Flip-Breakability: Combinatorial Characterizations of Monadically NIP Graph Classes

Jan Dreier[1], Nikolas Mählmann[2], Szymon Toruńczyk[3]

LoGAlg 2023

[1]TU Wien
[2]University of Bremen
[3]University of Warsaw

The FO Model Checking Problem

Problem: Given a graph G and an FO sentence φ, decide whether

$$G \models \varphi.$$

Example: G contains a dominating set of size k iff.

$$G \models \exists x_1 \ldots \exists x_k \forall y : \bigvee_{i \in [k]} (y = x_i \vee y \sim x_i).$$

Runtime: On the class of all graphs, FO model checking is $\mathrm{AW}[*]$-hard. We will assume $\mathrm{FPT} \neq \mathrm{AW}[*]$.

Question: On which classes is FO model checking fixed-parameter tractable, i.e., solvable in time $f(\varphi) \cdot n^c$?

Tractable Classes

Theorem [Grohe, Kreutzer, Siebertz, 2014]

Let $\mathcal{C}$ be a monotone class of graphs.

$\mathcal{C}$ admits fpt FO model checking if and only if $\mathcal{C}$ is nowhere dense.

Theorem [Dreier, Eleftheriadis, Mählmann, McCarty, Pilipczuk, Toruńczyk, 2023+]

Let $\mathcal{C}$ be a hereditary and orderless class of graphs.

$\mathcal{C}$ admits fpt FO model checking if and only if $\mathcal{C}$ is monadically stable.

Theorem [Bonnet, Giocanti, Ossona de Mendez, Simon, Thomassé, Toruńczyk, 2022]

Let $\mathcal{C}$ be a hereditary and ordered class of graphs.

$\mathcal{C}$ admits fpt FO model checking if and only if $\mathcal{C}$ has bounded twin-width.

Tractable Classes

Theorem [Grohe, Kreutzer, Siebertz, 2014]

Let $\mathcal{C}$ be a monotone class of graphs.

$\mathcal{C}$ admits fpt FO model checking if and only if $\mathcal{C}$ is **monadically NIP**.

Theorem [Dreier, Eleftheriadis, Mählmann, McCarty, Pilipczuk, Toruńczyk, 2023+]

Let $\mathcal{C}$ be a hereditary and orderless class of graphs.

$\mathcal{C}$ admits fpt FO model checking if and only if $\mathcal{C}$ is **monadically NIP**.

Theorem [Bonnet, Giocanti, Ossona de Mendez, Simon, Thomassé, Toruńczyk, 2022]

Let $\mathcal{C}$ be a hereditary and ordered class of graphs.

$\mathcal{C}$ admits fpt FO model checking if and only if $\mathcal{C}$ is **monadically NIP**.

A Unifying Theory

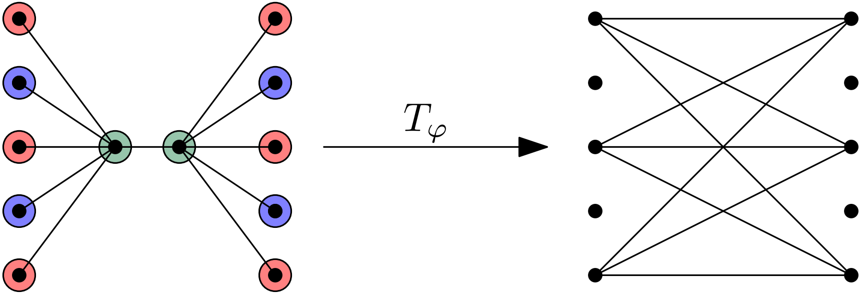
Conjecture

Let $\mathcal{C}$ be a **hereditary** *class of graphs.*

$\mathcal{C}$ admits fpt FO model checking if and only if $\mathcal{C}$ is **monadically NIP***.*

FO Transductions

Transductions $\hat{=}$ coloring + interpreting + taking an induced subgraph

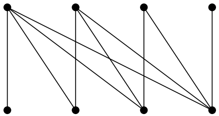


$\varphi(x, y) := \mathrm{Red}(x) \wedge \mathrm{Red}(y) \wedge \mathrm{dist}(x, y) = 3$

Monadic Stability and Monadic NIP

Definition

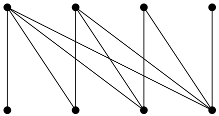
A class is *monadically stable*, if it does not transduce the class of *all half graphs*.



Monadic Stability and Monadic NIP

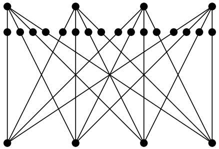
Definition

A class is *monadically stable*, if it does not transduce the class of *all half graphs*.



Definition

A class is *monadically NIP*, if it does not transduce the class of *all graphs*.
Equivalently, it does not transduce the class of all 1-subdivided bicliques.



Wanted: Combinatorial Characterizations

Monadically NIP classes are defined using logic.

Working towards algorithms we need tools that are combinatorial.

Wanted: Combinatorial Characterizations

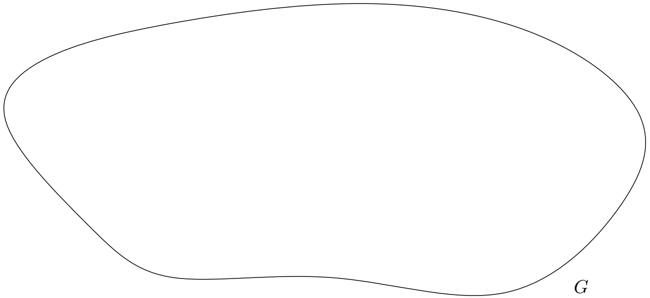
Monadically NIP classes are defined using logic.

Working towards algorithms we need tools that are combinatorial.

