when unparameterized, and is compositional, then it has no polykernel unless $\Sigma_3^p = PH$.

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- **Question:** is there a path of length $k$ in $G$?

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- Unfortunately, for many problems no composition algorithm can be seen at the first glance.

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- We denote $K \leqslant_{ptp} L$.
- If $K$ does not admit a polynomial kernel, then $L$ also does not admit (assuming $K, L$ are $NP$-complete).
- Otherwise we could apply the reduction, perform kernelization, and reduce $L$ to $K$ using $NP$-completeness.

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- both in $d$-degenerate graphs.

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- However, $d$-degenerate graphs do not have 'topological' structure.
  - Given a graph $G$ obtain $G'$ by inserting a vertex inside each edge. Then $G'$ is 2–degenerate.

# Motivation

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- However, FPT in $d$–degenerate graphs for bounded $d$.
- Dominating Set has a polynomial kernel in $K_{d+1,d+1}$–free graphs (a superclass).
- A natural question: does CDS have a polynomial kernel as well?

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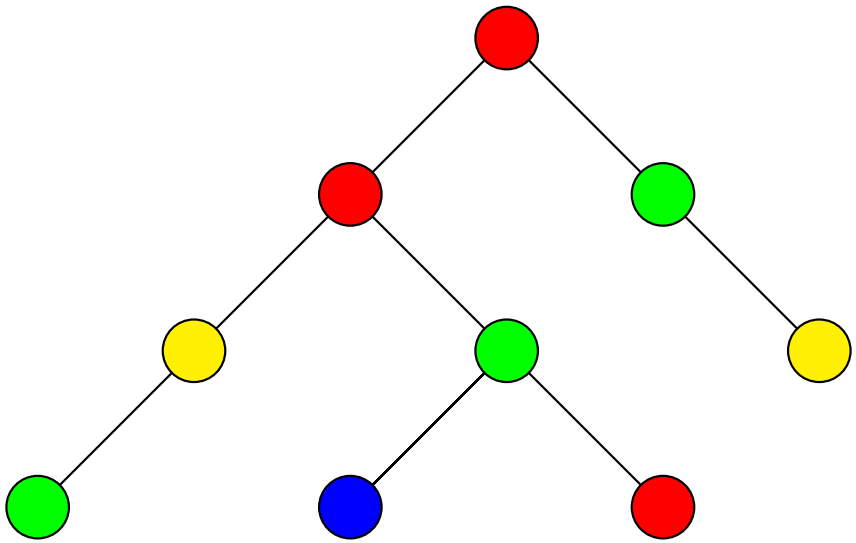
- Special case of Group Steiner Tree — $2^k|G|^{O(1)}$ FPT algorithm.

# Colourful Graph Motif – example



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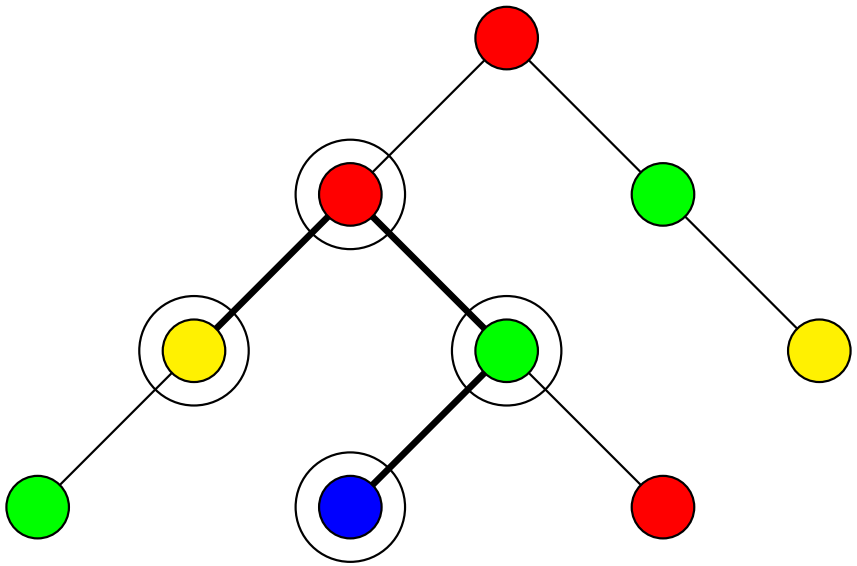

