LP-guided branching, part 1: The Multiway Cut problems

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Multiway Cut

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Question: Is there a set of edges F with $|F| \leq k$, such that every path between two terminals is hit by F?

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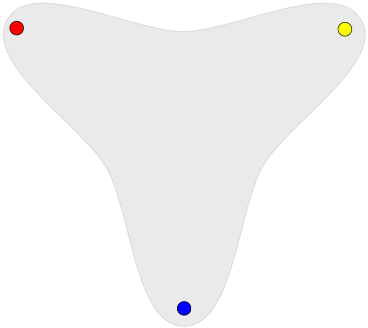
Question: Is there a set of vertices $X \subseteq V(G) \setminus T$ with $|X| \leq k$, s.t. every path between two terminals is hit by X?

On the picture



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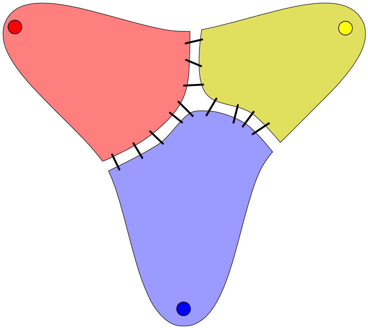

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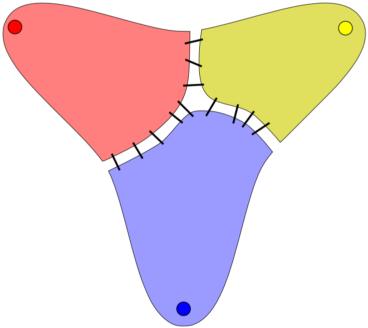

If G is connected and F is optimum,
then every vertex is reachable from some terminal.

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  - This algorithm is due to Xiao.

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- **Note**: If (A, B) is a minimum cut, then A and B are connected. Hence $A = \texttt{reach}(s, G \setminus F)$, where F is the cutset.

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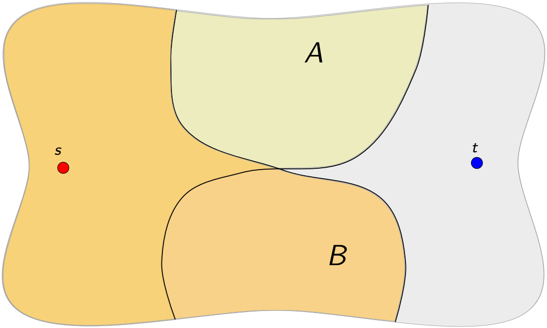

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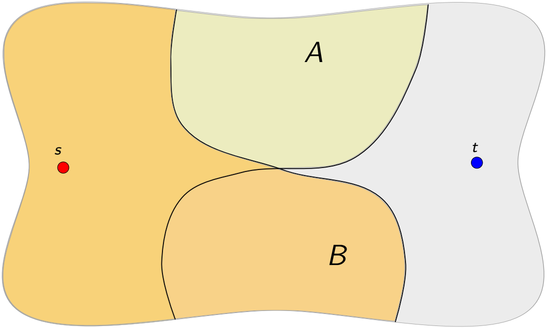

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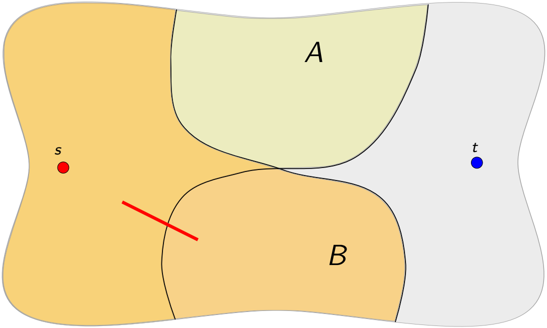

