A Study of Weisfeiler–Leman Colourings on Planar Graphs

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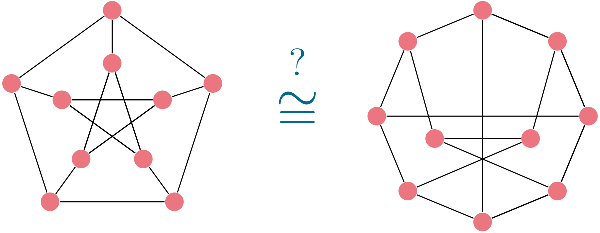
Graph Isomorphism Problem

Given two graphs G, H, decide whether $G \cong H$ or $G \ncong H$.



$$\{u, v\} \in E(G) \iff \{\varphi(u), \varphi(v)\} \in E(H)$$

The best known algorithm runs in quasipolynomial time. [Babai '16]

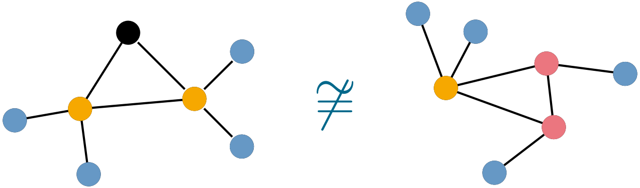
A central technique in GI approaches is the Weisfeiler-Leman algorithm.

The WL algorithm

…is a combinatorial iterative approach to finding symmetries in graphs.

It uses local information to restrict the search space for isomorphisms.

Goal: Assign different colours to $u \in V(G)$ and $v \in V(H)$
iff no isomorphism maps u to v.



k-WL colours vertex k-tuples. It has an $O(n^{k+1} \log n)$-time implementation.

[Immerman, Lander '90]

Distinguishability of graphs by k-WL gives bounds on the descriptive complexity of their difference.

Colour refinement

1-WL iteratively computes a vertex colouring.

1-WL

- *Initialisation*: All vertices have their initial colours.
- *Refinement*: Recolour vertices depending on colours in their neighbourhoods.
- *Stop* when colouring is stable.

1-WL has an $O((m+n)\log n)$ implementation.

[McKay '81; Cardon, Crochemore '82]

If two graphs result in different colourings, they are non-isomorphic.

Colour refinement

1-WL iteratively computes a vertex colouring.

1-WL

- *Refinement*: equally coloured v and w obtain different colours $\iff$ there is a colour c such that v and w have different numbers of c-coloured neighbours

If two graphs result in different colourings, they are non-isomorphic.

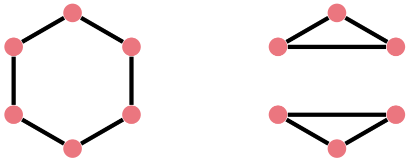
Planar graphs

1-WL identifies almost all graphs. [Babai, Erdös, Selkow ’80]

Theorem (K., Schweitzer, Selman 2015 *)

1-WL identifies G. $\iff$ The flip of G is a bouquet forest.

But it fails to identify, for example, all planar graphs.

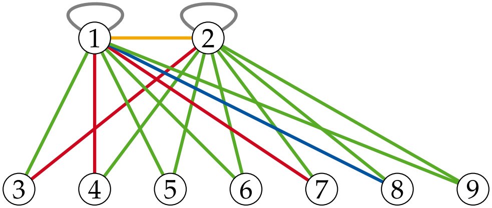


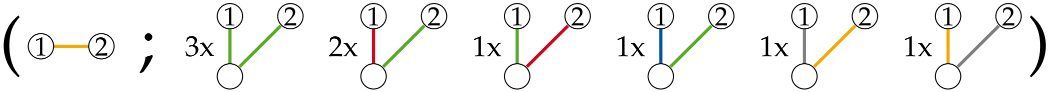
Theorem (K., Ponomarenko, Schweitzer 2017)

3-WL identifies all planar graphs.