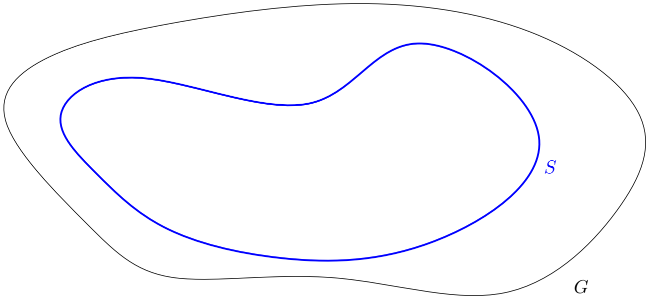
In this talk we will present:

- a combinatorial structure characterization: flip-breakability
- a combinatorial non-structure characterization: forbidden induced subgraphs

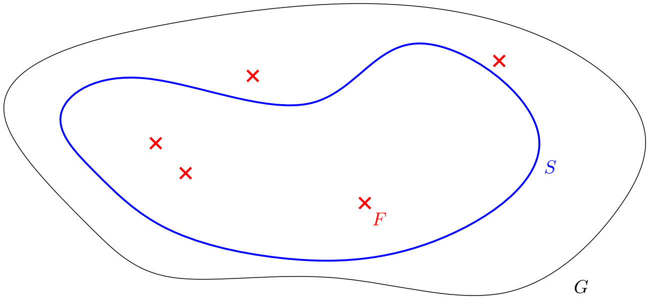
Characterizing Nowhere Denseness: Uniform Quasi-Wideness



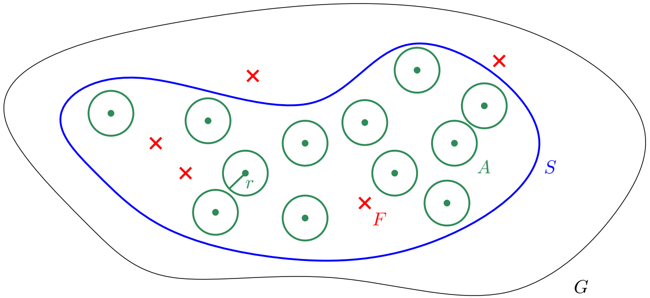
Characterizing Nowhere Denseness: Uniform Quasi-Wideness



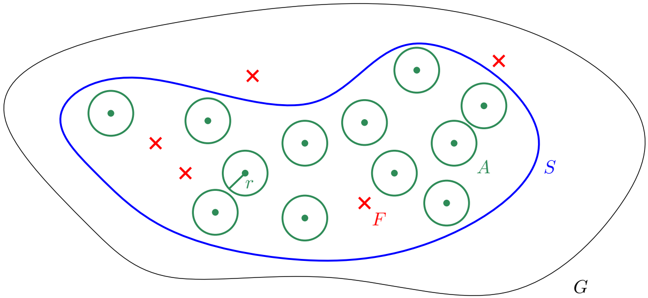
Characterizing Nowhere Denseness: Uniform Quasi-Wideness



Characterizing Nowhere Denseness: Uniform Quasi-Wideness



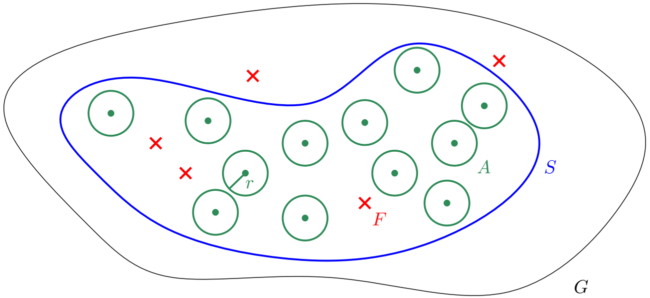
Characterizing Nowhere Denseness: Uniform Quasi-Wideness



Uniform Quasi-Wideness (slightly informal)

A class $\mathcal{C}$ is *uniformly quasi-wide* if for every radius r, in every large set S we find a still large set A that is r-independent after removing a set F of constantly many vertices.

Characterizing Nowhere Denseness: Uniform Quasi-Wideness



Uniform Quasi-Wideness (slightly informal)

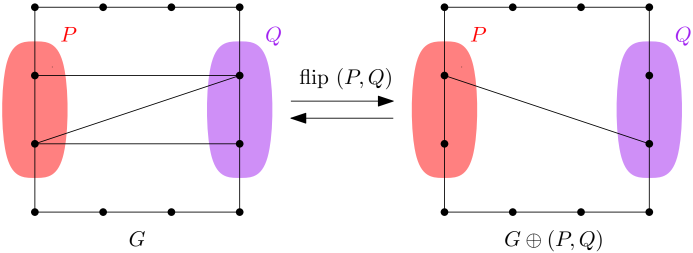
A class $\mathcal{C}$ is *uniformly quasi-wide* if for every radius r, in every large set S we find a still large set A that is r-independent after removing a set F of constantly many vertices.

Theorem [Něsetřil, Ossona de Mendez, 2011]

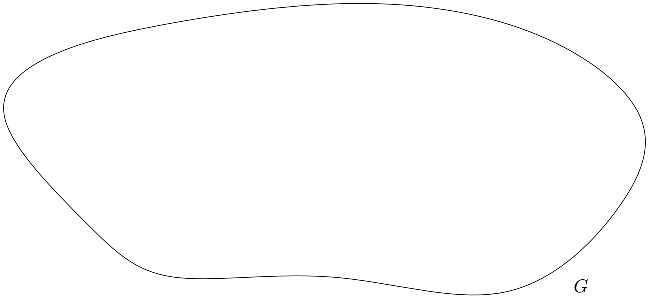
A class $\mathcal{C}$ is uniformly quasi-wide if and only if it is nowhere dense.

