Computing cutwidth and pathwidth of semi-complete digraphs via degree orderings

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Motivation

Containment testing

Input: (di)graphs G and H

Question: Is H contained in G as a minor/topological minor/immersion/...?

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- For undirected G and H and all these relations, there are $f(|H|) \cdot |G|^3$ algorithms.
- In directed setting, the problems are already hard for fixed graphs H of constant size.

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- Elegant mathematical containment theory.
- Immersion and minor orderings are WQO on semi-complete digraphs.
- Algorithmic applications: many FPT and XP algorithms for various versions of containment problems.

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- These theorems are foundations of algorithmic and WQO results.
- Approximation algorithms for both measures are hence crucial for the whole theory.

Comparison

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Cutwidth approximation	$O(n^3)$ time, width $O(k^2)$	
Cutwidth exact	$O(f(k) \cdot n^3)$ time, non-uniform, non-constructive	
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Immersion	$O(f(\lvert H\rvert) \cdot n^3)$ time	$O(2^{O(\lvert H\rvert^2 \log \lvert H\rvert)} \cdot n^2)$ time
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- *Cutwidth* of T, denoted $\mathbf{ctw}(T)$, is minimum width among orderings of V.

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- Pathwidth of T, denoted $\mathbf{pw}(T)$, is minimum width among the path decompositions of T.

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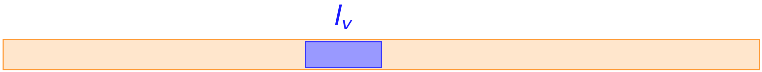

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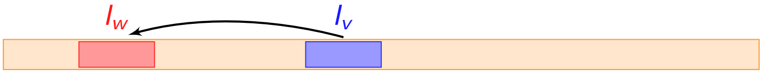

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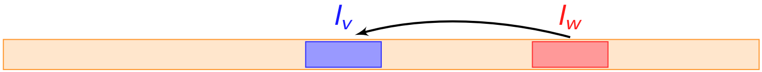

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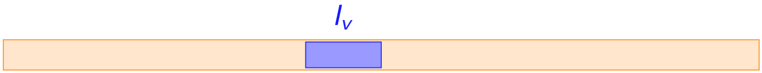

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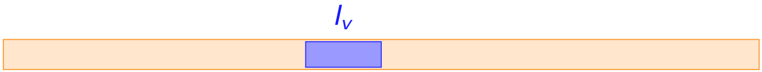

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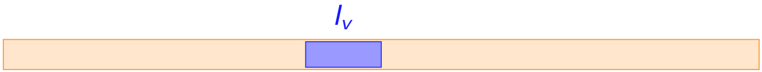

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  - Possible aberrations due to vertices sharing bags, but their number is limited by the width of the decomposition.
- In particular, there should not be many vertices with similar outdegrees.

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    - Trivial approximation algorithm: just sort.
- For pathwidth we pursue the direction of bounding the number of vertices with similar degrees.

Main lemma

Degree tangle

(k, ℓ)-*degree tangle* is a set $X \subseteq V(T)$ such that

(i) $|X| \geqslant k$,

(ii) for every $v, w \in X$, $|d^+(v) - d^+(w)| \leqslant \ell$.

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If a tournament T contains a $(5k+2, k)$-degree tangle, then $\mathbf{pw}(T) > k$.

- **Intuition**: The outdegrees in a low-pathwidth tournament must be really spread.

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- Let $A = \bigcup_{j=1}^{i} W_i$ and $B = \bigcup_{j=i+1}^{p} W_i$.
- We have that
  - $|A \cap B| \leqslant k$,
  - and there is no arc from $A \setminus B$ to $B \setminus A$.

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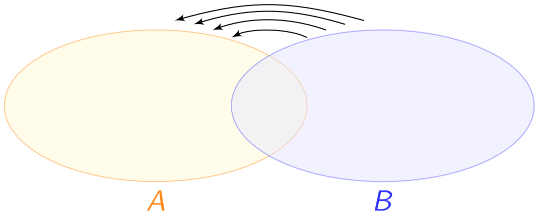

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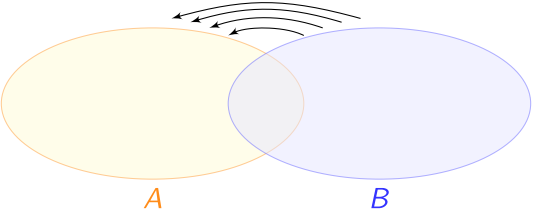

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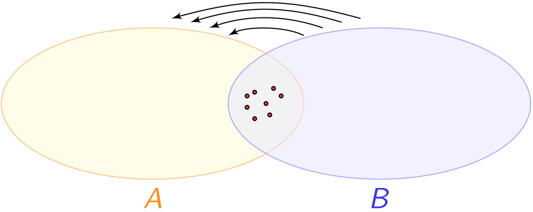

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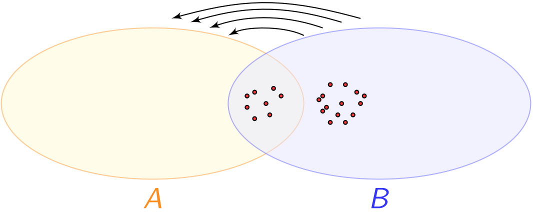

- Where vertices of X can be placed?
  - Not in $A \setminus B$, as the outdegrees there are at most $\alpha - 1$.
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  - Hence, $\geqslant 4k + 2$ vertices of X are placed in $B \setminus A$; denote them S.