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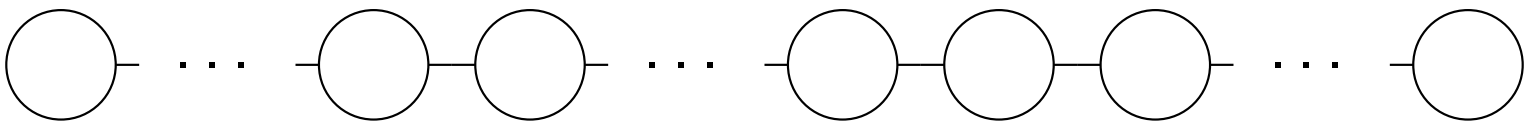

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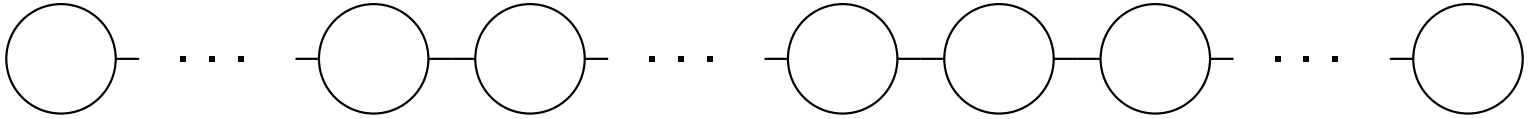

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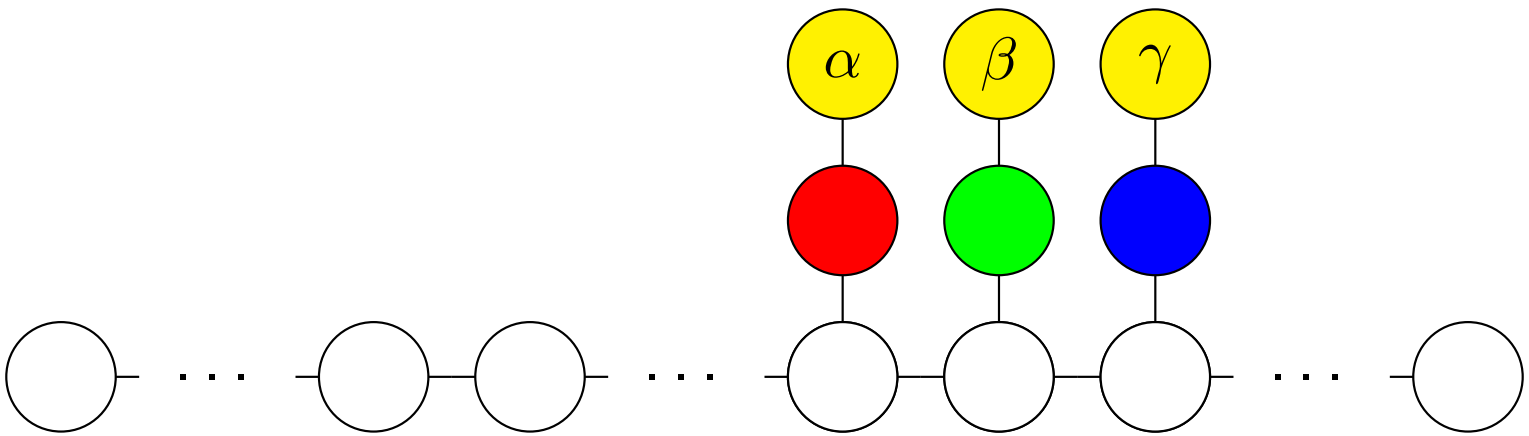

