Parameterized Approximation Algorithms in Network Design

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joint work with:

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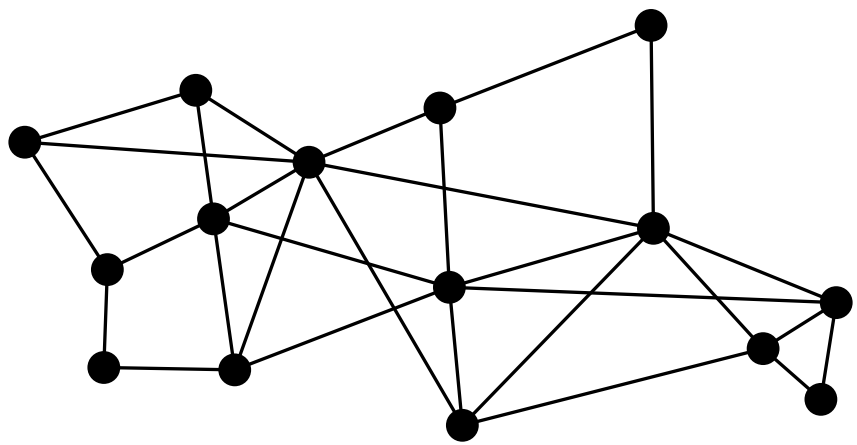
Optimization problems

An *optimization problem* is given by

- **instances**, e.g., connected graphs with positive edge weights
- **feasible solutions**
- **cost function**



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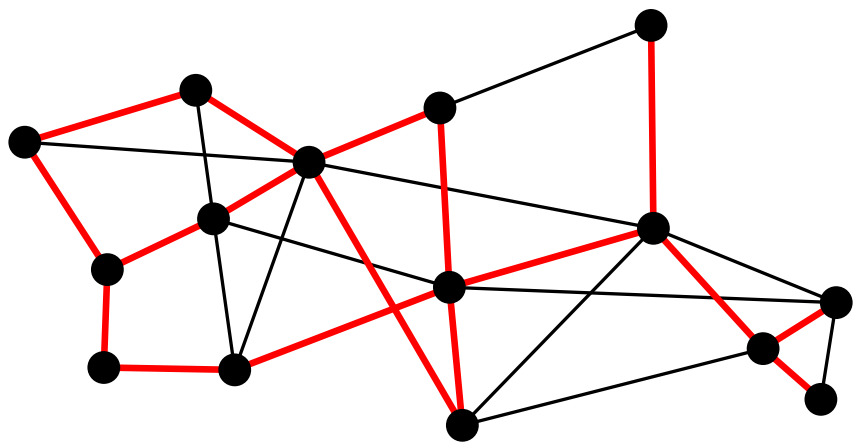

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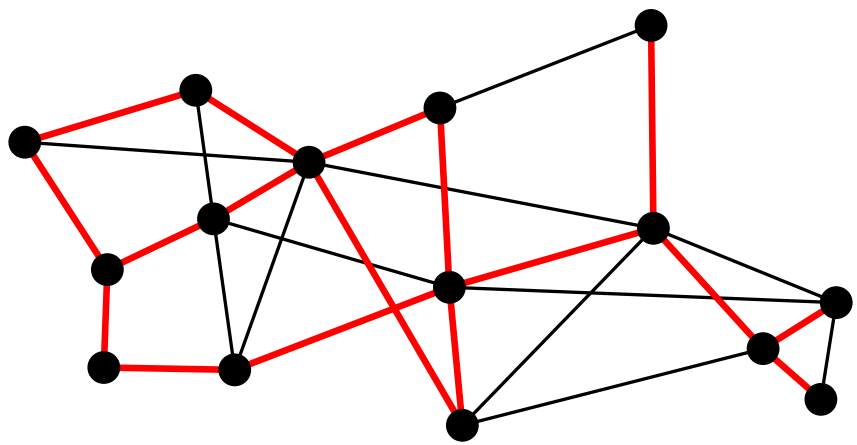

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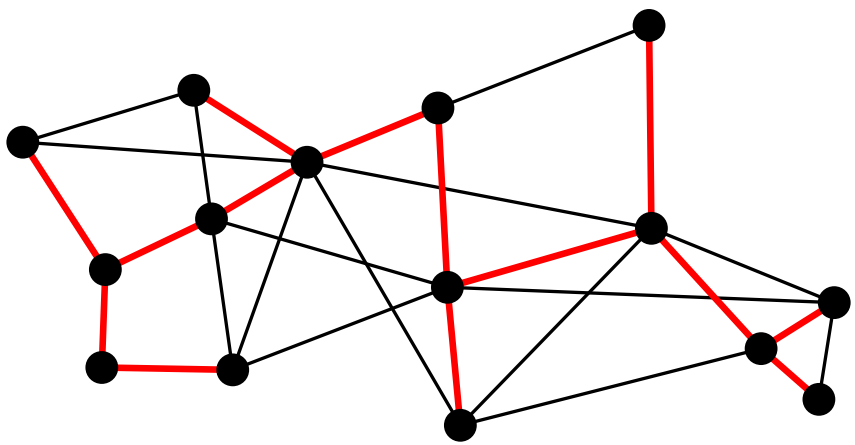
Aim: given instance, compute **feasible solution of smallest cost**, e.g., Minimum Spanning Tree (MST)



Cobham-Edmonds thesis

A problem can be *feasibly solved* if there is an algorithm satisfying

1. **accuracy:**
   the algorithm always produces the optimum solution
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Theorem [Jarník '30]

The MST problem can be **feasibly solved**.

NP-hard problems

STEINER TREE (ST) problem

- instances: connected graphs $G = (V, E)$ with positive edge weights and **set $R \subseteq V$ of terminals** ($V \setminus R$ are called *Steiner vertices*)
- feasible solutions: connected subgraphs $T \subseteq G$ containing **all terminals in R**
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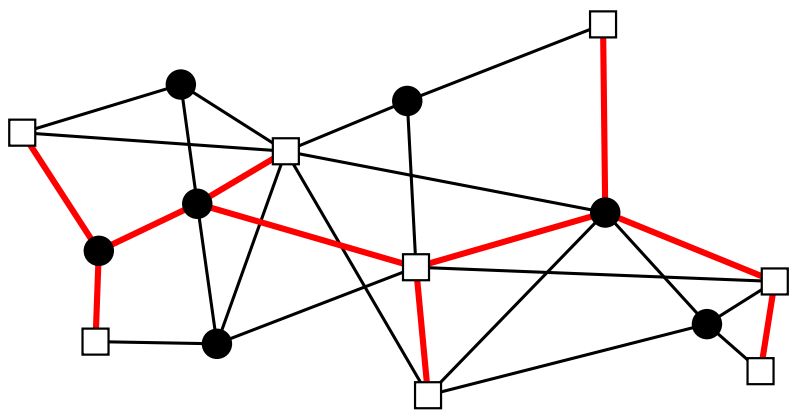

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Solution: give up at least one of **accuracy** or **efficiency**

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in practice: heuristics
hope: high quality output

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Definitions

An α-*approximation* is a solution with cost at most α times the optimum.

Polynomial time α-*approximation algorithm*: compute α-approximation in polynomial time.

Polynomial time approximation scheme (PTAS): for any constant $\varepsilon > 0$, compute $(1 + \varepsilon)$-approximation in polynomial time (e.g. $n^{O(1/\varepsilon)}$).

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Examples for the ST problem

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nr. of Steiner vertices $\lvert V(T) \setminus R\rvert$ in the optimum solution T	$n^{O(\lvert V(T)\setminus R\rvert)}$	[folklore]

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Theorem [Mölle, Richter, Rossmanith '06]

The ST problem has an **FPT** algorithm for **parameter** $|R|$ with runtime $(2+\delta)^{|R|}n^{O(1)}$ for any constant $\delta > 0$.

Theorem [folklore]

The ST problem is **W[2]-hard** for **parameter** $|V(T) \setminus R|$, where T is an optimum solution, i.e., no FPT algorithm, unless FPT=W[2].