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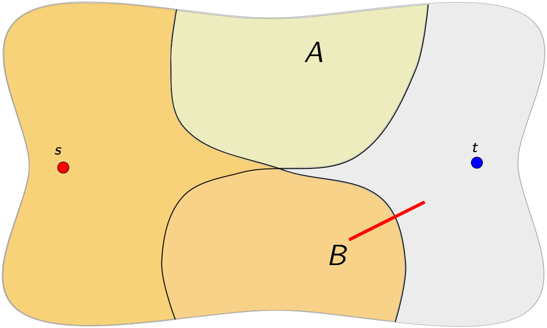

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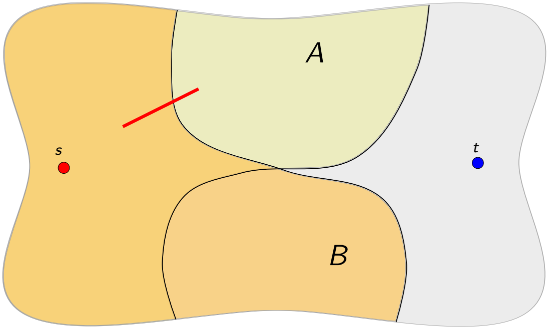

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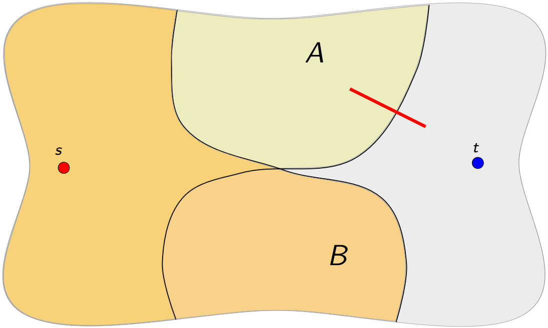

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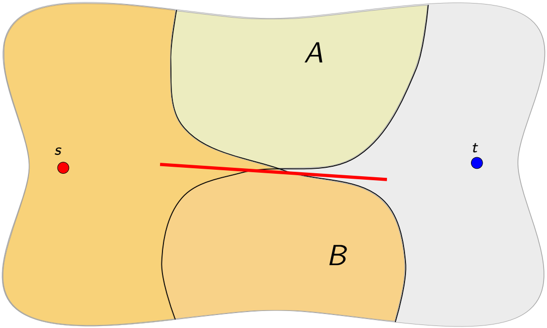

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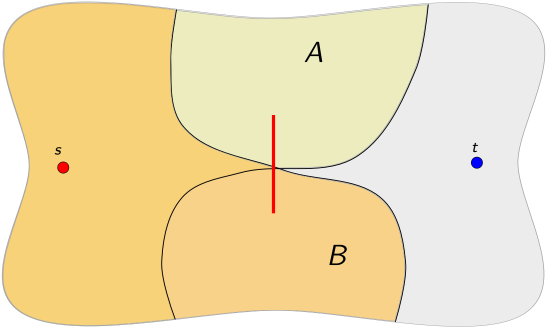

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Furthest and closest cuts

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- Symmetrical reasoning for $A \cap B$. □

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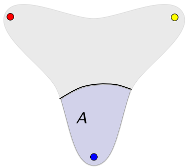
- Naturally generalizes to S and T being sets of sources and sinks.

Min-cut reduction for EDGE MULTIWAY CUT

- Pick a terminal t, and let A be the $(\{t\}, T \setminus \{t\})$ min-cut that is furthest from t.



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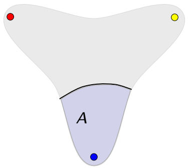
There exists an optimal solution to the instance that does not include any edge with both endpoints in A.



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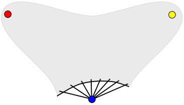
- **Note**: If the lemma is true, then it is safe to contract the whole set A onto t.