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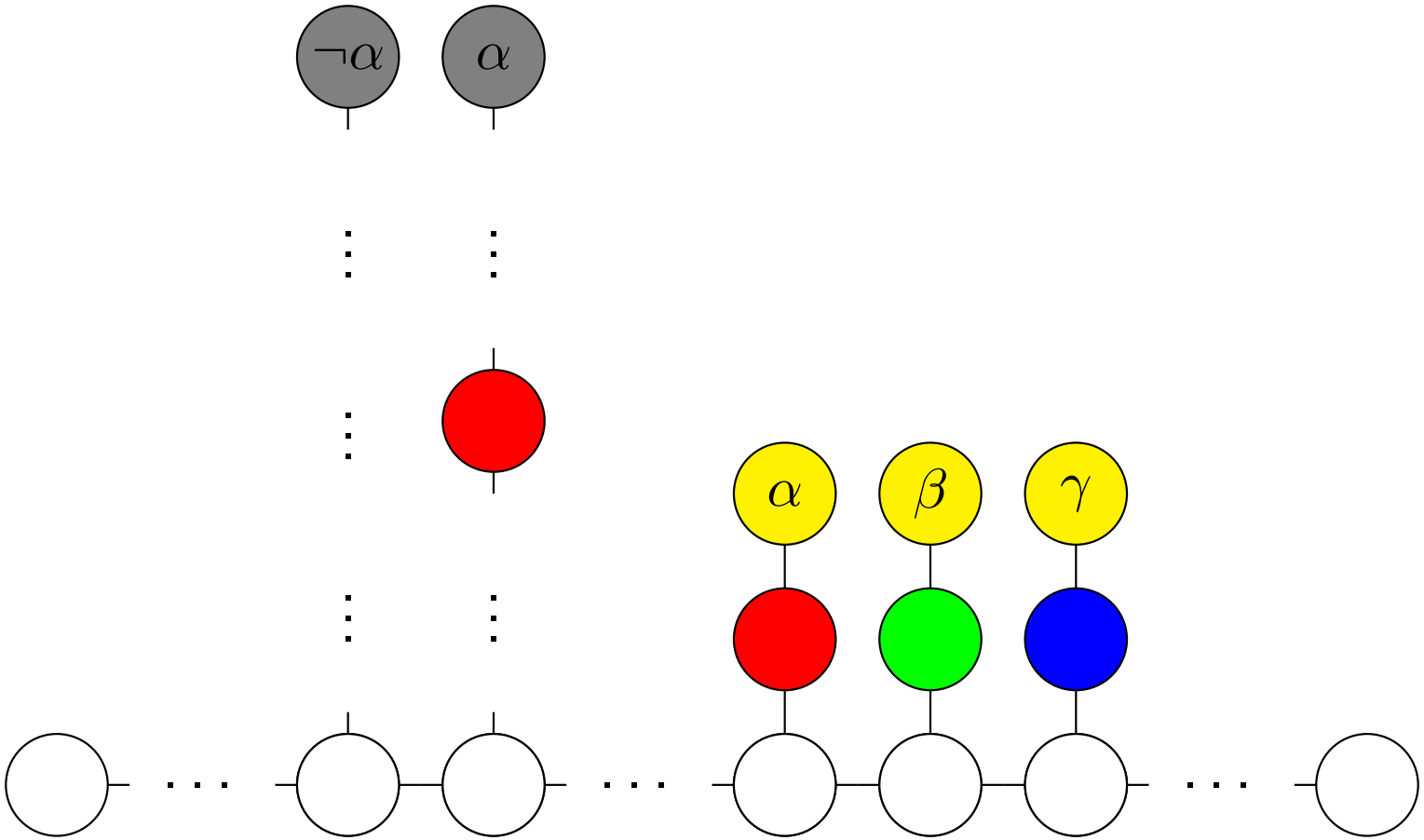

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- Composition algorithm: disjoint sum.
- But **reusing** the colours — the parameter does not increase!
- Thus, Colourful Graph Motif does not admit polynomial kernel in sets of disjoint combs (unless $\Sigma_3 = PH$).
  - **Remark:** There is a simple reduction to a case of one comb (technical).

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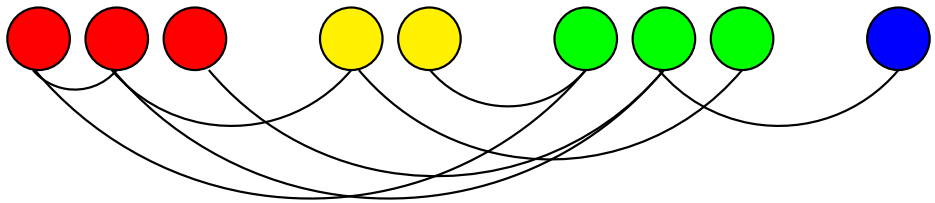