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Theorem [Dvořák, F, Knop, Masařík, Toufar, Veselý '18]

The ST problem has a **PAS** for **parameter** $k = |V(T) \setminus R|$, where T is an optimum Steiner tree, with runtime $2^{O(k^2/\varepsilon^4)}n^{O(1)}$

Technique: Preprocessing

Preprocessing algorithm

in practice: data reduction

1. remove “easy parts” to obtain “core”
2. solve “core” (IP/SAT solvers or heuristics)

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Definition [Lokshtanov, Panolan, Ramanujan, Saurabh '17]

An α-*approximate preprocessing algorithm* $\mathcal{A}$ for parameter k is a pair of **polynomial time** algorithms:

- *reduction alg:* given instance (I, k), compute new instance (I', k')
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Lemma [Lokshtanov, Panolan, Ramanujan, Saurabh '17]

A problem with parameter k has a **parametrized α-approximation algorithm** if and only if it admits an **α-approximate kernel**.

Reduction algorithm for Steiner Tree

In each step:

add a tree to the solution, which

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For star S with all leaves terminals, the *ratio* is

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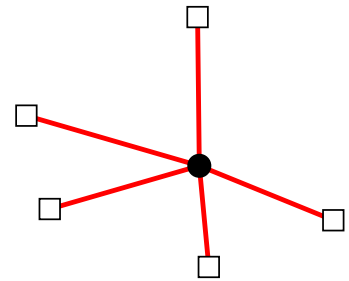

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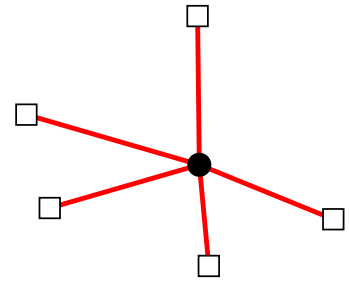

Lemma

A star of smallest ratio can be found in polynomial time.

A parametrized approximation scheme

The algorithm for parameter $k = |V(T) \setminus R|$

1. reduction: as long as $|R| \geq \tau$ $(= \Theta(k^2/\varepsilon^4))$
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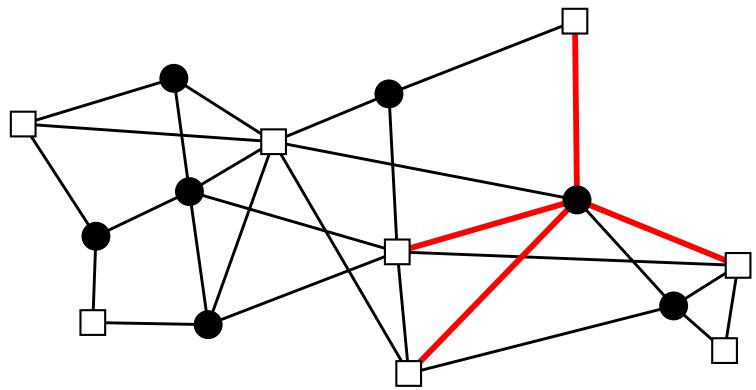

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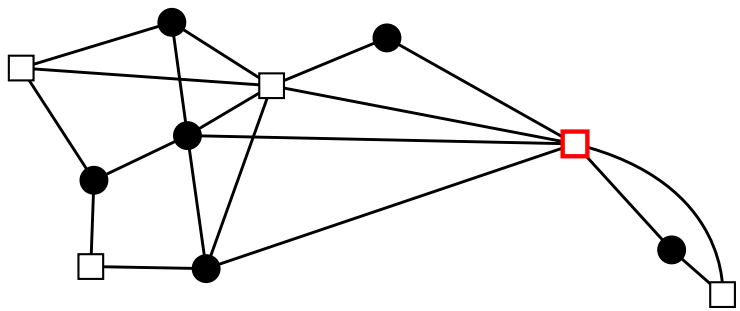

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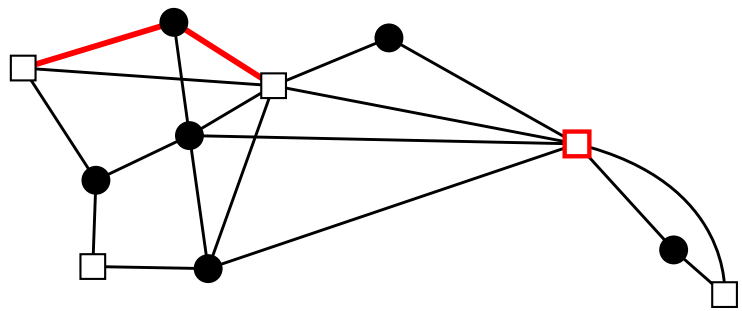

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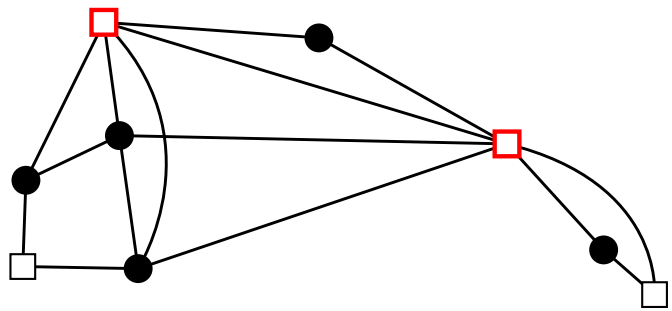

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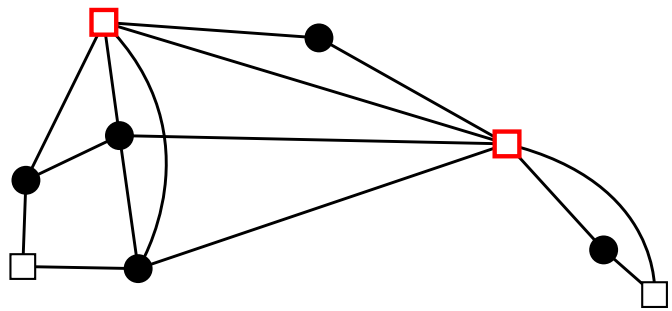

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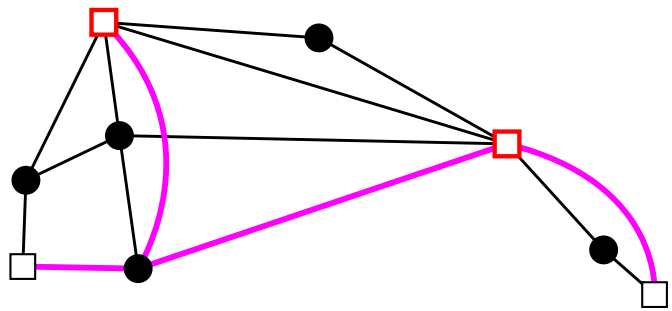

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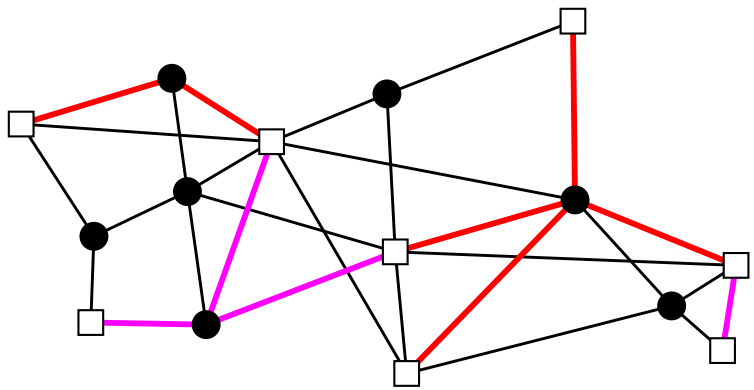

