LP branching, part III: using CSP tools

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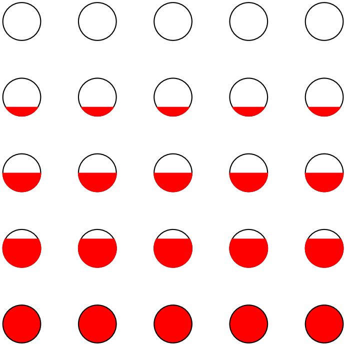
Finse, March 2014

Vertex Cover LP: recap



Vertex Cover LP:

$$\min \sum_{v \in V(G)} x_v$$

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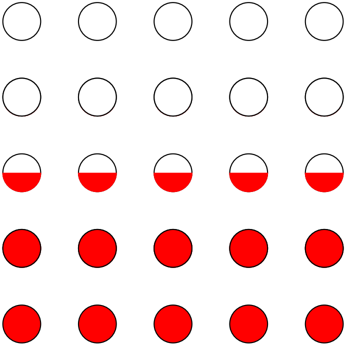

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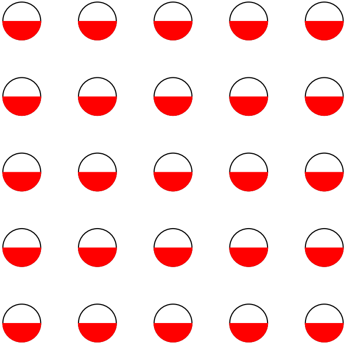

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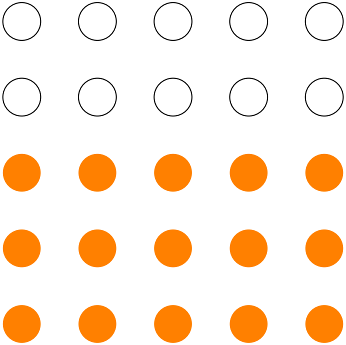

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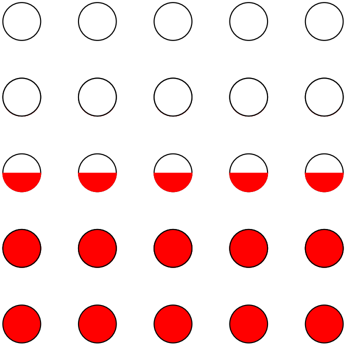

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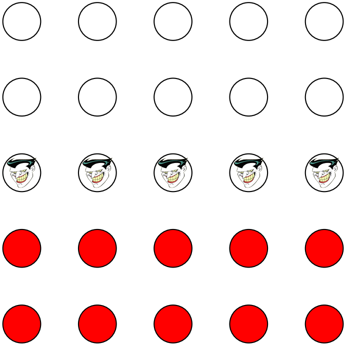

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Pay 1 for • and 0.5 for .

Edge covered by one • or two .

Valued CSP

A Valued Constraint Satisfaction Problem Γ consists of

- a finite domain Σ; and
- a set of functions $f : \Sigma^r \to \mathbb{R}$; r is the arity of f.

An input consists of

- variables $X = \{x_1, x_2, \ldots, x_n\}$; and
- a set of applications $f(x_{i_1}, x_{i_2}, \ldots, x_{i_r})$ of functions of Γ.

Goal: find $\phi : X \to \Sigma$ minimizing

$$\sum_{\text{application } f} f(\phi(x_{i_1}), \ldots, \phi(x_{i_r})).$$

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1. $f_1 : \Sigma \to \mathbb{R}$, $f_1(\bullet) = 1$, $f_1(\circ) = 0$;
   take $f_1(x_v)$ for each $v \in V(G)$.
2. $f_2 : \Sigma^2 \to \mathbb{R}$, $f_2(\circ, \circ) = \infty$, and $f_2 = 0$ otherwise;
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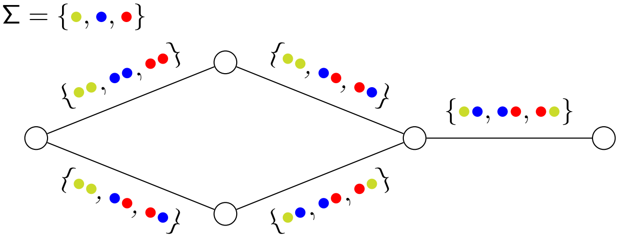
Unique Label Cover

Unique Label Cover (ULC)

Input: Graph G, alphabet Σ, and for each $e \in E(G)$, $v \in e$ a permutation $\psi_{e,v}$ of Σ such that $\psi_{uv,v} = \psi_{uv,u}^{-1}$ for each $uv \in E(G)$.
Task: Find an assignment $\Psi : V(G) \to \Sigma$ that minimizes the number of edges uv for which $\psi_{uv,v}(\Psi(v)) \neq \Psi(u)$.