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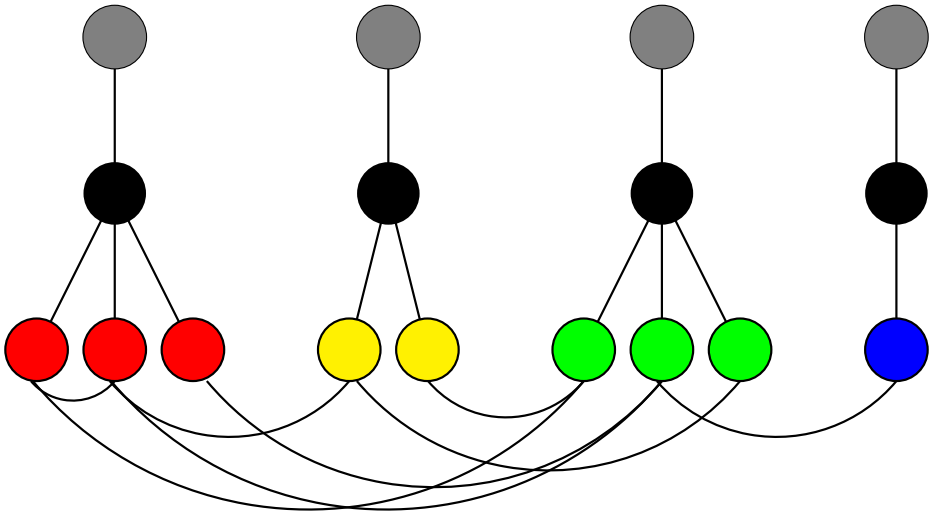

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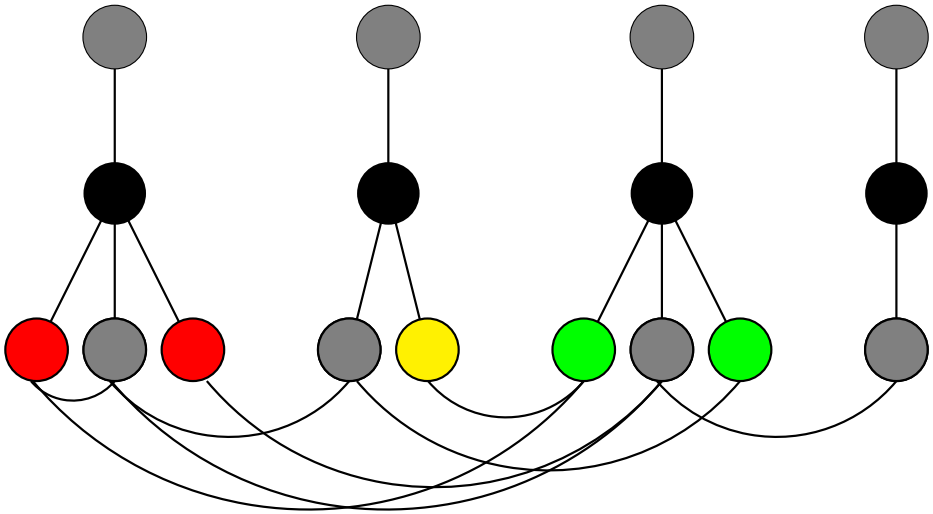
- Need to use black nodes and one node of each colour.

# Connected Dominating Set

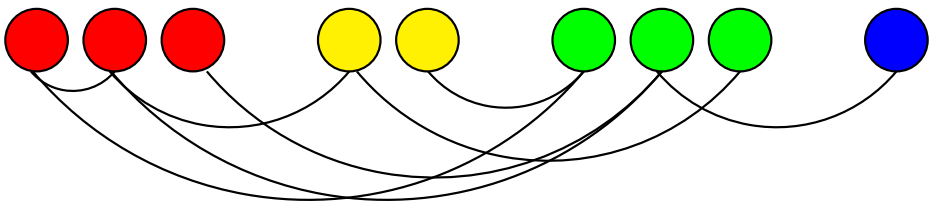
$$1\text{-deg-CGM} \leqslant 2\text{-deg-CDS}$$



- Ask for Connected Dominating Set with $2k = 8$ nodes.
- Need to use black nodes and one node of each colour.

# Steiner Tree

$$1\text{-deg-CGM} \leqslant 2\text{-deg-ST}$$



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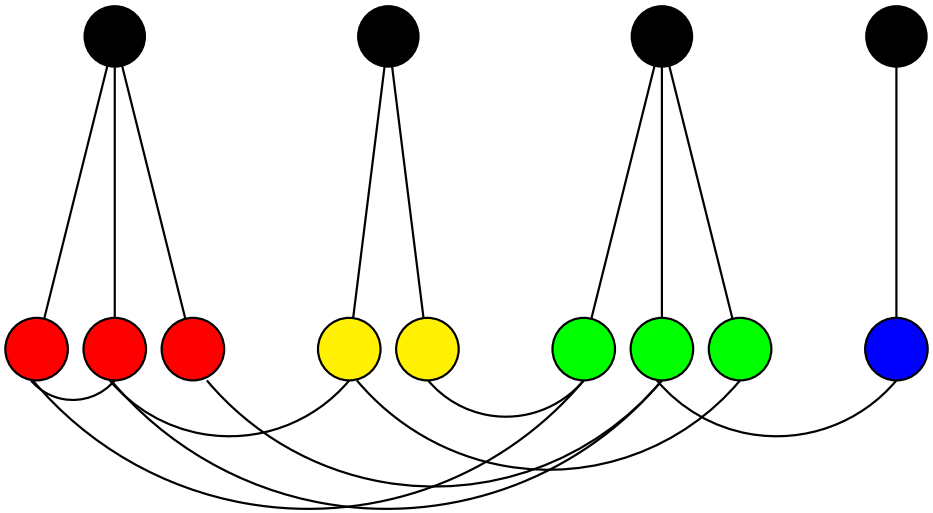