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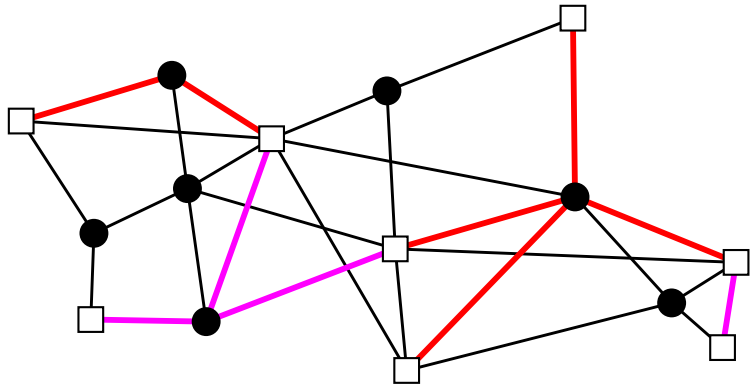
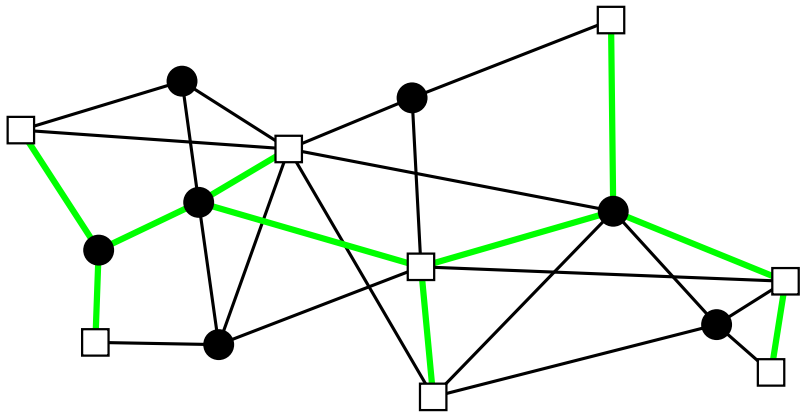
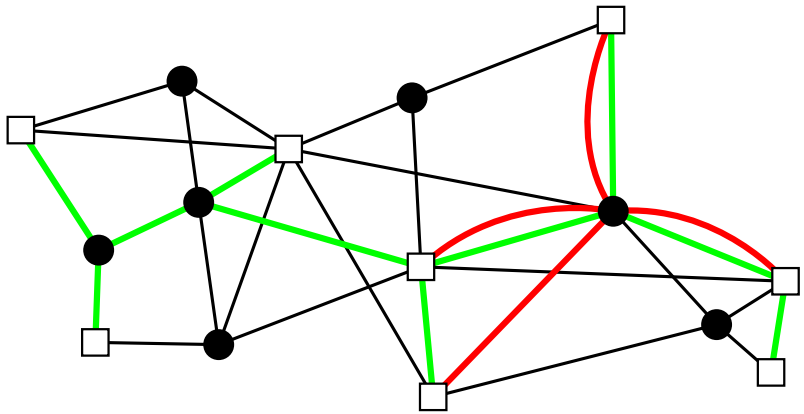


$\rightarrow$ runtime: $2^{O(k^2/\varepsilon^4)} n^{O(1)}$ as $|R| = O(k^2/\varepsilon^4)$ in step 2
due to FPT algorithm with runtime $(2+\delta)^{|R|} n^{O(1)}$
[Mölle, Richter, Rossmanith '06]

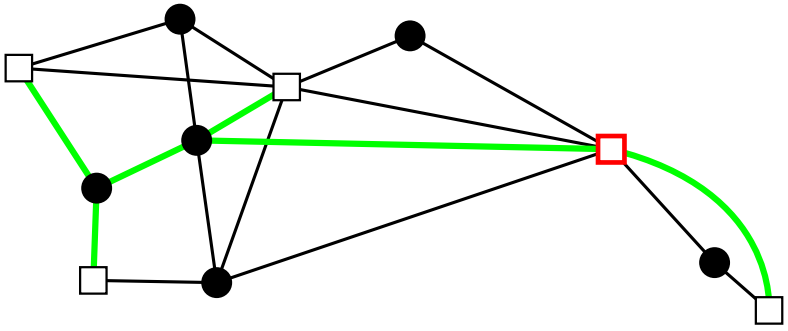
Charging stars against the optimum



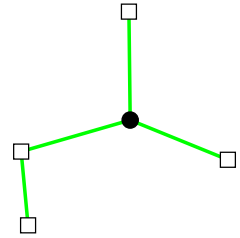
Charging stars against the optimum



Charging stars against the optimum

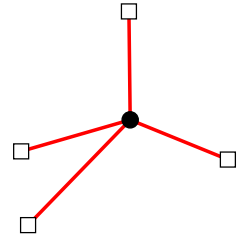


D_i:



vs

S_i:



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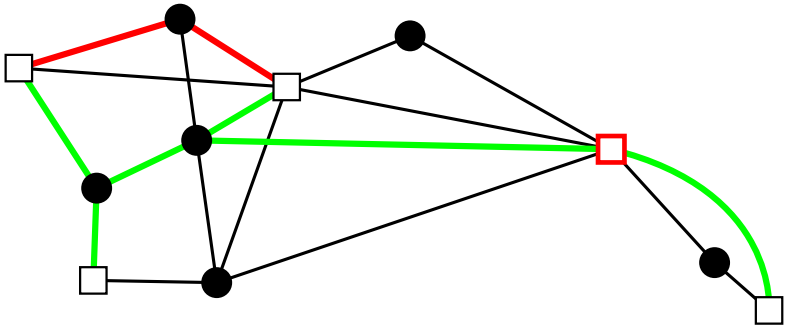
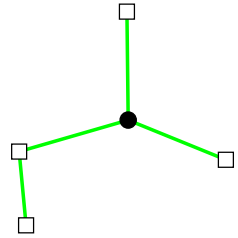
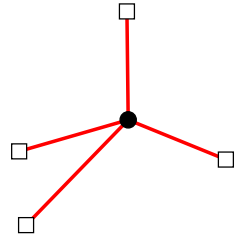


D_i:



vs

S_i:



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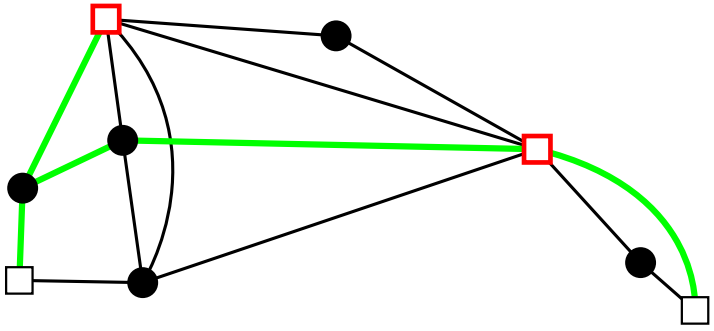
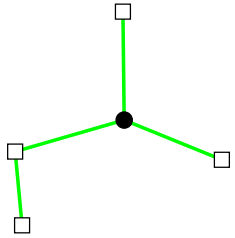


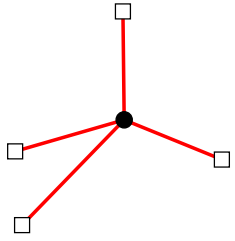
D_i:

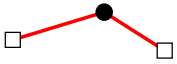


vs

vs

S_i:





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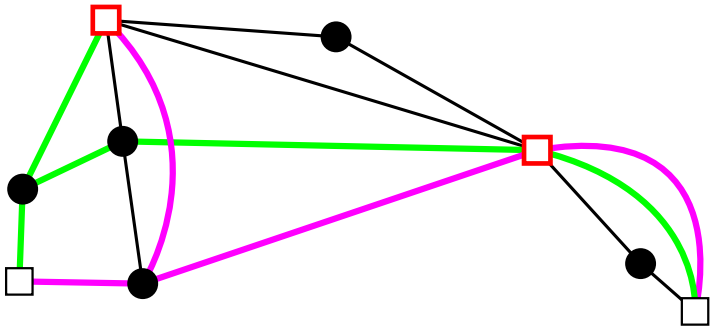
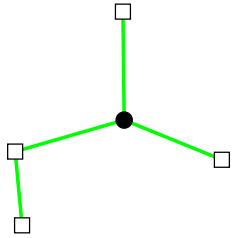


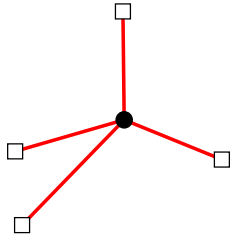
D_i:

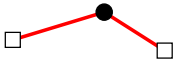


vs

vs

S_i:





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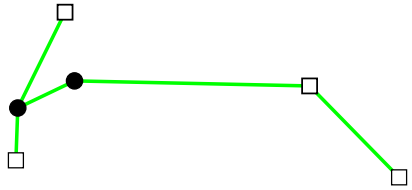
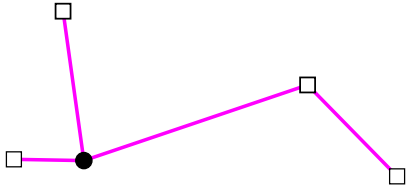
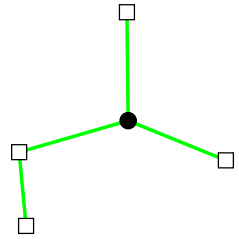


IV



$\Rightarrow (1+\varepsilon)$-approximation :-)

D_i:



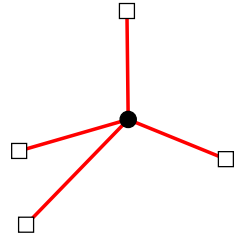
$\times(1+\varepsilon)$

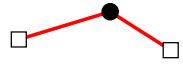
IV

$\times(1+\varepsilon)$

IV

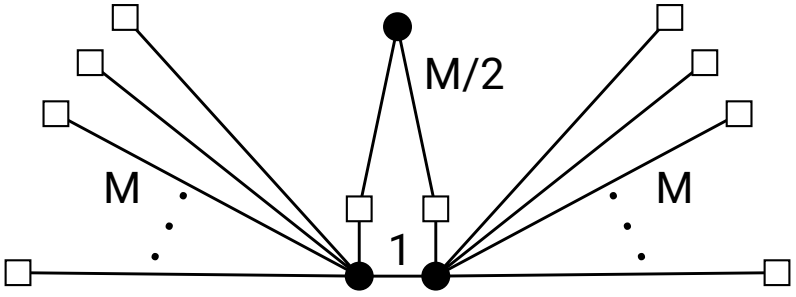
S_i:



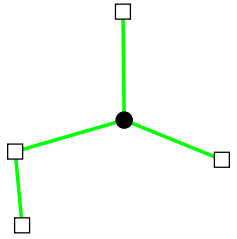


Charging stars against the optimum

A bad example: :-(



D_i:



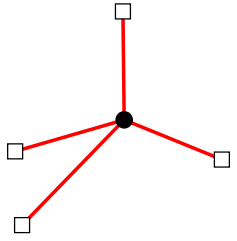
$\times(1+\varepsilon)$

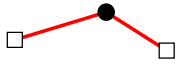
IV

$\times(1+\varepsilon)$

IV

S_i:





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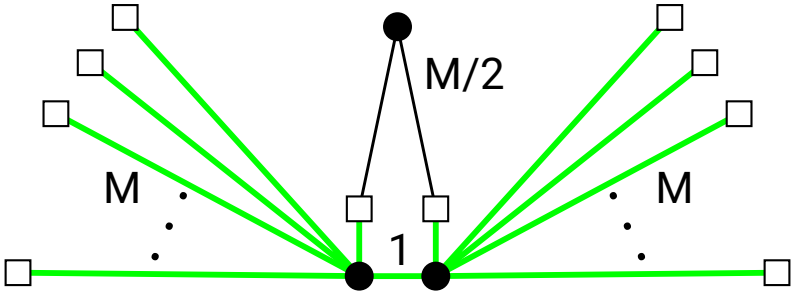
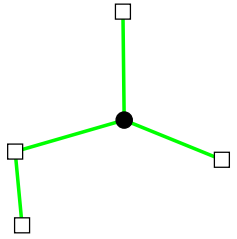


D_i:



$\times(1+\varepsilon)$

IV

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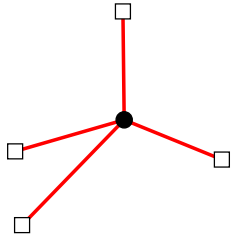
IV

S_i:



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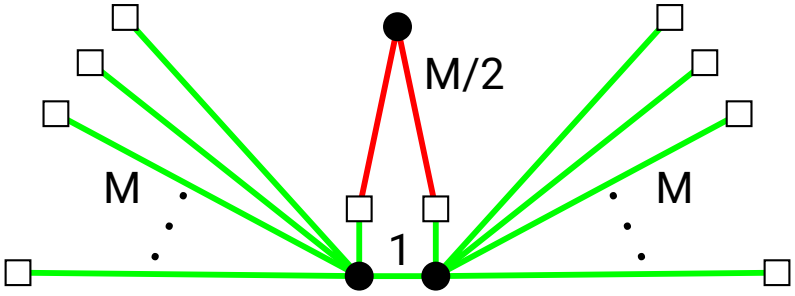
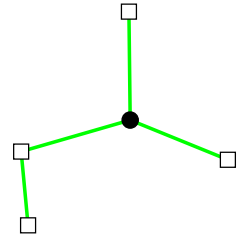


D_i:



$\times(1+\varepsilon)$

IV

$\times(1+\varepsilon)$

IV

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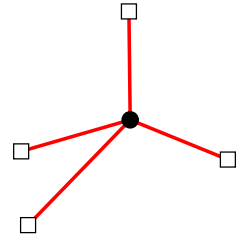

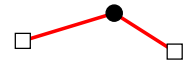

min ratio star:

$$\frac{w(S)}{|V(S)\cap R|-1} = \frac{2\cdot M/2}{2-1} = M$$

$$< \frac{\ell M}{\ell-1} \qquad \text{and} \qquad < \frac{\ell M+1}{\ell+1-1}$$

Charging stars against the optimum

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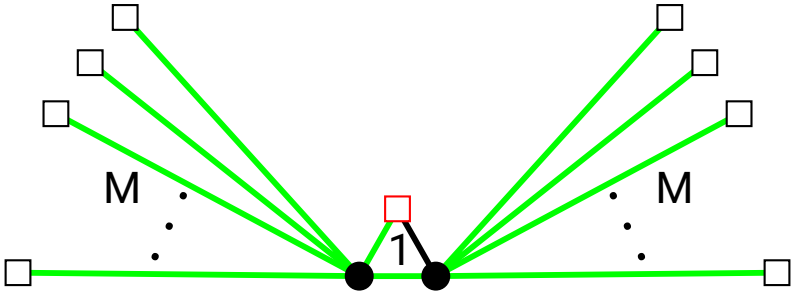
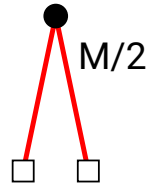


D_i:

vs

S_i:



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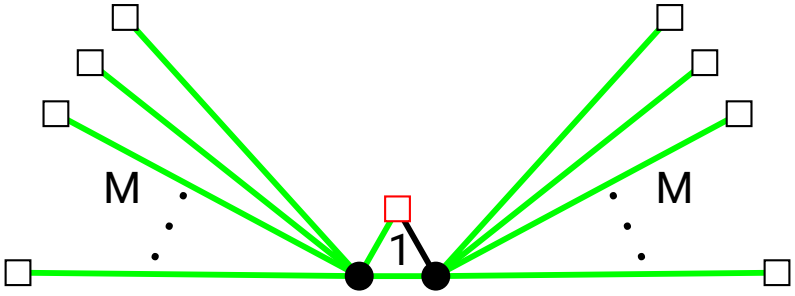

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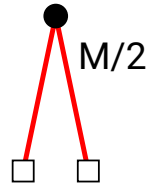
vs

S_i:



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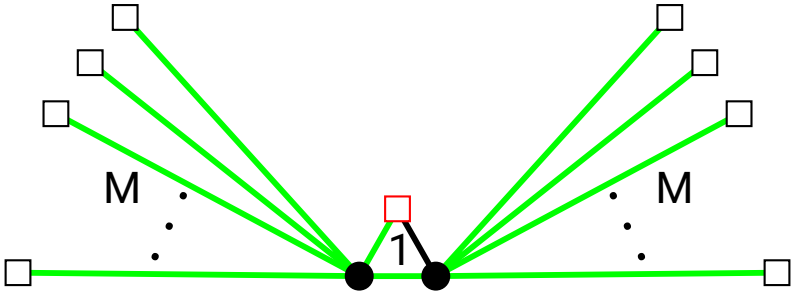

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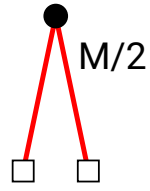

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Definition

A contracted star S_i is *bad* if $w(S_i) > (1+\varepsilon)w(D_i)$.
Otherwise it is *good*.