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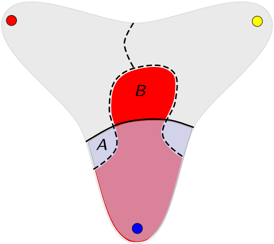

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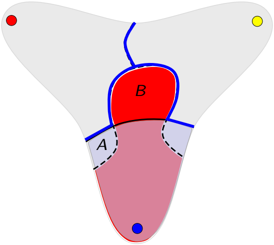

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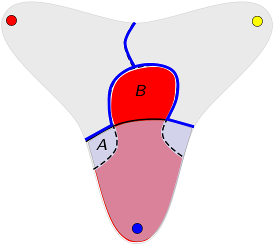

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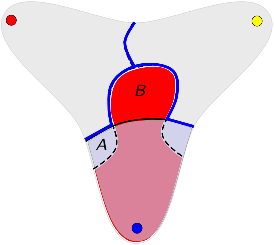
  - **Check (2)**. F' is still a solution.



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- $P[t', u]$ avoids $F' \supseteq \Delta(A \cup B)$.
- Ergo there is a t'-t path avoiding $\Delta(A \cup B)$, a contradiction.

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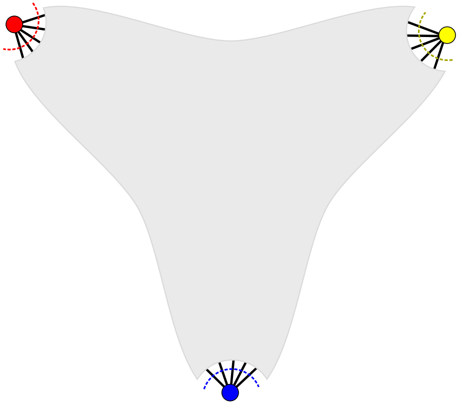

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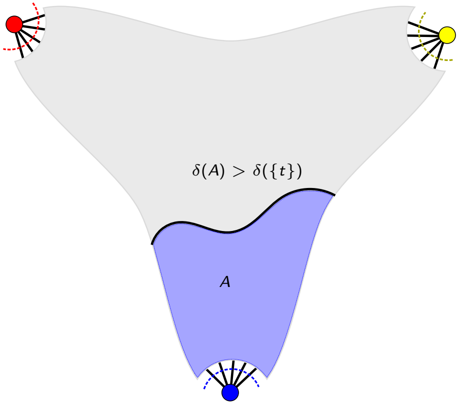
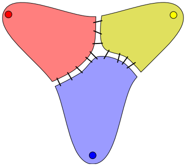


Digression: approximation

- Let F be a solution and $d_i = \delta(\mathtt{reach}(t_i, G \setminus F))$. Then $|F| = \frac{1}{2}\sum_i d_i$.



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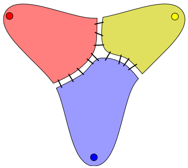

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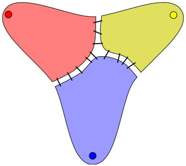

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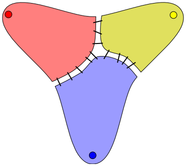

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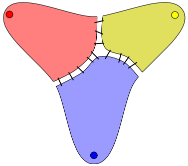
- If we omit the largest C_i, we get $(2 - \frac{2}{|T|})$-approximation.



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- Let $\phi(G, T, k) := k - \lambda(G, T) = k - \frac{1}{2}\sum_i c_i$. In a nontrivial YES-instance we have $0 \le \phi(G, T, k) < \frac{k}{2}$

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- **Step 3**. Proceed with Steps 1 and 2 up to the point when every terminal becomes isolated (YES), or $\phi(G, T, k)$ becomes negative (NO).

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- Hence, the potential decreases by exactly $1 - \frac{1}{2} = \frac{1}{2}$.

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- Hence we get an $\mathcal{O}^\star(2^k)$ algorithm.
- **Crucial point**: A branching rule can lead to some progress, even if this progress is not visible in the budget.

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  - **Min-cut rule**: Contractions can only increase min-cuts, so the potential can only decrease.
  - **Second rule**: k decreases by 1, $\sum_i c_i$ decreases by 2 $\Rightarrow$ the potential stays the same.