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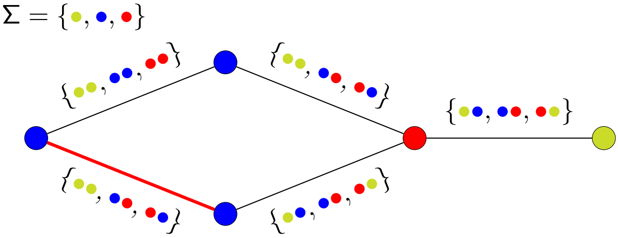

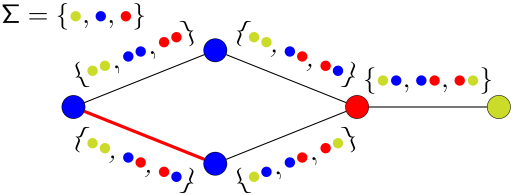

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- The alphabet Σ is the domain.
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- Add  to Σ.
- Pay $\frac{1}{2}$ for each .
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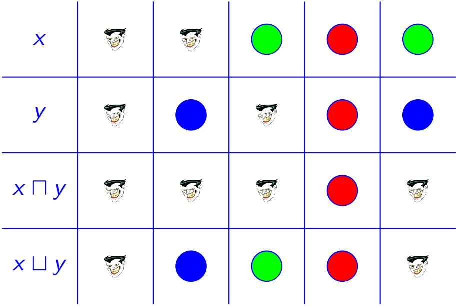

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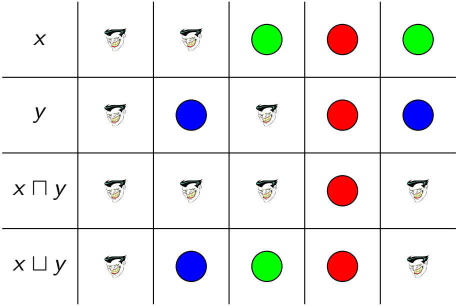

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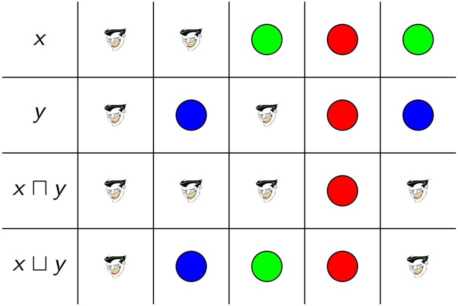

- Extend $\sqcap$ and $\sqcup$ to r-dimensional vectors coordinate-wise.
- A function $f : (\Sigma \cup \{$$\})^r \to \mathbb{R}$ is $|\Sigma|$-submodular if for any $x, y \in (\Sigma \cup \{$$\})^r$ we have

$$f(x) + f(y) \geq f(x \sqcap y) + f(x \sqcup y).$$

Relaxed ULC is in P

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Question: What else do we need for branching?

Answer: Persistence.
If $\textsc{Relaxed ULC}$ sets $x_v \neq$ , we want greedily to fix the value x_v.

Persistence of s-submodular constraints

Persistence is for free from $|\Sigma|$-submodularity!

Let ϕ be integral optimum and ϕ^* the relaxed one.

$$\begin{aligned} f_\pi(\phi) + 2f_\pi(\phi^*) &\geq f_\pi(\phi \sqcap \phi^*) + f_\pi(\phi \sqcup \phi^*) + f(\phi^*) \\ &\geq f_\pi(\phi \sqcap \phi^*) + f_\pi((\phi \sqcup \phi^*) \sqcap \phi^*) + f_\pi((\phi \sqcup \phi^*) \sqcup \phi^*) \\ &\geq 2f(\phi^*) + f((\phi \sqcup \phi^*) \sqcup \phi^*) \end{aligned}$$

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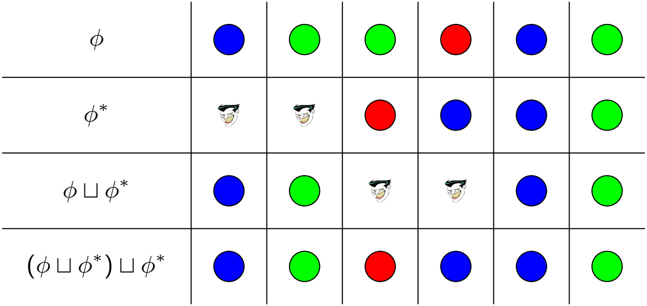

ULC branching

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Theorem (Wahlström 2013)

UNIQUE LABEL COVER *can be solved in time* $\mathcal{O}^*(|\Sigma|^{2k})$*, where* k *is the number of unsatisfied constraints.*

Extensions

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Theorem (Wahlström 2013)

The vertex-deletion variant of UNIQUE LABEL COVER *is solvable in time* $\mathcal{O}^*(|\Sigma|^{2k})$.

GROUP FEEDBACK VERTEX SET *is solvable in time* $\mathcal{O}^*(4^k)$.