Towards Dense Graphs

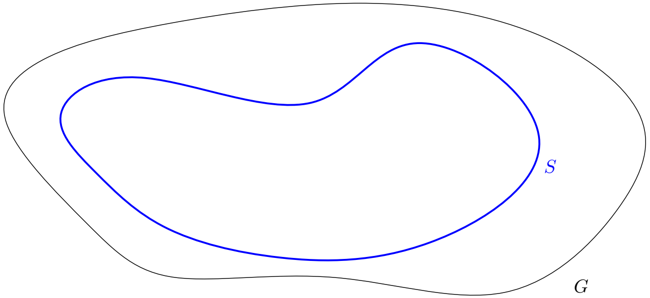
Denote by $G \oplus (P, Q)$ the graph obtained from G by complementing edges between pairs of vertices from $P \times Q$.



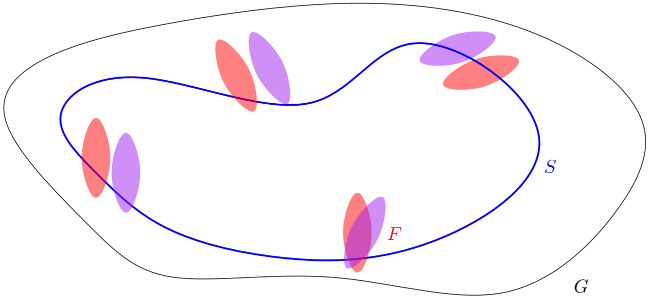
Characterizing Monadic Stability: Flip-Flatness



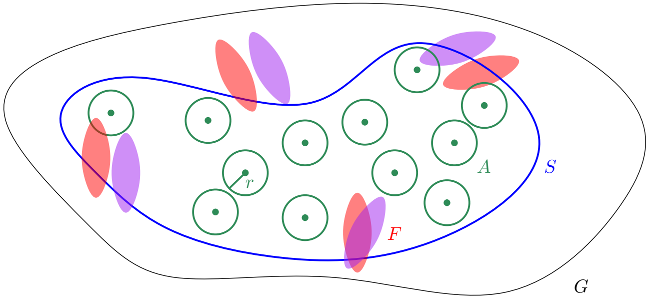
Characterizing Monadic Stability: Flip-Flatness



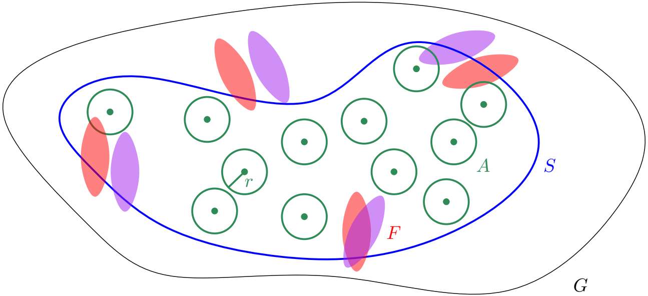
Characterizing Monadic Stability: Flip-Flatness



Characterizing Monadic Stability: Flip-Flatness



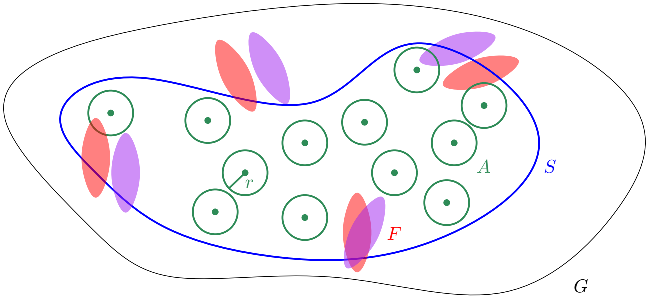
Characterizing Monadic Stability: Flip-Flatness



Flip-Flatness (slightly informal)

A class $\mathcal{C}$ is *flip-flat* if for every radius r, in every large set S we find a still large set A that is r-independent after performing a set F of constantly many flips.

Characterizing Monadic Stability: Flip-Flatness



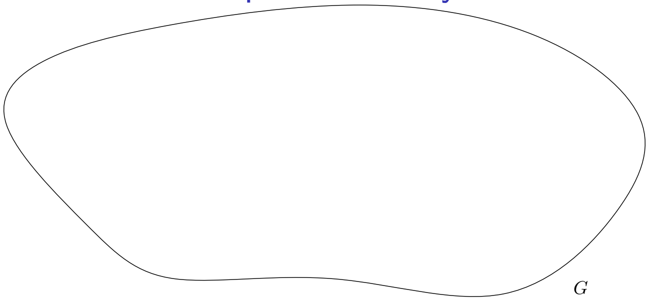
Flip-Flatness (slightly informal)

A class $\mathcal{C}$ is *flip-flat* if for every radius r, in every large set S we find a still large set A that is r-independent after performing a set F of constantly many flips.

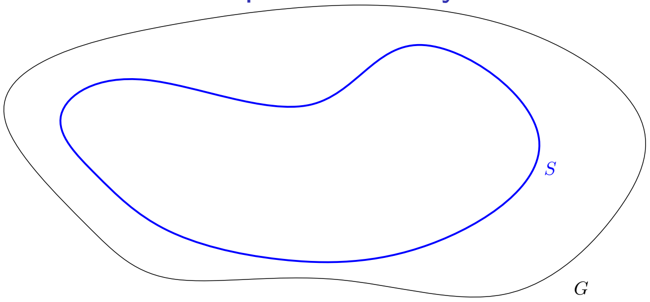
Theorem [Dreier, Mählmann, Siebertz, Toruńczyk, 2022]

A class $\mathcal{C}$ is flip-flat if and only if it is monadically stable.