2-WL

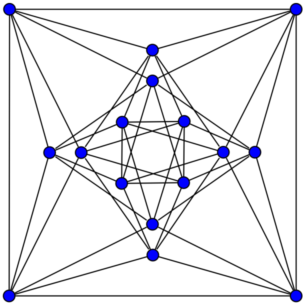
- *Refinement*: (v, w) and (v', w') obtain different colours. $\Longleftrightarrow$
  A certain local refinement criterion holds.





Hard Examples

A *strongly regular graph* $\mathrm{srg}(n, d, \lambda, \mu)$ is a d-regular graph with n vertices such that every two adjacent vertices have exactly λ common neighbors and every two non-adjacent vertices have exactly μ common neighbors.



The Shrikhande graph and the line graph of $K_{4,4}$ are non-isomorphic examples for $\mathrm{srg}(16, 6, 2, 2)$.

Facts about 2-WL

2-WL is the original algorithm by Weisfeiler and Leman. [Weisfeiler, Leman '68]

2-WL does not distinguish strongly regular graphs with equal parameters.

2-WL identifies all graphs of colour class size at most 3. [Immerman, Lander '90]

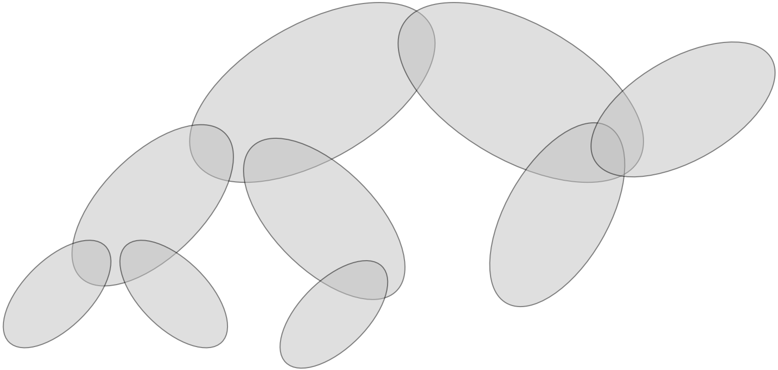
2-WL identifies

- interval graphs. [Evdokimov, Ponomarenko, Tinhofer '00]
- distance-hereditary graphs. [Gavrilyuk, Nedela, Ponomarenko '20]
- almost all regular graphs. [Bollobás '82]

2-WL detects

- (and counts) certain small subgraphs. [Fürer '17]
- 2-separators. [K., Neuen '19]

Decompositions



Reduction scheme:

1. planar $\leq$ vertex-coloured 2-connected planar **2-WL**
2. vertex-col. 2-connected planar $\leq$ arc-col. 3-connected planar **2-WL**

Our results

We investigate the colourings that 2-WL computes on planar graphs.

Understanding 2-WL on planar graphs amounts to studying it on 3-connected ones.

Theorem

For every 3-connected planar graph G, one of the following holds.

1. *2-WL identifies G, or*
2. *2-WL detects a matching in G, or*
3. *2-WL detects a well-understood connected subgraph in G.*

As a main tool, we use the following classification.