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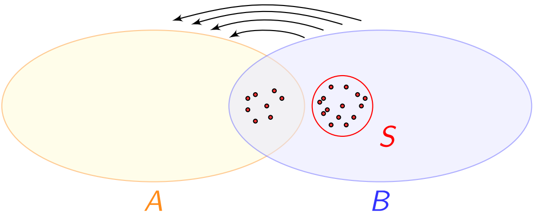

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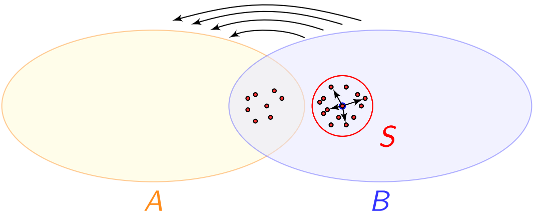

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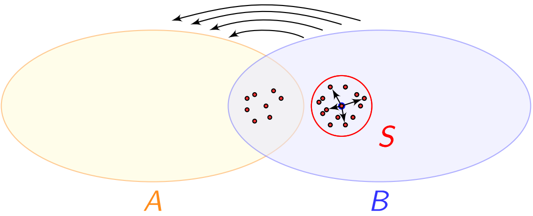

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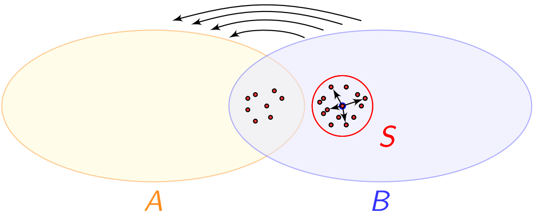

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- Contradiction with $w \in X$.

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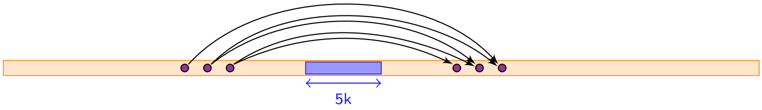

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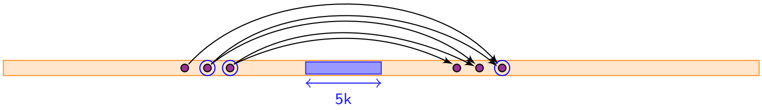

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- To separate left from right, we can pick:
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  - a vertex cover of forward arcs between left and right.

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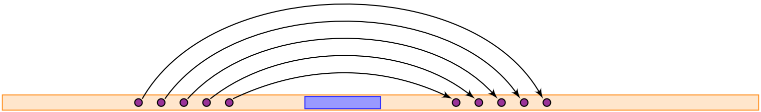

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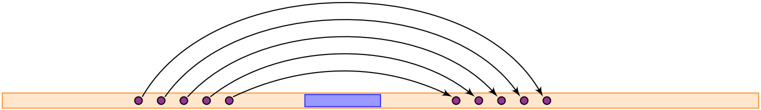

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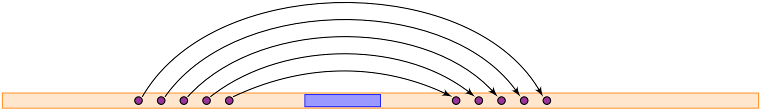

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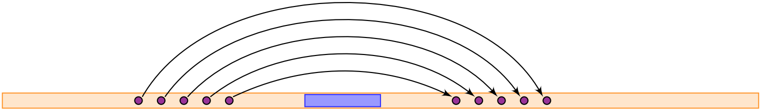
- **Problem**: we cannot pick the vertex covers arbitrarily, as the choice must be consistent as the window slides.

Approximating pathwidth



- **Observation**: A matching of size $k+1$ of forward edges is an obstruction for pathwidth k (*matching tangle*).
- Hence there is a vertex cover of size at most k.
- **Problem**: we cannot pick the vertex covers arbitrarily, as the choice must be consistent as the window slides.
- **Solution**: use 2-approximation of vertex cover — set of all vertices that are matched in **every** maximum matching.

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  - Use simple Buss kernelization instead of 2-approximation based on matchings.

Future work

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Cutwidth approximation	$O(n^2)$ time, width $O(k^2)$	
Cutwidth exact	$O(2^{O(k)} \cdot n^2)$ time	
Pathwidth approximation	$O(kn^2)$ time, width $7k$	
Pathwidth exact	$O(2^{O(k \log k)} \cdot n^2)$ time	
Immersion	$O(2^{O(\lvert H\rvert^2 \log \lvert H\rvert)} \cdot n^2)$ time	
Topological containment	$O(2^{O(\lvert H\rvert \log \lvert H\rvert)} \cdot n^2)$ time	

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Pathwidth approximation	$O(kn^2)$ time, width $7k$	better approximation factor?
Pathwidth exact	$O(2^{O(k \log k)} \cdot n^2)$ time	$O(2^{O(k)} \cdot n^c)$?
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Open problems

- Constant factor approximation of cutwidth.
- Better approximation factor for pathwidth.
- Improving $2^{O(k \log k)}$ to $2^{O(k)}$ in the exact algorithm for pathwidth.
- Single-exponential algorithm for rooted immersion testing.
- FPT algorithm for vertex-disjoint paths.