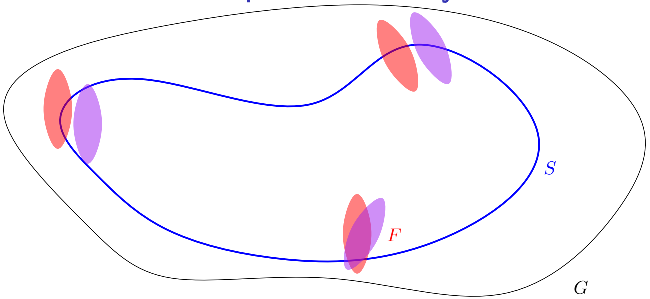
Characterizing Monadic NIP: Flip-Breakability



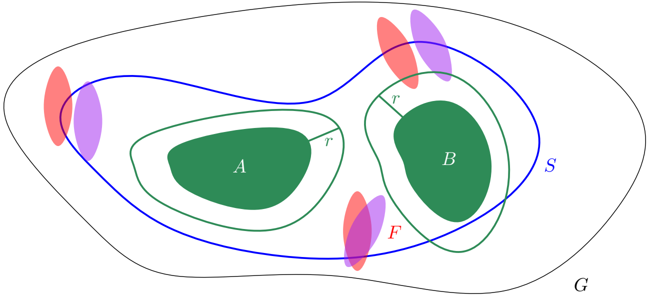
Characterizing Monadic NIP: Flip-Breakability



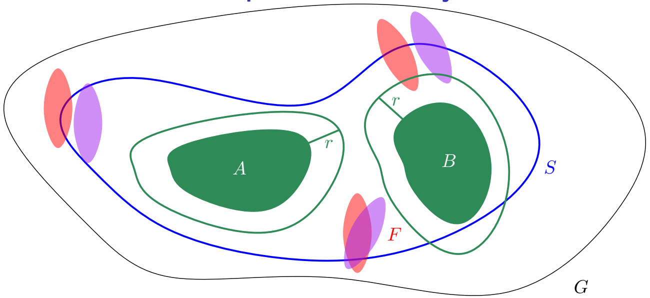
Characterizing Monadic NIP: Flip-Breakability



Characterizing Monadic NIP: Flip-Breakability



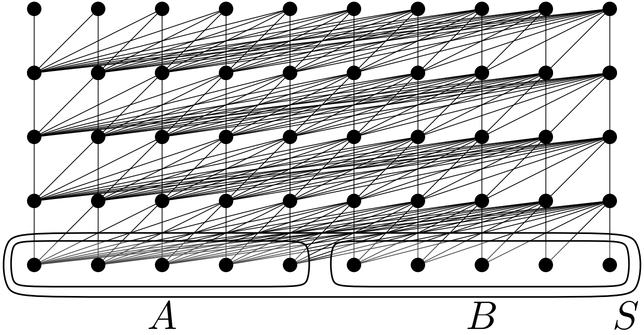
Characterizing Monadic NIP: Flip-Breakability



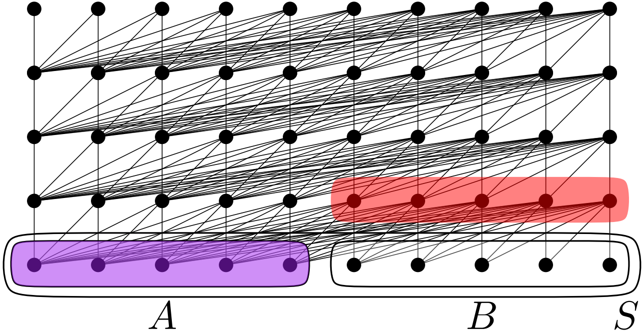
Flip-Breakability (slightly informal)

A class $\mathcal{C}$ is *flip-breakable* if for every radius r, in every large set S we find two large sets A and B that and a flip F of bounded size such that $N^r_{G \oplus F}(A) \cap N^r_{G \oplus F}(B) = \varnothing$.