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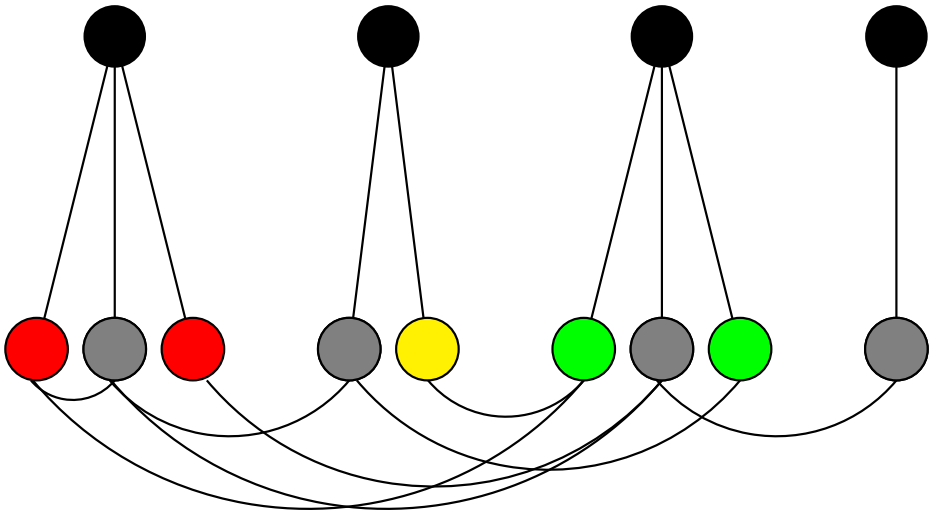

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- Firstly, randomly colour the vertices of the graph in $k$ colours (each vertex independently, with uniform distribution).
- Then, using dynamic programming, determine, whether there exists a path with the first vertex of the first colour, the second of the second colour etc.

# FPT algorithm for Longest Path: analysis

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- We can independently run the algorithm $k^k \ln 2$ times, and the probability that the solution was not found is bounded by $\frac{1}{2}$.
- **Remark:** We could look for a multi-coloured $k$-path, relaxing the condition about the order of the colours. The dynamic program will now run in $2^k|V|^{O(1)}$ time, but the probability of finding a solution in a single run will be around $e^{-k}$. Thus we have $(2e)^k|V|^{O(1)}$ algorithm.

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- Derandomize the $(2e)^k$ algorithm: take a smarter hashing family $[k^2] \longrightarrow [k]$ than all the functions.