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  - Based on a joint work with Cygan and Wojtaszczyk.

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- We need a smarter lower bound.

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- The goal is to find a vector x that minimizes/maximizes $\sum_{i=1}^{n} c_i x_i$ while satisfying all the constraints.

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- **Usage**: Model a problem as an integer program, and relax the integer constraints to linear ones. The solution to the relaxation is a lower bound for the solution to the integer program.

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- **Separation oracle**: Dijkstra with vertex weights.

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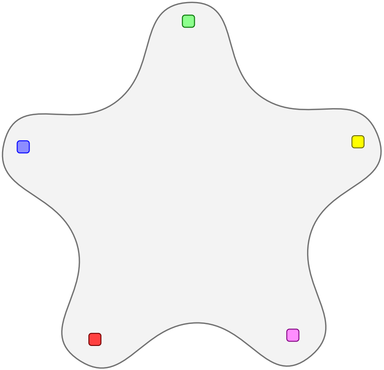
- **Note**: Self-reducibility $\Rightarrow$ We can find half-integral solution in polynomial time.
- **Note**: Obviously $OPT_{\mathrm{LP}} \leq OPT$, but also $OPT \leq 2 \cdot OPT_{\mathrm{LP}}$, since we can round all the halves up to ones. So this rounding yields a 2-approximation.

Structure of the solution



Take any half-integral optimum solution F.

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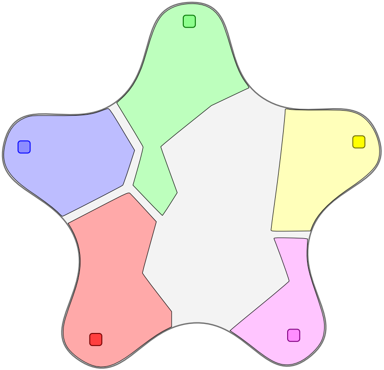

For $t \in T$, the *zero-region* U_t comprises vertices reachable from t using paths of weight 0.

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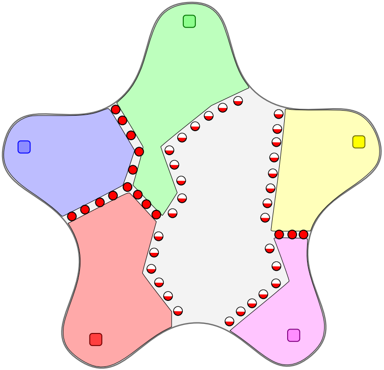

Define F' by putting 1-s on vertices that see ≥ 2 zero-regions, and $\frac{1}{2}$ on those seeing 1 region.

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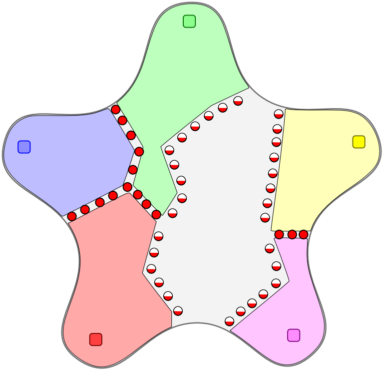

Observe: F' is still a solution,
and $F'(u) \leq F(u)$ for each u.

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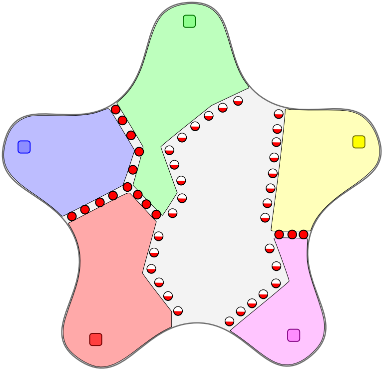

Conclusion: $F = F'$

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Structure of the solution to the relaxation

Every optimum half-integral solution F has the following form:

- $F(u) = 1$ if u is in the neighbourhood of two or more zero-regions.
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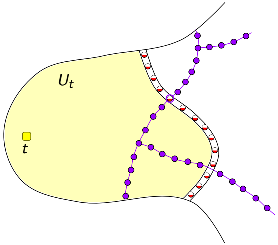
Reduction of Guillemot

There is always an optimum solution of NMWC that does not touch any U_t. Hence, it is safe to contract every region U_t onto t.