Theorem

A planar graph is edge-transitive. $\iff$ *All edges have the same 2-WL colour.*

Edge-coloured planar graphs of min deg 3

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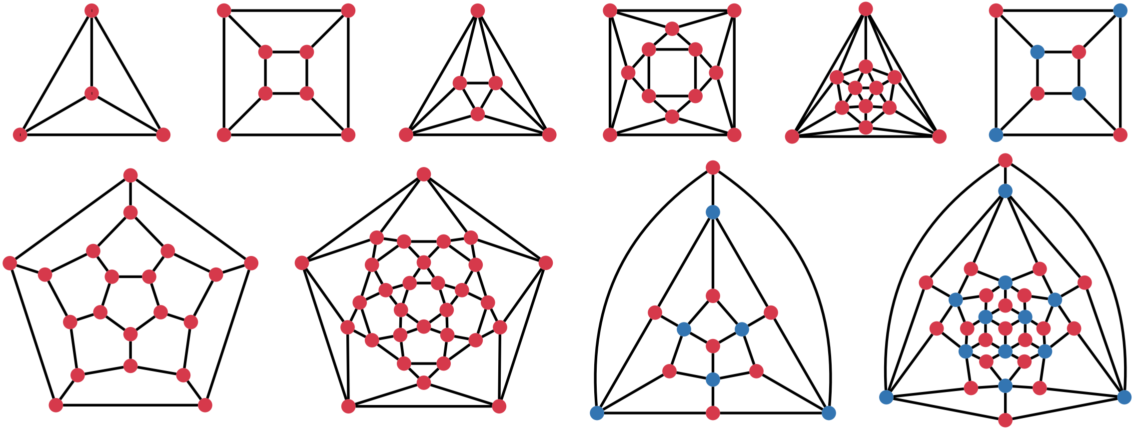
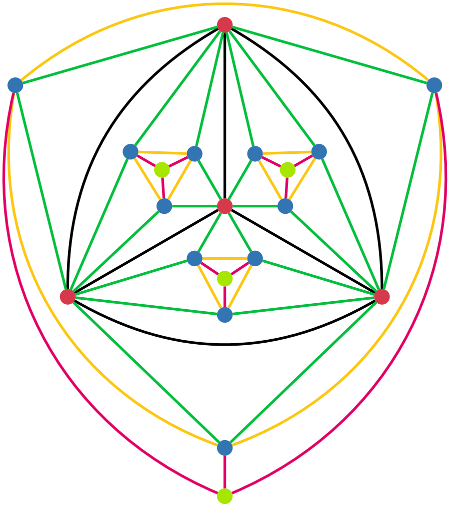


Graphs induced by one edge colour

How do graphs induced by a single edge colour look?



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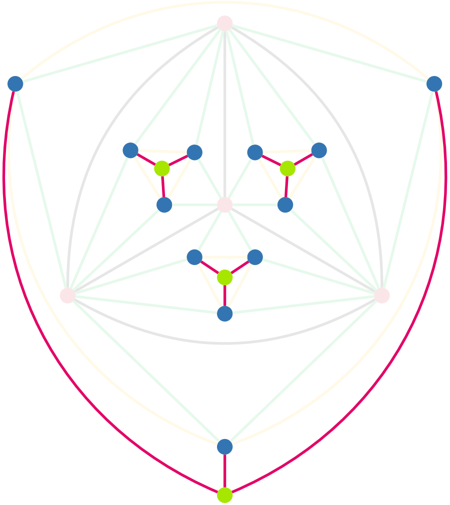

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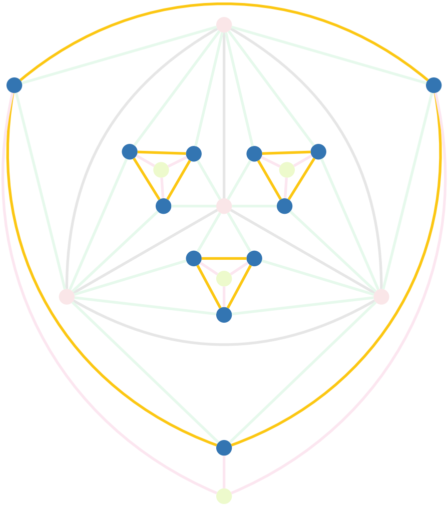

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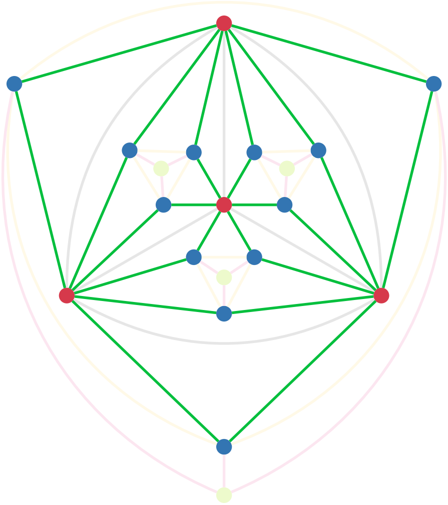