Charging bad stars against the optimum

Lemma

For the optimum Steiner tree T, parameter $k = |V(T) \setminus R|$, and threshold $\tau = \Theta(k^2/\varepsilon^4)$ we have

$$\sum_{\text{bad } S} w(S) \leq \varepsilon \cdot w(T)$$

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Theorem [Dvořák, F, Knop, Masařík, Toufar, Veselý '18]

The ST problem has a **PAS** for **parameter** $k = |V(T) \setminus R|$, where T is an optimum Steiner tree, with runtime $2^{O(k^2/\varepsilon^4)} n^{O(1)}$

Kernelization

A kernel for parameter $k = |V(T) \setminus R|$

1. reduction: as long as $|R| \geq \tau$ $(= \Theta(k^2/\varepsilon^4))$
   1.1 find star S with smallest ratio (which exists if $|R| > k$)
   1.2 contract S and make resulting vertex a terminal
2. **find a kernel for parameter** $|R|$
3. lifting: uncontract all stars in the solution computed by FPT alg

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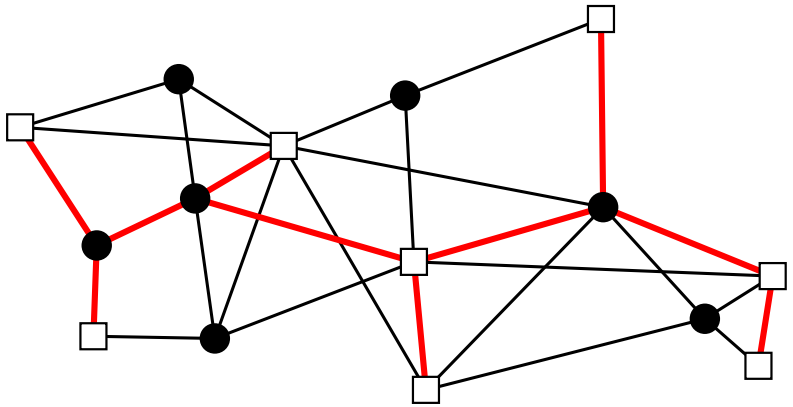
The Borchers & Du Theorem

Approximation of ST:
decomposition into small
instances is near-optimal



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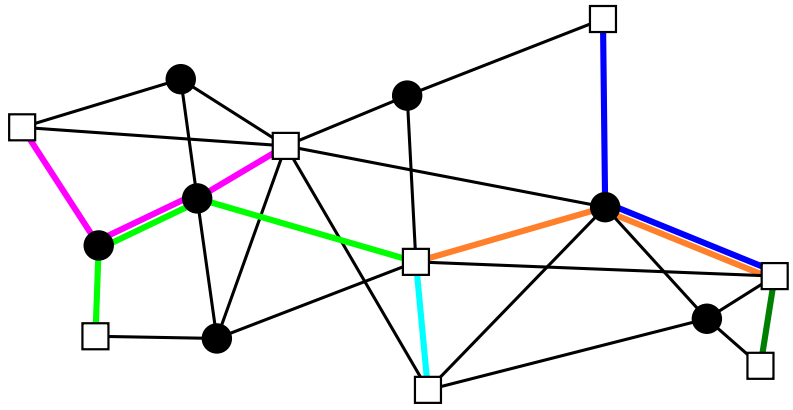

Definition

A *full-component* of a Steiner tree T is a subtree $T' \subseteq T$ for which all leaves are terminals. Let $R(T') = V(T') \cap R$ denote the terminals of T'.

The Borchers & Du Theorem

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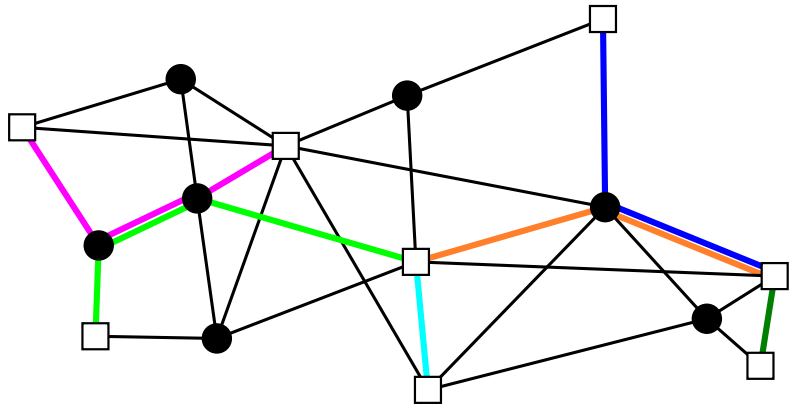

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Theorem [Borchers, Du '97]

For every $\varepsilon > 0$ and optimum Steiner tree T there exists a set $\mathcal{T}$ of full-components of T, such that

- $|R(T')| \leq 2^{O(1/\varepsilon)}$ for every $T' \in \mathcal{T}$
- $\bigcup_{T' \in \mathcal{T}} T' = T$
- $\sum_{T' \in \mathcal{T}} w(T') \leq (1+\varepsilon) w(T)$

Algorithmic applications

Simplifying the input

- for every $R' \subseteq R$ for which $|R'| \leq 2^{O(1/\varepsilon)}$: compute optimum Steiner tree in time $2^{O(|R'|)}n^{O(1)}$ via FPT algorithm

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$\rightarrow$ by Thm: set of precomputed Steiner trees contains a subset $\mathcal{T}$, such that their union is a $(1+\varepsilon)$-approximation

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Consequences

- find a good subset in polynomial time, e.g., via iterative rounding $\rightarrow$ $(\ln(4)+\varepsilon)$-approximation [Byrka, Grandoni, Rothvoss, Sanità '13]

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- sparsify union of precomputed Steiner trees: shortcut degree 2 Steiner vertices & encode edge lengths using $O(\log |R|/\varepsilon)$ bits $\rightarrow$ PSAKS of size $|R|^{2^{O(1/\varepsilon)}}$ [Lokshtanov, Panolan, Ramanujan, Saurabh '17] (vs: ST has no poly sized kernel for $|R|$)

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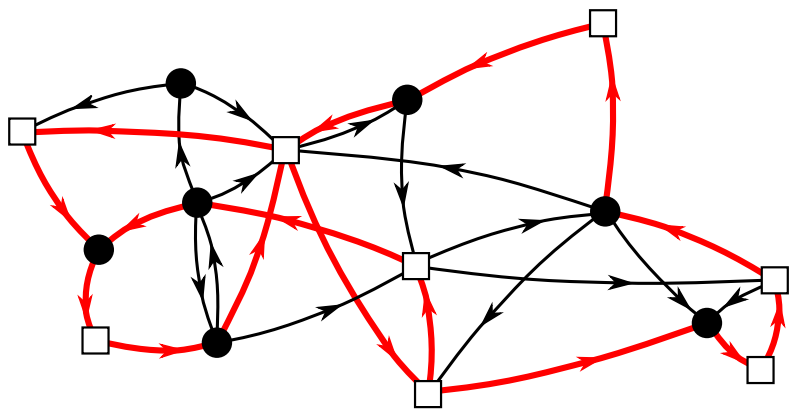
  $\rightarrow$ PSAKS of size $|R|^{2^{O(1/\varepsilon)}}$ [Lokshtanov, Panolan, Ramanujan, Saurabh '17]
  (vs: ST has no poly sized kernel for $|R|$)
- preprocess instance to reduce number of terminals
  $\rightarrow$ PSAKS of size $k^{2^{O(1/\varepsilon)}}$ for $k = |V(T) \setminus R|$ and opt tree T
  [Dvořák, F, Knop, Masařík, Toufar, Veselý '18]
  (vs: ST is W[2]-hard for k)