Proof of the reduction

- Assume for simplicity that the LP solution does not use any ones.



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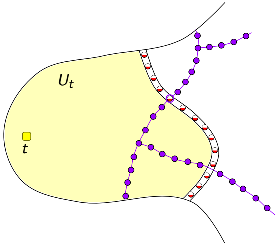

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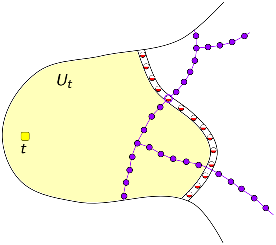

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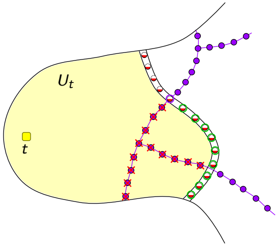
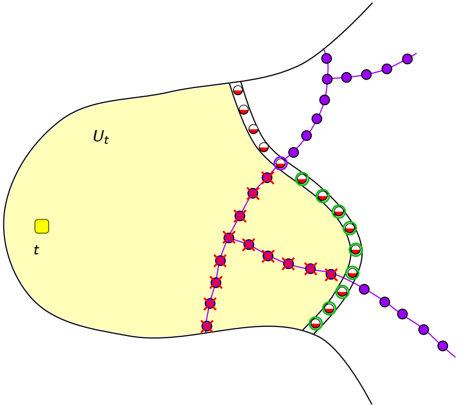
- Replace B with C, i.e., consider $X' := (X \setminus B) \cup C$.



Verifying X'

- Again, we check two things.



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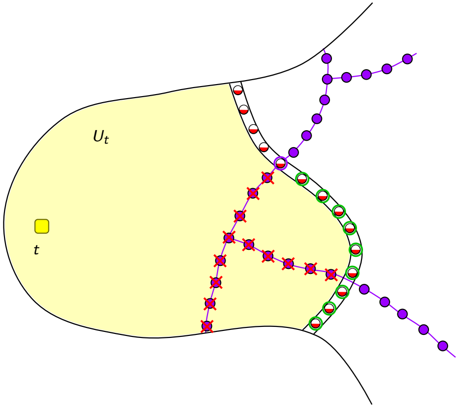

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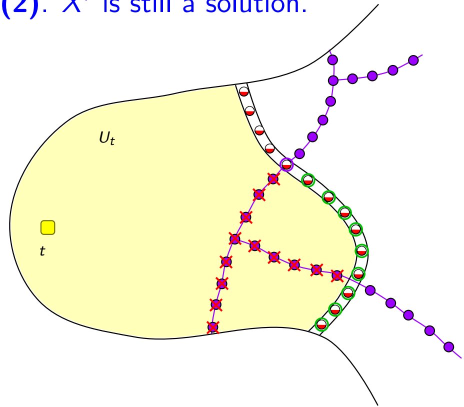

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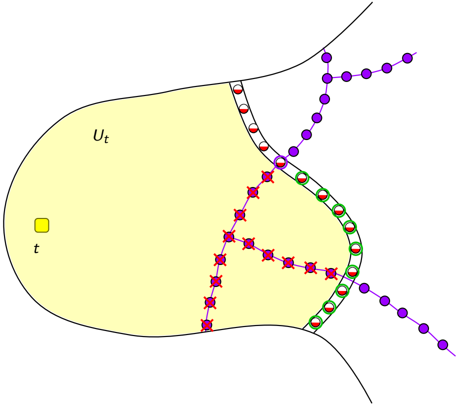