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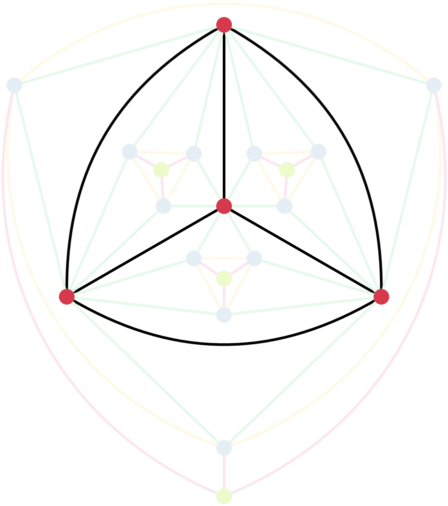


Theorem

A planar graph is edge-transitive.

$\iff$

All edges have the same 2-WL colour.

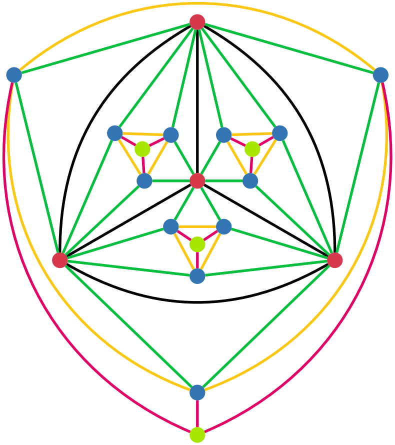
Edge types

Assume G is 3-connected.

Consider $G[c]$ for a 2-WL edge color c.

Then all components of $G[c]$ have the same numbers of vertices, edges, and faces.

We distinguish between three types for c.



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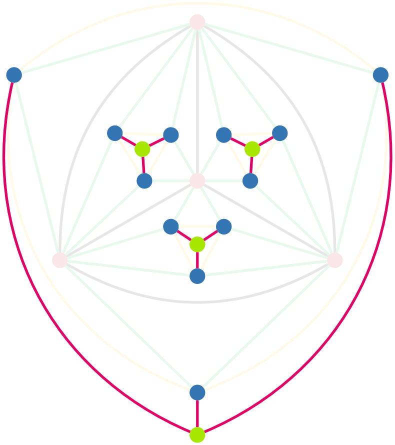
1. $G[c]$ has only one face.
   $\rightsquigarrow$ *$G[c]$ is a disjoint union of stars.*



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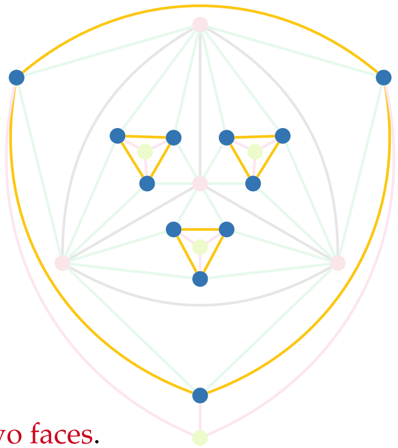

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1. $G[c]$ has only one face.
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2. Every connected component of $G[c]$ has exactly two faces.
   $\rightsquigarrow$ *$G[c]$ is a disjoint union of ℓ-cycles for some ℓ.*

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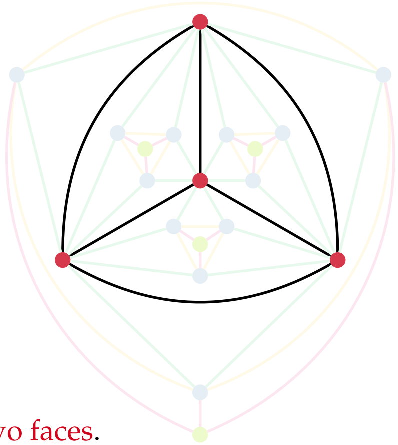

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3. Every connected component of $G[c]$ has at least three faces.