Technique: Taking Two Easier Solutions

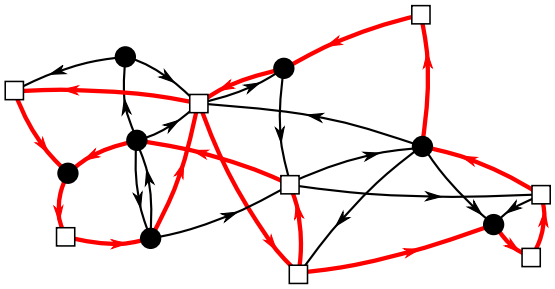
Strongly connected solutions

Strongly Connected Steiner Subgraph (SCSS) problem

- instances: **directed graphs** $G = (V, E)$ with positive edge weights and **terminal set** $R \subseteq V$
- feasible solutions: **strongly connected** subgraphs $N \subseteq G$ containing R
- cost function: total weight of directed subgraph
  $\Rightarrow N$ is **Steiner network**

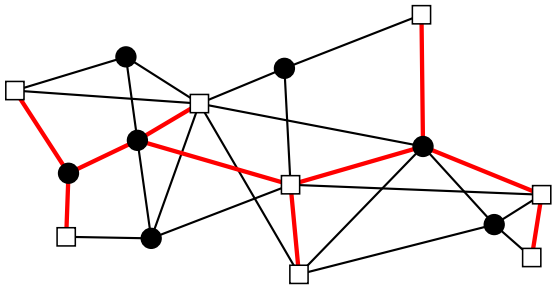


Parameterized vs approximation complexity



SCSS

harder than



Steiner Tree

Parametrized complexity:

Theorem [Guo, Niedermeier, Suchý '11]

SCSS is **W[1]-hard** for **parameter** $|R|$, i.e., no FPT algorithm, unless FPT=W[1].

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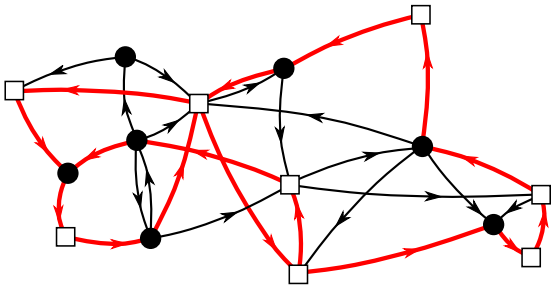

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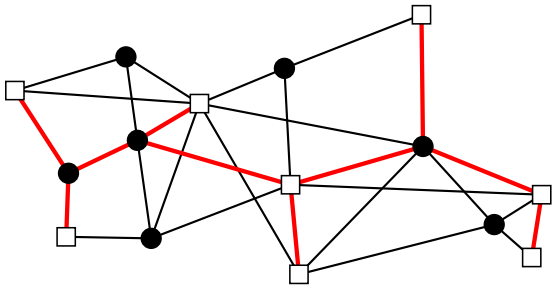

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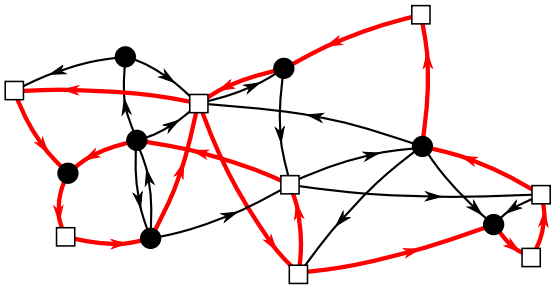

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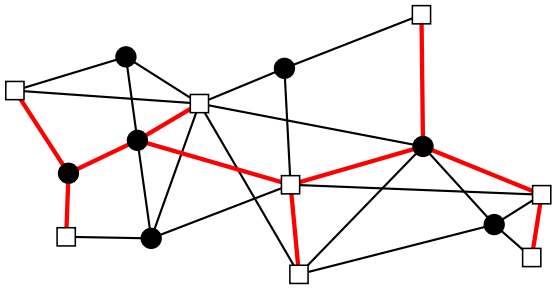
harder than



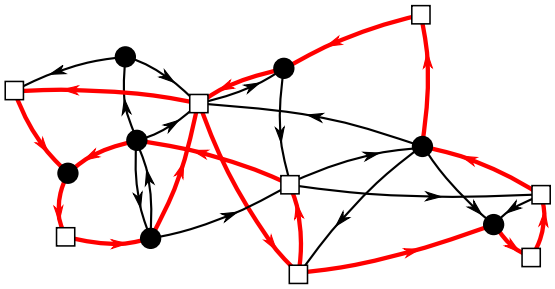
Steiner Tree

Parameterized approximation complexity:

Theorem [Chitnis, Hajiaghayi, Kortsarz '13]

SCSS has a **parametrized** 2-**approximation** algorithm with **runtime** $2^{O(|R|)}n^{O(1)}$.

Parameterized vs approximation complexity



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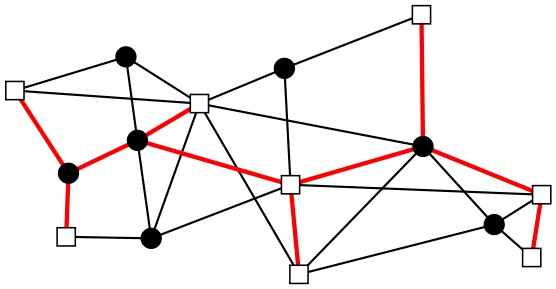

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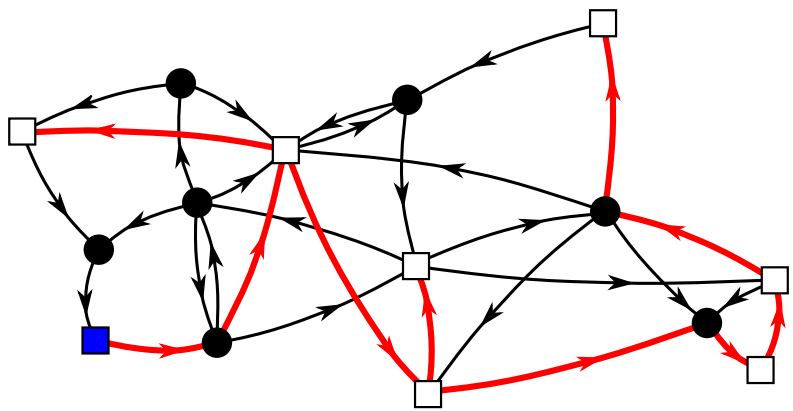
Theorem [Chitnis, F, Manurangsi '18]

SCSS has **no parametrized** $(2-\varepsilon)$-**approximation** algorithm with **runtime** $f(|R|)n^{O(1)}$, for any function f, under Gap-ETH.

Arborescences as solutions

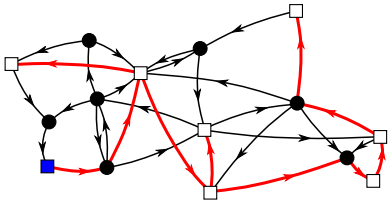
Directed Steiner Tree (DST) problem

- instances: **directed graphs** $G = (V, E)$ with positive edge weights, **terminal set** $R \subseteq V$, and **root terminal** $r \in R$
- feasible solutions: subgraphs $T \subseteq G$ containing $r \to t$ path for every $t \in R$
- cost function: total weight of directed subgraph
  $\Rightarrow T$ is **Steiner arborescence**



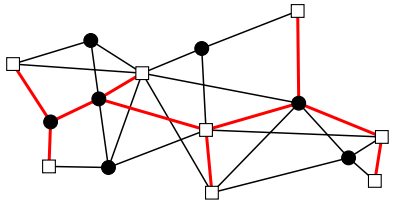
Directed vs undirected graphs

Parametrized complexity:



DST

similar to



Steiner Tree

Theorem [Mölle, Richter, Rossmanith '06]

The DST problem has an **FPT** algorithm for **parameter** $|R|$ with runtime $(2+\delta)^{|R|} n^{O(1)}$ for any constant $\delta > 0$.