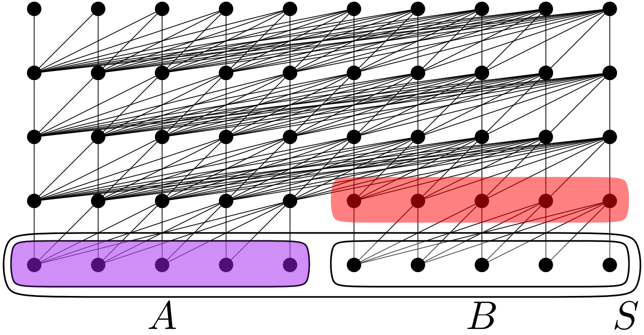
Flip-Breakability: Example



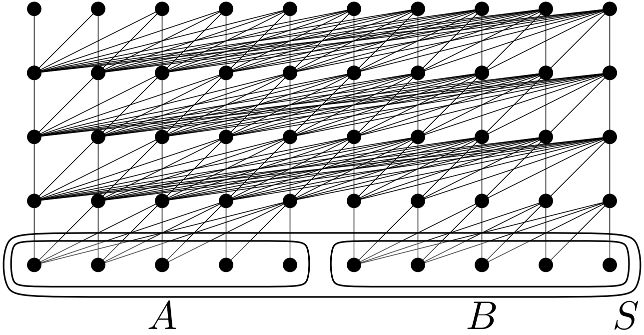
Flip-Breakability: Example



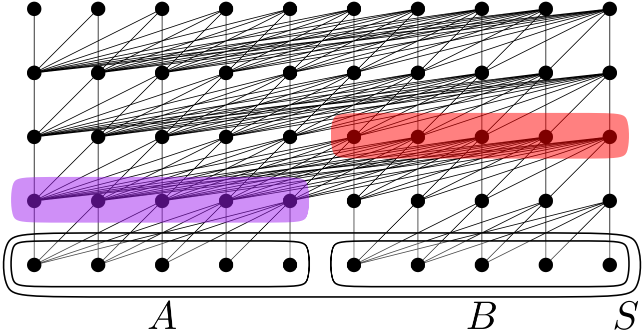
Flip-Breakability: Example



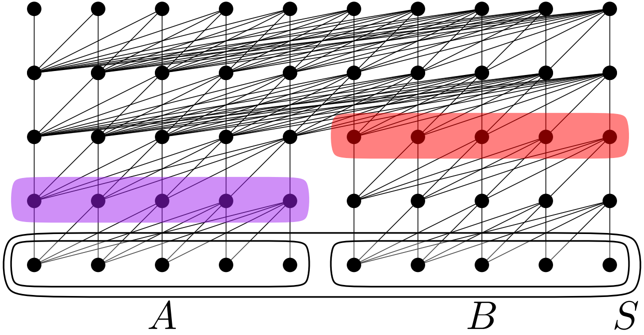
Flip-Breakability: Example



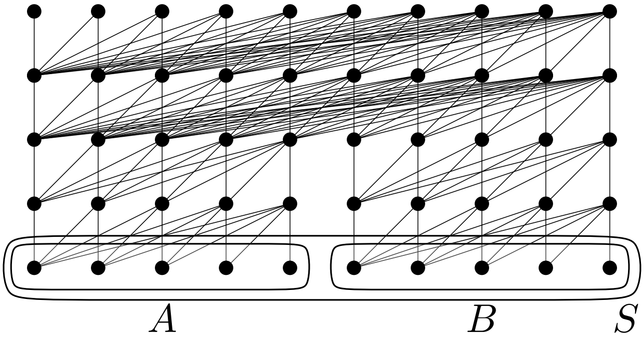
Flip-Breakability: Example



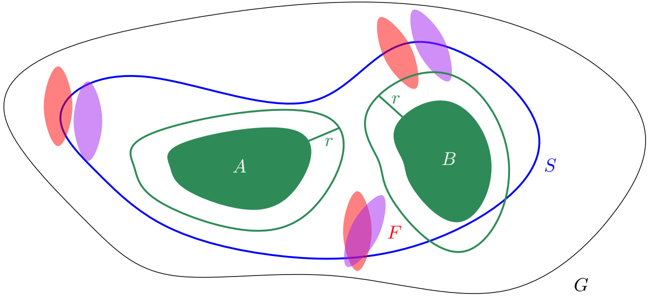
Flip-Breakability: Example



Flip-Breakability: Example



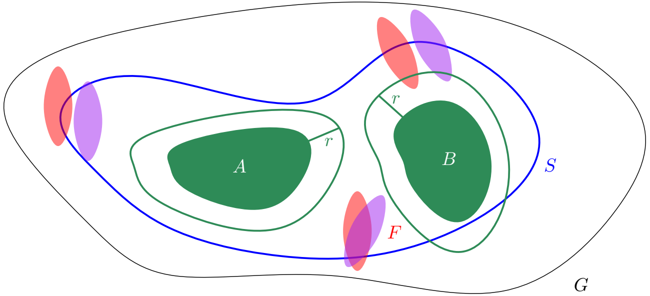
Characterizing Monadic NIP: Flip-Breakability



Flip-Breakability (slightly informal)

A class $\mathcal{C}$ is *flip-breakable* if for every radius r, in every large set S we find two large sets A and B that and a flip F of bounded size such that $N^r_{G \oplus F}(A) \cap N^r_{G \oplus F}(B) = \varnothing$.

Characterizing Monadic NIP: Flip-Breakability



Flip-Breakability (slightly informal)

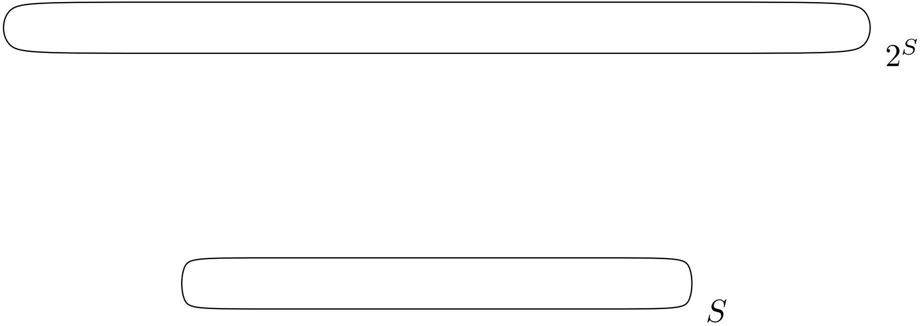
A class $\mathcal{C}$ is *flip-breakable* if for every radius r, in every large set S we find two large sets A and B that and a flip F of bounded size such that $N^r_{G\oplus F}(A) \cap N^r_{G\oplus F}(B) = \varnothing$.

Theorem [Dreier, Mählmann, Toruńczyk]

A class $\mathcal{C}$ is flip-breakable if and only if it is monadically NIP.

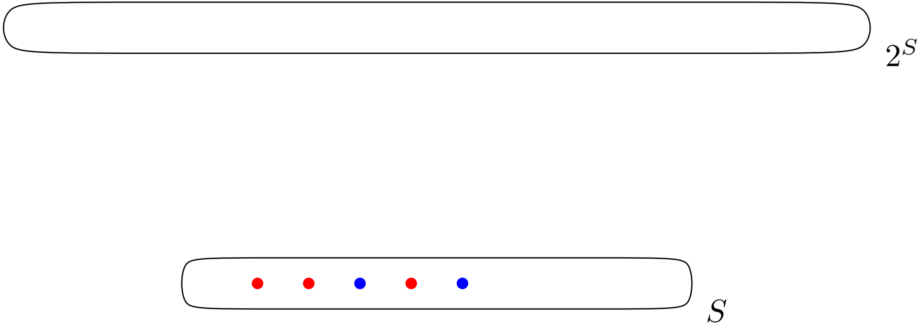
Flip-Breakability ⇒ Monadic NIP

Assume towards a contradiction a class $\mathcal{C}$ is not monadically NIP but flip-breakable.