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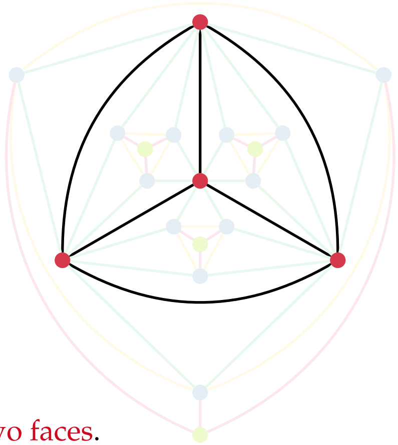

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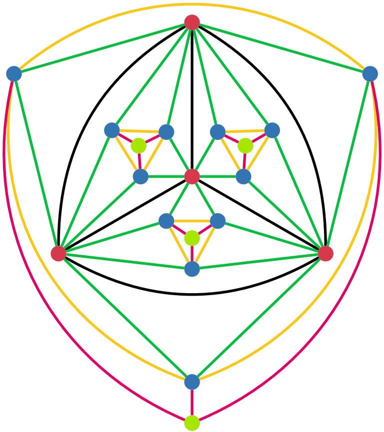
3. **Every connected component of $G[c]$ has at least three faces.**

Graphs of type 3

Assume there is a c such that every component of $G[c]$ has at least three faces.



Lemma

Let G be a 3-connected planar graph and c be an edge colour of Type 3.

Then $G[c]$ is connected.

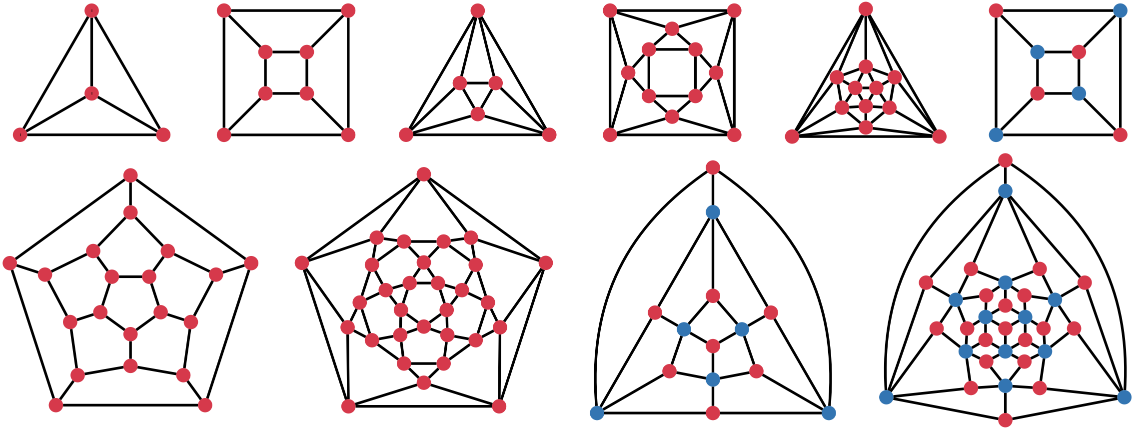
We obtain a precise classification of the graphs $G[c]$ for c of Type 3.

It includes the connected edge-transitive planar graphs of minimum degree 3 – in particular, all Platonic solids.

Graphs induced by one edge colour

Theorem

A planar graph is edge-transitive. $\iff$ *All of its edges have the same* 2*-WL colour.*



Edge types

Assume G is 3-connected.

Consider $G[c]$ for a 2-WL edge color c.

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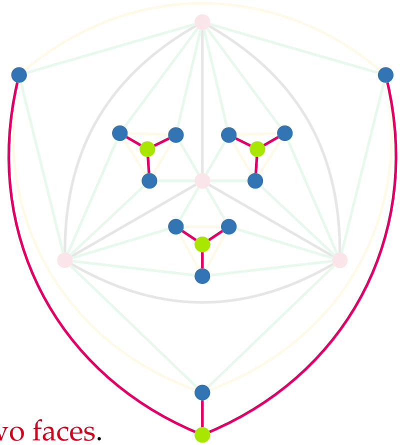
We distinguish between three types for c.