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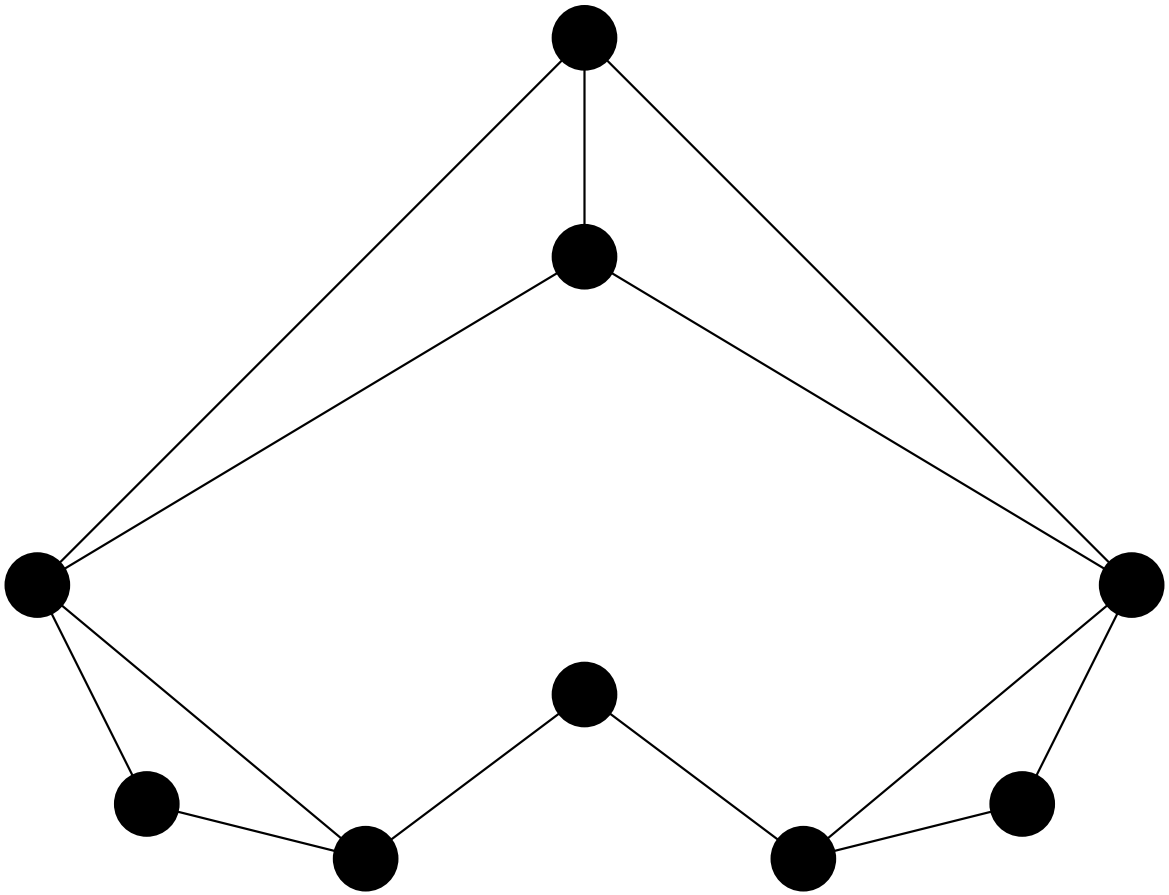
- **Input:** connected undirected graph $G = (V, E)$, integer $k$
- **Parameter:** $k$
- **Question:** can one delete at most $k$ edges from $G$, so that $G$ becomes Eulerian, i.e., is connected and all the degrees are even?

# Eulerian Edge Deletion— example



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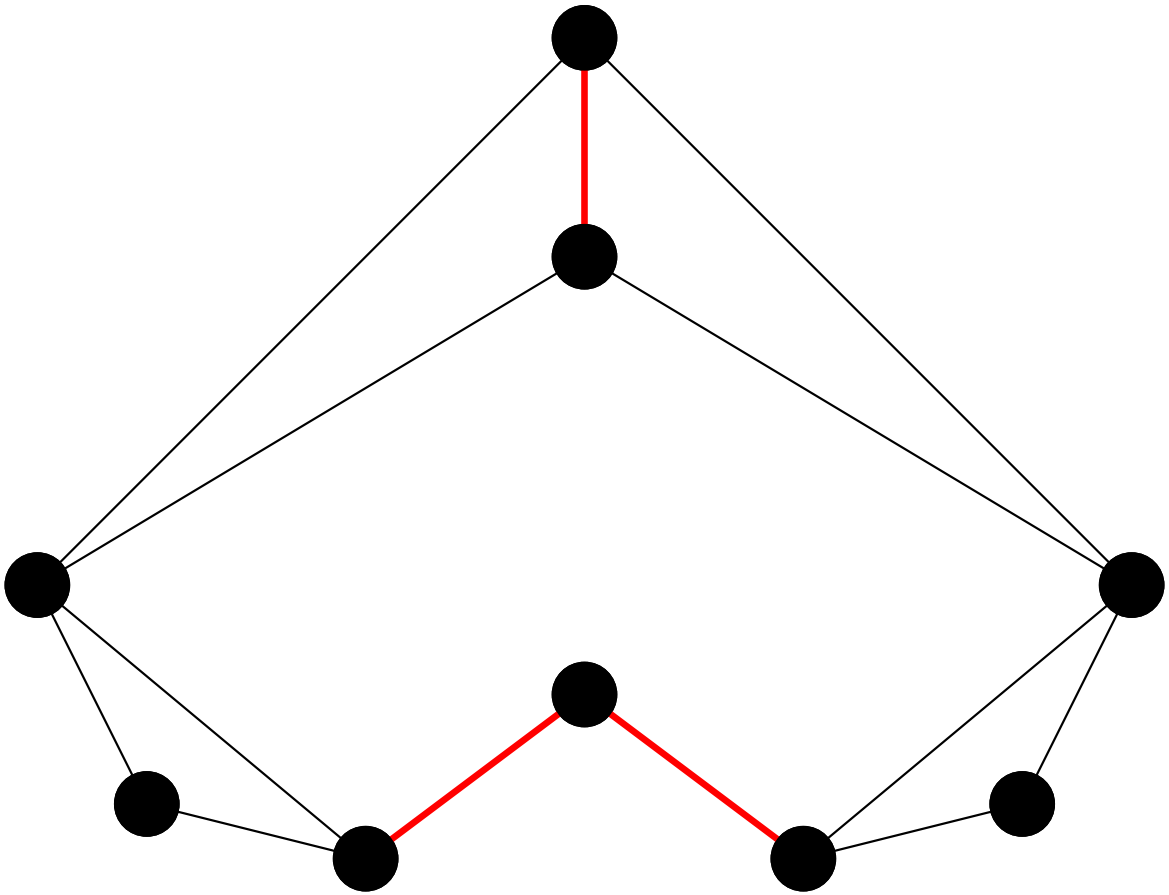