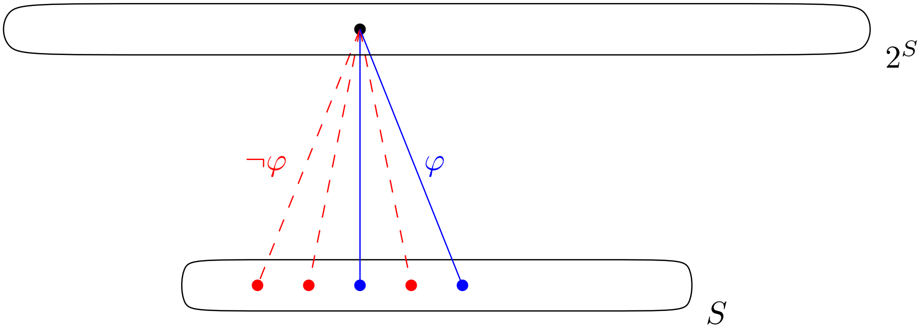
Flip-Breakability $\Rightarrow$ Monadic NIP

Assume towards a contradiction a class $\mathcal{C}$ is not monadically NIP but flip-breakable.



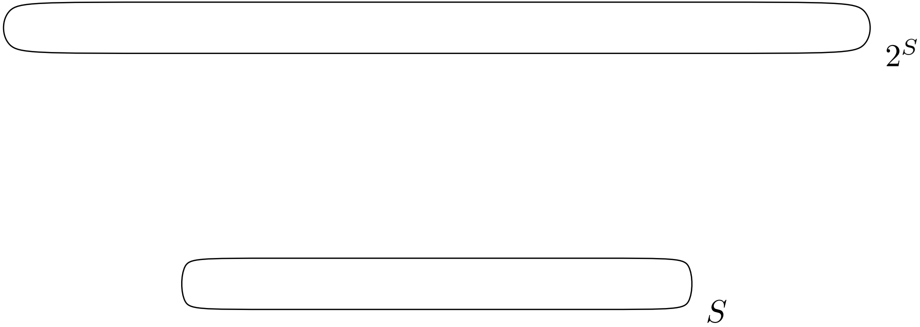
Flip-Breakability ⇒ Monadic NIP

Assume towards a contradiction a class $\mathcal{C}$ is not monadically NIP but flip-breakable.



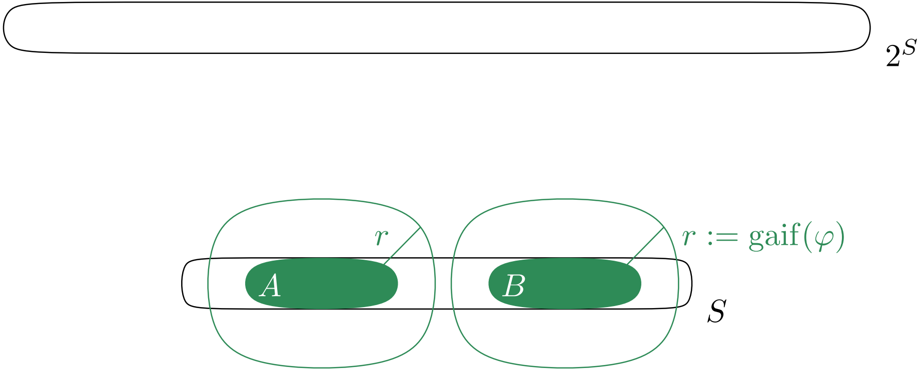
Flip-Breakability $\Rightarrow$ Monadic NIP

Assume towards a contradiction a class $\mathcal{C}$ is not monadically NIP but flip-breakable.



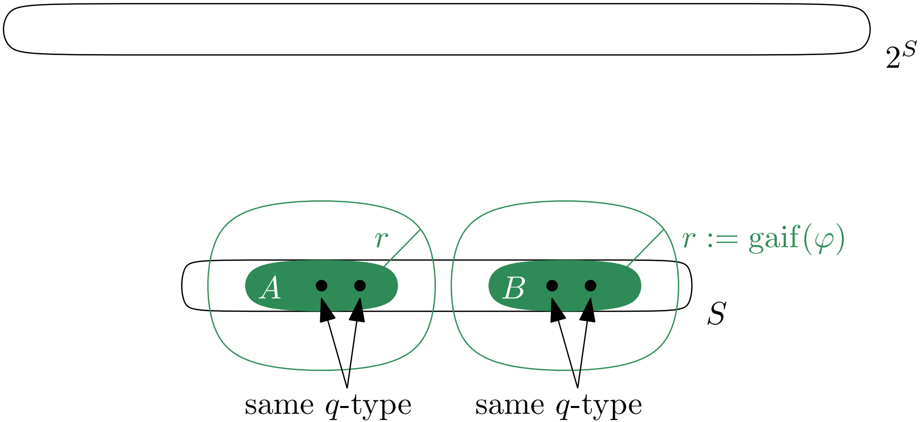
Flip-Breakability $\Rightarrow$ Monadic NIP

Assume towards a contradiction a class $\mathcal{C}$ is not monadically NIP but flip-breakable.