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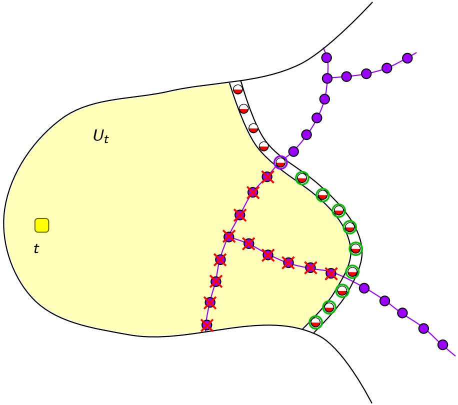

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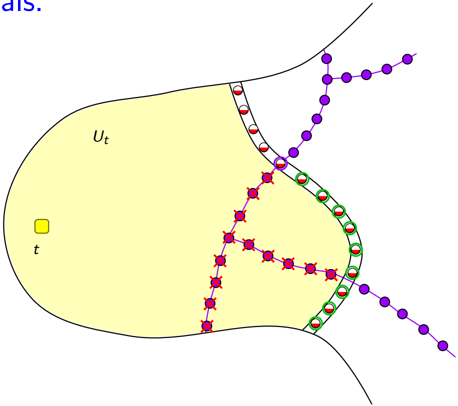
- Contradiction with $N(U_t \cup \texttt{reach}(t, G \setminus X))$ separating t from other terminals.



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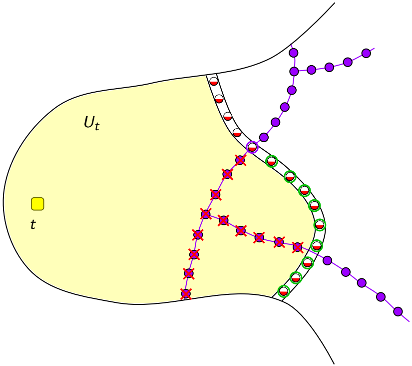

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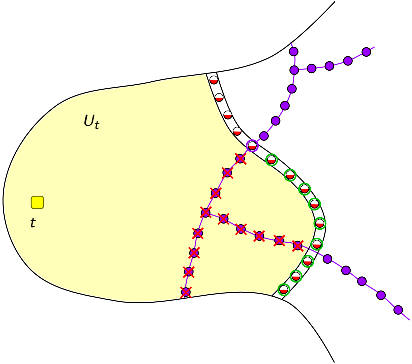

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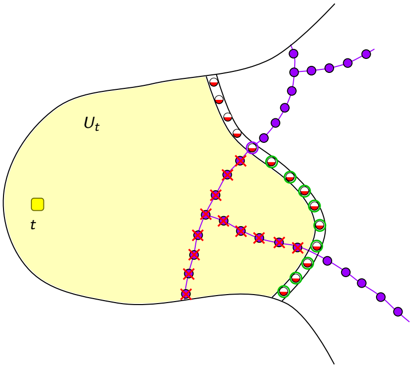
- The obtained F' is still a solution, and has strictly lower cost.



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- Such non-terminals must be always chosen.
- **Corollary**: Safe to greedily choose vertices assigned 1 by the LP.
- After applying both rules exhaustively, we can assume that the **only** half-integral optimum solution to LP assigns $\frac{1}{2}$ on each $N(t)$, for $t \in T$.

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- **Step 3**. Proceed with Steps 1 and 2 up to the point when every terminal becomes isolated (YES), or the LP solution exceeds the budget (NO).

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- Hence, potential ϕ decreases by exactly $1 - \frac{1}{2} = \frac{1}{2}$.

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- Progress is achieved not only by decreasing the budget, but also by increasing the lower bound.
- Solution to an LP relaxation may be a more robust lower bound than a combinatorial object.

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- **Tomorrow**: Applications of the same concept to Vertex Cover above Maximum Matching and other problems.