1. **$G[c]$ has only one face.**
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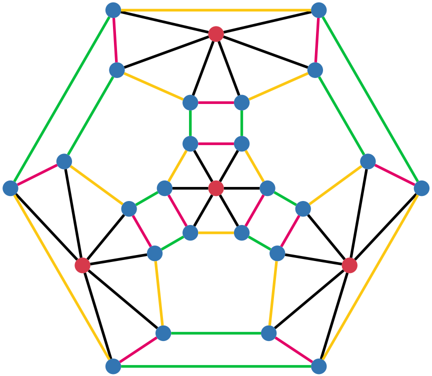
3. Every connected component of $G[c]$ has at least three faces.

Graphs of type 1

Assume $G[c]$ has only one face for every 2-WL edge colour c.



Theorem (K., Ponomarenko, Schweitzer 2017)

Let G be a 3-connected planar graph and suppose v_1, v_2, v_3 are distinct vertices on a common face of G.

Then 1-WL computes a discrete colouring on G_{v_1,v_2,v_3}.

Lemma

Let G be a 3-connected planar graph with edge colours only of Type 1.

Then there is a $v \in V$ such that $\mathrm{Singles}(v) = V$.

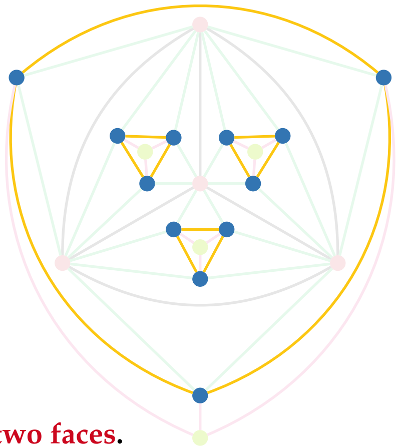
In particular, 2-WL identifies G.

Edge types

Assume G is 3-connected.

Consider $G[c]$ for a 2-WL edge color c.

Then all components of $G[c]$ have the same numbers of vertices, edges, and faces.



We distinguish between three types for c.

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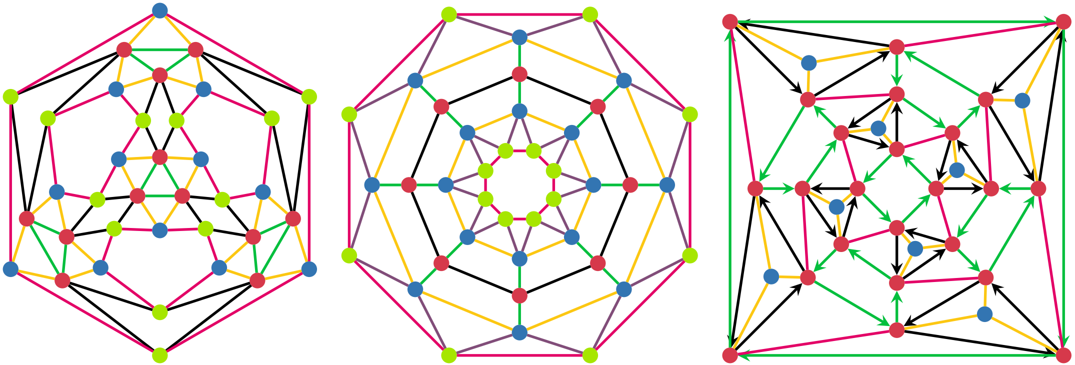
3. Every connected component of $G[c]$ has at least three faces.

Graphs of type 2

Assume there is a c such that every component of $G[c]$ has exactly two faces, and there is no c' such that every component of $G[c']$ has at least three faces…



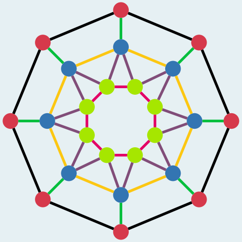
Examples of graphs of types IIa, IIb, IIc

Conclusion

Theorem

For every 3-connected planar graph G, one of the following holds.

1. *2-WL identifies G, or*
2. *2-WL detects a matching in G, or*
3. *2-WL detects a connected subgraph that*
    - a. *is essentially a Platonic or Archimedean solid, or*
    - b. *stems from a small number of infinite families of connected graphs.*



Future project:

Determine the WL-dimension of planar graphs.

To this end, study interactions with the subgraphs from Case 3.