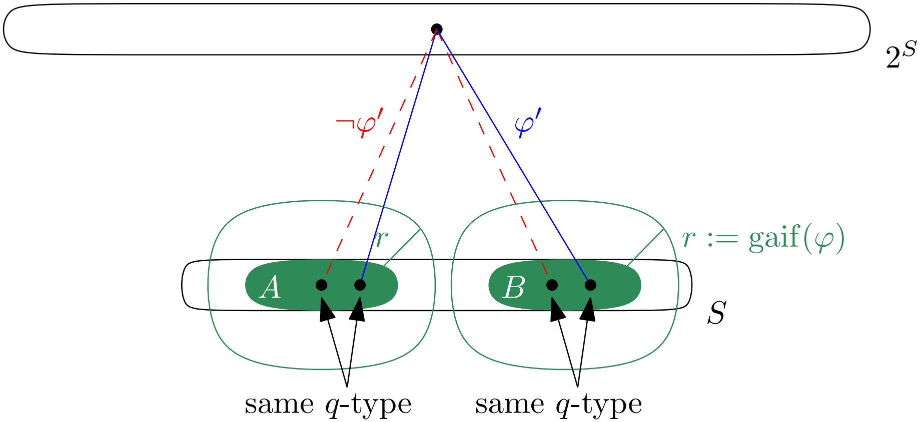
Flip-Breakability ⇒ Monadic NIP

Assume towards a contradiction a class $\mathcal{C}$ is not monadically NIP but flip-breakable.



Flip-Breakability ⇒ Monadic NIP

Assume towards a contradiction a class $\mathcal{C}$ is not monadically NIP but flip-breakable.



Variants of Flip-Breakability

1. We modify a graph using either flips or vertex deletions.

Variants of Flip-Breakability

1. We modify a graph using either flips or vertex deletions.
2. We demand our resulting set is either flat or broken.

   flat: pairwise separated; broken: separated into two large sets

Variants of Flip-Breakability

1. We modify a graph using either flips or vertex deletions.
2. We demand our resulting set is either flat or broken.

   flat: pairwise separated; broken: separated into two large sets
3. Separation means either distance-r or distance-∞.

Variants of Flip-Breakability

1. We modify a graph using either flips or vertex deletions.
2. We demand our resulting set is either flat or broken.

   flat: pairwise separated; broken: separated into two large sets
3. Separation means either distance-r or distance-∞.

		flatness	breakability
dist-r	flip-	**monadic stability**	**monadic NIP**
	deletion-	**nowhere denseness**	
dist-∞	flip-		
	deletion-		

Variants of Flip-Breakability

1. We modify a graph using either flips or vertex deletions.
2. We demand our resulting set is either flat or broken.

   flat: pairwise separated; broken: separated into two large sets
3. Separation means either distance-r or distance-∞.

		flatness	breakability
dist-r	flip-	**monadic stability**	**monadic NIP**
	deletion-	**nowhere denseness**	**nowhere denseness**
dist-∞	flip-		
	deletion-		

Variants of Flip-Breakability

1. We modify a graph using either flips or vertex deletions.
2. We demand our resulting set is either flat or broken.

   flat: pairwise separated; broken: separated into two large sets
3. Separation means either distance-r or distance-∞.

		flatness	breakability
dist-r	flip-	**monadic stability**	**monadic NIP**
	deletion-	**nowhere denseness**	**nowhere denseness**
dist-∞	flip-	**bd. shrubdepth**	**bd. cliquewidth**
	deletion-		

Variants of Flip-Breakability

1. We modify a graph using either flips or vertex deletions.
2. We demand our resulting set is either flat or broken.

   flat: pairwise separated; broken: separated into two large sets
3. Separation means either distance-r or distance-∞.

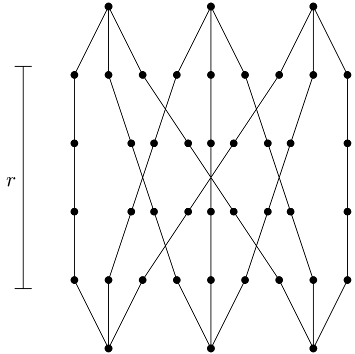
		flatness	breakability
dist-r	flip-	**monadic stability**	**monadic NIP**
	deletion-	**nowhere denseness**	**nowhere denseness**
dist-∞	flip-	**bd. shrubdepth**	**bd. cliquewidth**
	deletion-	**bd. treedepth**	**bd. treewidth**

Wanted: Combinatorial Characterizations

In this talk we will present:

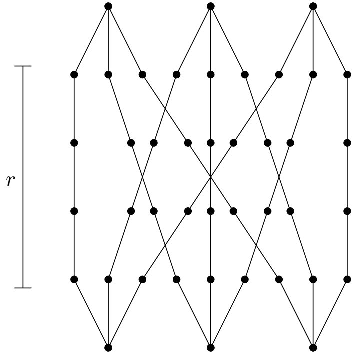
- a combinatorial structure characterization: flip-breakability ✓
- a combinatorial non-structure characterization: forbidden induced subgraphs