Directed vs undirected graphs

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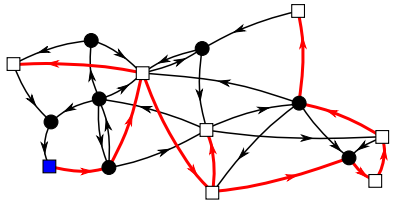
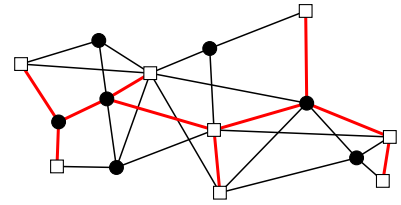


DST

similar to



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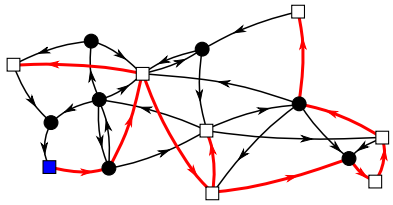

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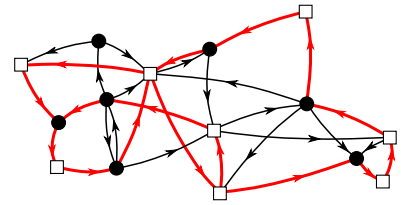
similar to



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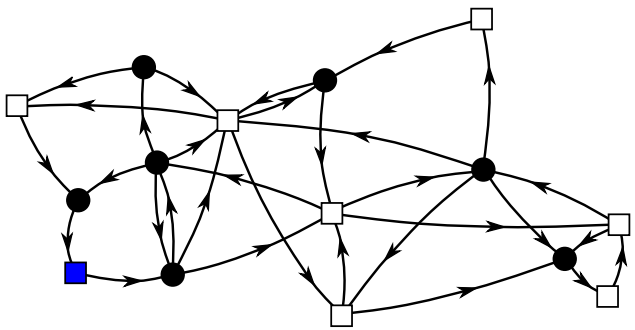
Approximating SCSS via two solutions to DST

Algorithm for directed graph G and terminal set R

1. declare arbitrary terminal to be root r



G with root $r \in R$

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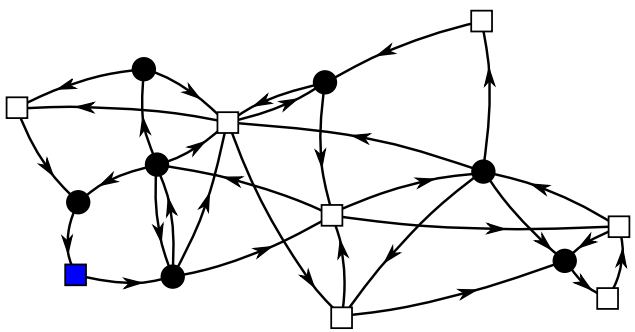
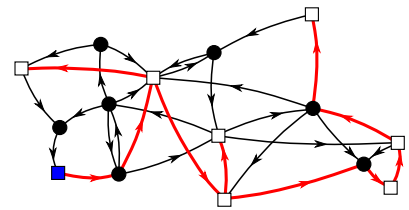


G with root $r \in R$

$\longrightarrow$

T_{out}



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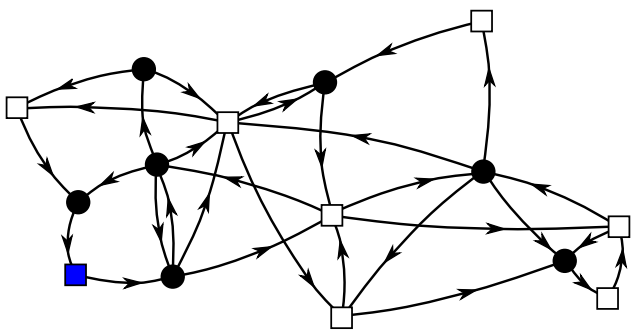

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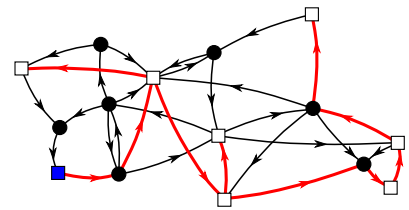
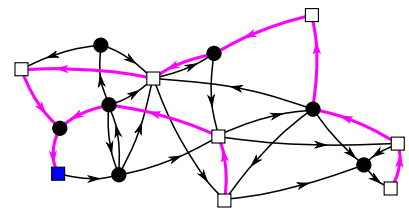


T_{in}



Approximating SCSS via two solutions to DST

Algorithm for directed graph G and terminal set R

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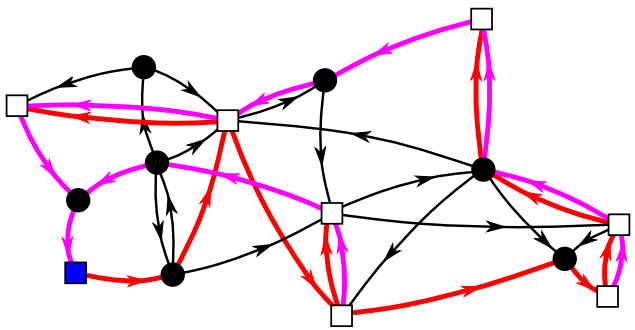
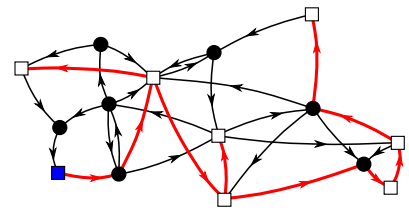
4. output union N of T_{in} and T_{out}



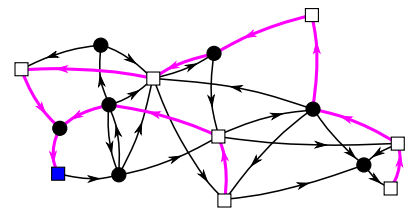
$N = T_{in} \cup T_{out}$

$\longleftarrow$

T_{out}



T_{in}



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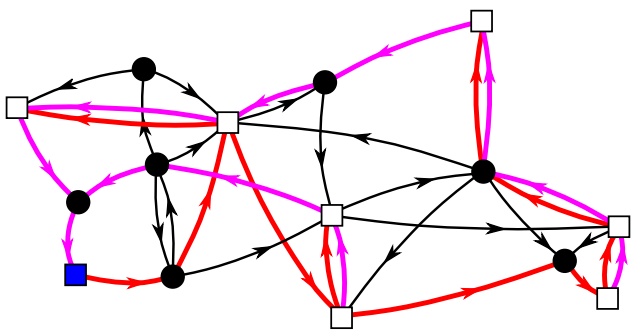

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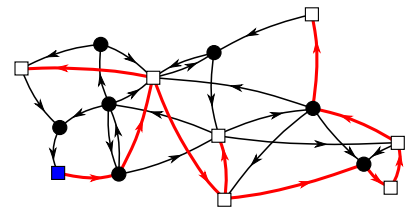
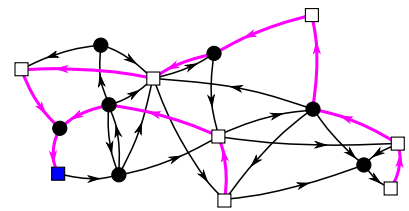


T_{in}



Approximation factor

$w(N) \leq w(T_{in}) + w(T_{out})$

Approximating SCSS via two solutions to DST

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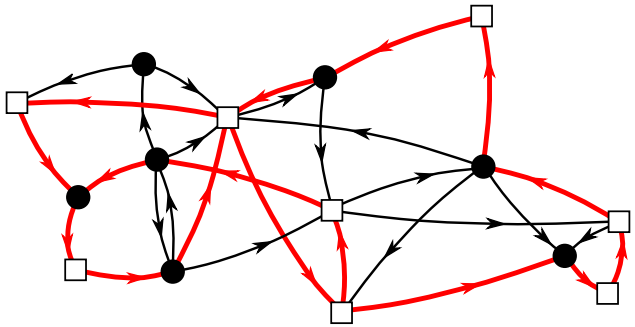
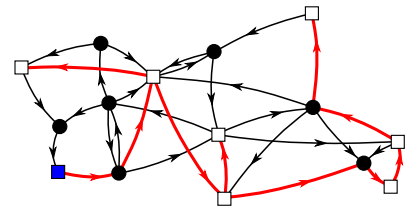