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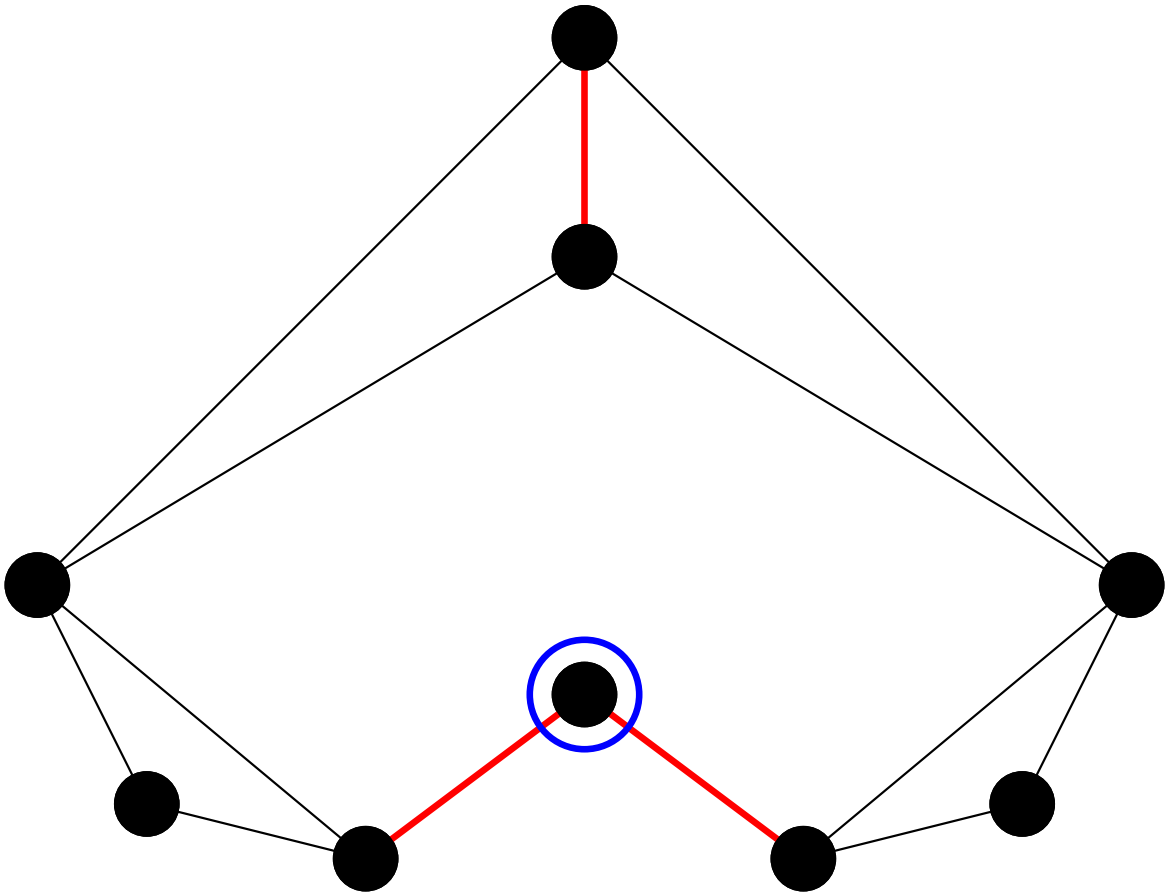