Characterizing Monadic NIP by Forbidden Induced Subgraphs

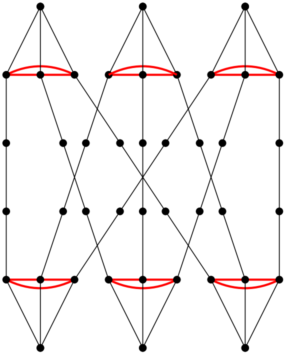


star r-crossing
$= r$-subdivided biclique

Characterizing Monadic NIP by Forbidden Induced Subgraphs

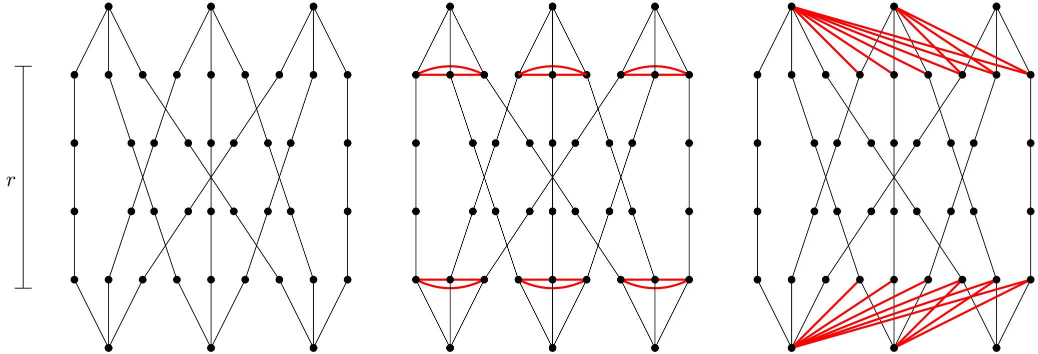


star r-crossing
$= r$-subdivided biclique



clique r-crossing

Characterizing Monadic NIP by Forbidden Induced Subgraphs

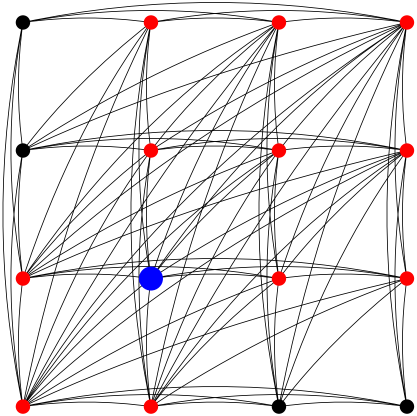


star r-crossing
$= r$-subdivided biclique

clique r-crossing

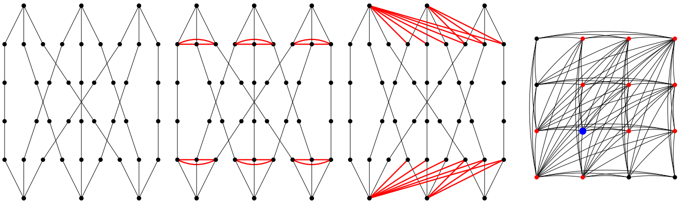
half-graph r-crossing

Characterizing Monadic NIP by Forbidden Induced Subgraphs



comparability grid

Characterizing Monadic NIP by Forbidden Induced Subgraphs

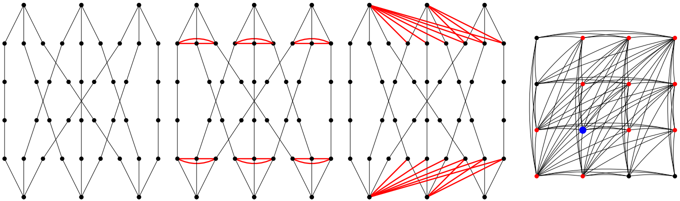


Theorem [Dreier, Mählmann, Toruńczyk]

Let $\mathcal{C}$ be a graph class. Then $\mathcal{C}$ is monadically NIP if and only if for every $r \geq 1$ there exists $k \in \mathbb{N}$ such $\mathcal{C}$ excludes as induced subgraphs

- all layerwise flipped star r-crossings of order k, and
- all layerwise flipped clique r-crossings of order k, and
- all layerwise flipped half-graph r-crossings of order k, and
- the comparability grid of order k.

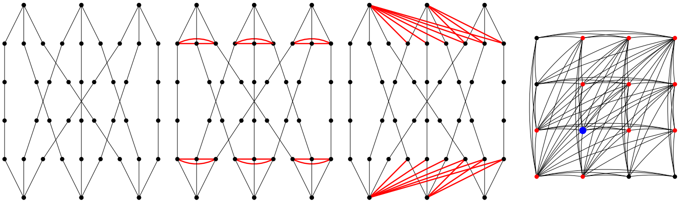
Forbidden Induced Subgraphs: Applications



Theorem [Dreier, Mählmann, Toruńczyk]

1. FO Model checking is $\mathrm{AW}[*]$-hard on every hereditary class that is not mon. NIP.

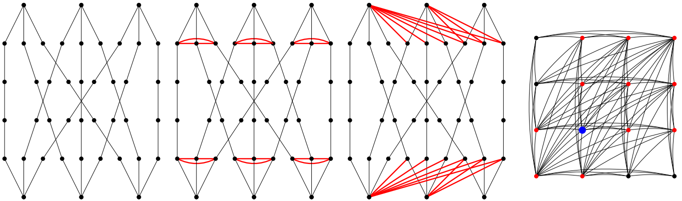
Forbidden Induced Subgraphs: Applications



Theorem [Dreier, Mählmann, Toruńczyk]

1. FO Model checking is $\mathrm{AW}[*]$-hard on every hereditary class that is not mon. NIP.
2. Every small hereditary class is monadically NIP.

Forbidden Induced Subgraphs: Applications

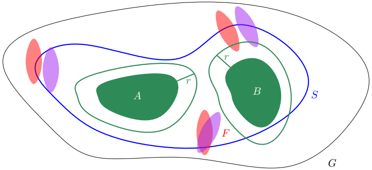


Theorem [Dreier, Mählmann, Toruńczyk]

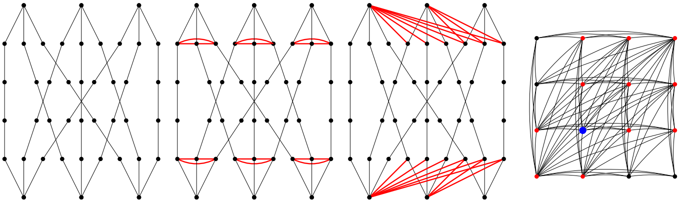
1. FO Model checking is $\mathrm{AW}[*]$-hard on every hereditary class that is not mon. NIP.
2. Every small hereditary class is monadically NIP.
3. Every class with almost bounded flip-width is monadically NIP.

Summary: We give two combinatorial characterizations of mon. NIP graph classes.

A structure characterization called flip-breakability:



A non-structure characterization by forbidden induced subgraphs:



FO model checking is $\mathrm{AW}[*]$-hard on hereditary graph classes that are not mon. NIP.