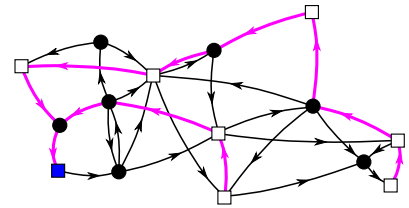
optimum solution N^*

$\geq$

T_{out}



T_{in}



Approximation factor

$$w(N) \leq w(T_{in}) + w(T_{out}) \leq w(N^*) + w(N^*) = 2 \cdot w(N^*)$$

Approximating SCSS via two solutions to DST

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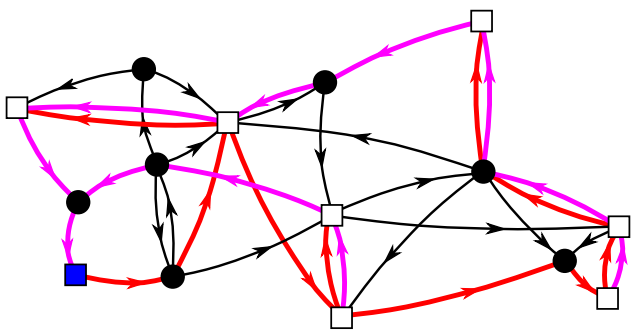

$N = T_{in} \cup T_{out}$

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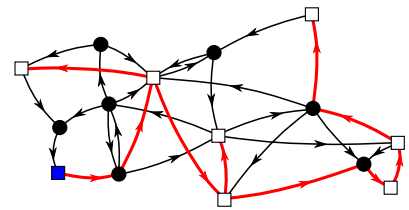
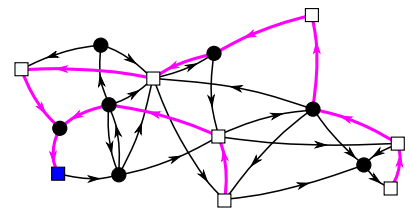
T_{out}



T_{in}



Theorem [Chitnis, Hajiaghayi, Kortsarz '13]

SCSS has a **parametrized** 2-**approximation** algorithm with **runtime** $2^{O(|R|)} n^{O(1)}$.

Technique: Reductions from Densest k-Subgraph

A tight lower bound for SCSS

Theorem [Guo, Niedermeier, Suchý '11]

SCSS is **W[1]-hard** for **parameter** $|R|$.

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The Multicoloured Clique problem

- given: graph $G = (V, E)$ with vertex partition $V_1, \ldots, V_k$
- find: clique on $v_1, \ldots, v_k \in V$ such that $v_i \in V_i$ for every $i \in [k]$
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Reduction to SCSS

- reduce to unweighted graph
- parameter: $|R| = O(k^2)$ (one terminal for each pair $(i, j) \in \binom{k}{2}$)

$\Rightarrow$ no $f(|R|)n^{o(\sqrt{|R|})}$ time alg for SCSS,
since Clique has no $f(k)n^{o(k)}$ time alg under ETH,
i.e., assuming 3SAT cannot be solved in $2^{o(n)}$ time.

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Theorem [Guo, Niedermeier, Suchý '11]

SCSS has **no** $f(|R|)n^{o(|R|/\log|R|)}$ **time** algorithm for any function f, under ETH, i.e., assuming 3SAT cannot be solved in $2^{o(n)}$ time.

The Subgraph Isomorphism problem

- given: graph $G = (V, E)$ with vertex partition $V_1, \ldots, V_k$, and graph H on k vertices
- find: $H' \subseteq G$ isomorphic to H s.t. $\forall i \in [k] : |V(H') \cap V_i| = 1$
- parameter: $p = |E(H)|$

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Reduction to SCSS: (almost) the same as from Clique

- reduce to unweighted graph
- parameter: $|R| = O(p)$ (one terminal for each edge of H)

$\Rightarrow$ under ETH: no $f(|R|)n^{o(|R|/\log|R|)}$ time alg for SCSS, since Subgraph Isomorphism has no $f(p)n^{o(p/\log p)}$ time alg [Marx '07] (open: get rid of $\log p$)

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Theorem [Chitnis, F, Manurangsi '18]

SCSS has **no parametrized** $(2-\varepsilon)$**-approximation** algorithm with **runtime** $f(|R|)n^{O(1)}$, for any function f, under Gap-ETH, i.e, assuming no $2^{o(n)}$ time algorithm can distinguish if all or at most a $1-\delta$ fraction of 3SAT clauses are satisfiable, for some constant $\delta > 0$.

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The Densest k-Subgraph (DkS) problem

- given: graph $G = (V, E)$ with vertex partition $V_1, \ldots, V_k$
- find: subgraph $H \subseteq G$ with max nr of edges s.t. $\forall i \in [k] : |V(H) \cap V_i| = 1$
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also known as: Binary CSP or Label Cover

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Theorem [Dinur, Manurangsi '18]

Under Gap-ETH, no $f(q)n^{O(1)}$ time algorithm can distinguish between the following two cases for DkS:

- (completeness) there is H with $|E(H)| = q$
- (soundness) every H has $|E(H)| \leq q/k^{1-o(1)}$

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Reduction to SCSS: similar as from Subgraph Isomorphism

- reduce to **weighted** graph
- parameter: $|R| = O(q)$ (one terminal for each potential edge of H)

$\Rightarrow$ under Gap-ETH: no parameterized $2-\varepsilon$-approximation alg with runtime $f(|R|)n^{O(1)}$ for SCSS

A tight lower bound for SCSS

Theorem [Chitnis, F, Manurangsi '18]

SCSS has no parametrized $(2-\varepsilon)$-approximation algorithm with **runtime** $f(|R|)n^{o(|R|)}$, for any function f, under

randomized Gap-ETH, i.e, assuming no **randomized** $2^{o(n)}$ time algorithm can distinguish if all or at most a $1-\delta$ fraction of 3SAT clauses are satisfiable, for some constant $\delta > 0$.

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Theorem [Manurangsi '20]

Under randomized Gap-ETH, for **any constant** $\alpha > 0$ **no** $f(q)n^{o(q)}$ **time** algorithm can distinguish between the following two cases for DkS:

- (completeness) there is H with $|E(H)| = q$
- (soundness) every H has $|E(H)| \leq \alpha q$

Outlook

Approximation factors

Tight polynomial time approximation factors:

1. approx schemes: e.g. KNAPSACK, ...
2. constant factors: e.g. STEINER TREE, ...
3. (poly-)logarithmic factors: e.g. SET COVER, ...
4. polynomial factors: e.g. CLIQUE, ...

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2. constant factors: (i) SCSS param $|R|$, (ii) Directed Odd Cycle Transversal param solution size
3. (poly-)logarithmic factors: ?
4. polynomial factors: ?
5. no reasonable factor: e.g. Dominating Set param solution size, ... (no $g(k)$-approx in $f(k)n^{o(k)}$ time for any g, f under ETH)

Running times

Tight parameterized runtimes:

1. sub-exponential $2^{O(\sqrt{k})}n^{O(1)}$: e.g. Planar Vertex Cover, ...
2. single exponential $2^{O(k)}n^{O(1)}$: e.g. Vertex Cover, ...
3. double exponential $2^{2^{O(k)}}n^{O(1)}$: e.g. Edge Clique Cover, ...
4. sub-exponential $n^{O(\sqrt{|R|})}$: e.g. Planar SCSS, ...
5. exponential $n^{O(|R|)}$: e.g. SCSS, ...
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none ... (only some for algorithms with runtime exponential in n)

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Example of open problem: SCSS

- 2-approximation in $2^{O(|R|)}n^{O(1)}$ time
- no $(2-\varepsilon)$-approximation in $f(|R|)n^{O(1)}$ time, under Gap-ETH

$\rightarrow$ can you show: no $2^{o(|R|)}n^{O(1)}$ time alg to compute 2-approximation?

Thanks for your attention!

Summary

- **preprocessing algorithm:** parametrized approximation scheme for ST with parameter $|V(T) \setminus R|$, where T is opt Steiner tree
- **take two easier solutions:** parametrized 2-approximation for SCSS with parameter $|R|$
- **reduction from Densest k-Subgraph:** no $(2-\varepsilon)$-approximation for SCSS with parameter $|R|$, under Gap-ETH
- **outlook:**
  1. tight approximation factors
  2. tight runtime bounds