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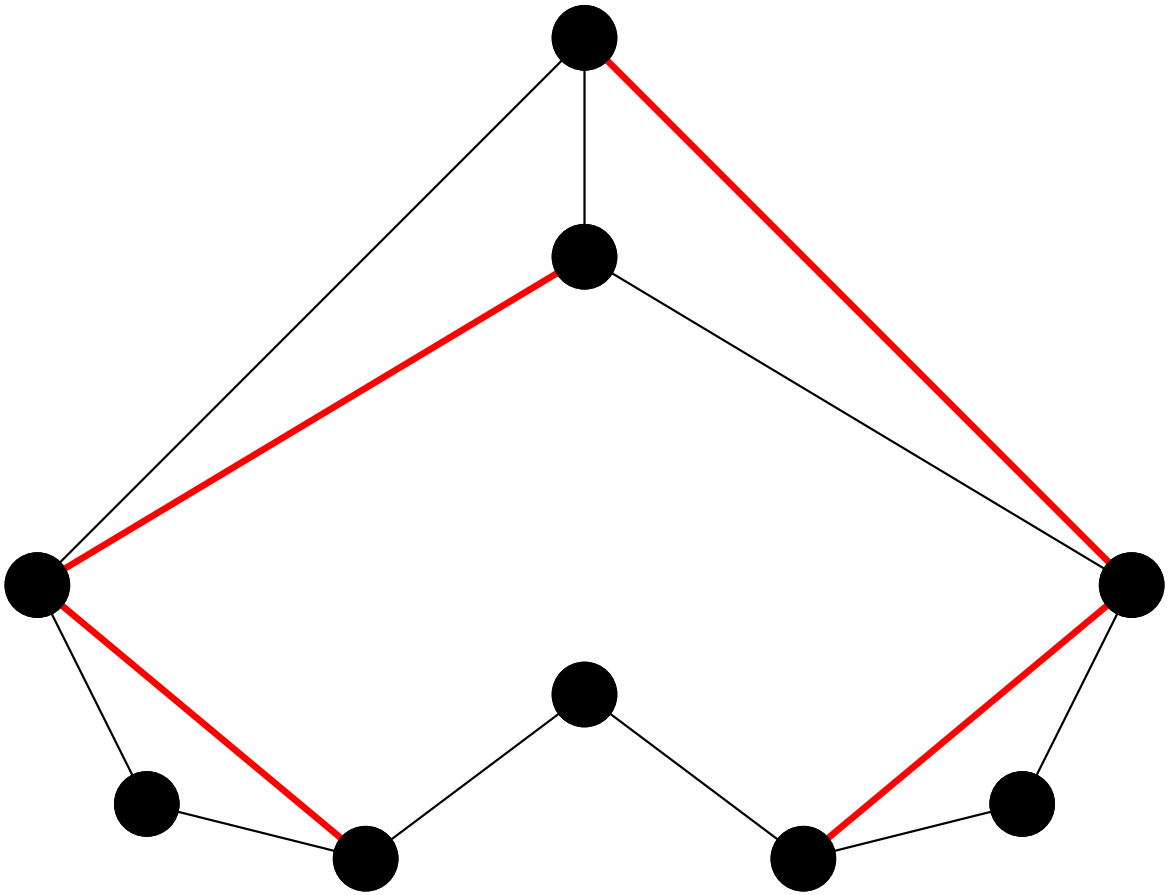

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- We need to find a subset of edges of cardinality at most $k$, inducing a subgraph with exactly vertices from $T$ having odd degrees.
- This is known as Minimal T-Join and can be done in polynomial time (using weighted matching in general graphs).

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- $W$ will be called a *witness*.
- **Idea:** reformulate the problem — ask for a pair $(S, W)$, such that $S$ is a T-JOIN and $W$ is its witness.

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- **Observation:** Solution can use only close edges.
- Crucial Lemma:

**Witness Lemma**

If $S$ is a solution of size $k$, then $S$ has a witness $W$ with at most $(2k-1)(2k+2)$ close edges.

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- **Claim:** in this shortest path $P$ only first and last $k+1$ edges can be close.

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- Hence, in each step we add at most $2k + 2$ close edges and the claim follows.

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- Solve Minimal T-Join in $G_R$, the graph induced by all the red edges with both endpoints in $K$. Return the answer.

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- **Derandomization:** similar to Longest Path example.

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  Problem: Is there a subgraph $K_{t,t}$ of the given graph?
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- Odd Cycle Transversal
  Problem: delete at most $k$ vertices in order to obtain a bipartite graph.
  Parameter: $k$
  Is there a polykernel?

# Thank you for your attention

Questions?