Game comonads – a new kid in the finite model theory playground

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Existential k-pebble game

The game from A to B, rules in each round:

- Spoiler places/moves one of pebbles $p_1, \ldots, p_k$ on A,
- Duplicator moves the corresponding $q_1, \ldots, q_k$ on B.

Duplicator wins if after every round the mapping $p_i \mapsto q_i$ is a partial homomorphism $A \to B$.

Proposition (Kolaitis, Vardi 1990)

For relational structures A, B, the following are equivalent:

- *Duplicator has a winning strategy in the existential k-pebble game from A to B.*
- $A \Rrightarrow_{\exists^+ \mathcal{L}^k} B$, *i.e.* $\forall$ *positive existential k-variable sentence* φ,

$$A \models \varphi \quad \textit{implies} \quad B \models \varphi.$$

Semantic reformulation

Proposition (Dawar, Severini, Zapata 2017)

The following are equivalent:

- *Duplicator has a winning strategy in the existential k-pebble game from A to B.*
- *There exists a homomorphism $\mathbb{P}_k(A) \to B$.*

For a graph/relational structure A,

$\mathbb{P}_k(A) =$ the structure of Spoiler's plays on A <u>with k pebbles</u>

Formally

$\mathbb{P}_k(A)$ given by

- universe of words $\overline{x} = [\,(p_1, a_1),\ \ldots,\ (p_n, a_n)\,]$

  where $a_1, \ldots, a_n \in A$, $p_1, \ldots, p_n \in \{1, \ldots, k\}$

  $\Rightarrow$ define projections

  $p(\overline{x}) = p_n$ and $\varepsilon(\overline{x}) = a_n$

- $R^{\mathbb{P}_k(A)}(\overline{x}_1,\ \ldots,\ \overline{x}_n)$ if
  - $R^A(\varepsilon(\overline{x}_1),\ \ldots,\ \varepsilon(\overline{x}_n))$
  - for all i, j:
    - either $\overline{x}_i = \overline{x}_j \cdot \overline{y}$ and $p(x_i) \notin \overline{y}$
    - or $\overline{x}_j = \overline{x}_i \cdot \overline{y}$ and $p(x_j) \notin \overline{y}$

A comonad is born

Theorem (Abramsky, Dawar, Wang 2017)

$(\mathbb{P}_k, \varepsilon, \overline{(-)})$ *is a comonad.*

A comonad $(\mathbb{C}, \varepsilon, \overline{(-)})$ given by

- operation on structures/graphs $A \;\mapsto\; \mathbb{C}(A)$
- a homomorphism $\varepsilon_A \colon \mathbb{C}(A) \to A$ for each A
- extension operation $f \colon \mathbb{C}(A) \to B \quad \mapsto \quad \overline{f} \colon \mathbb{C}(A) \to \mathbb{C}(B)$
  satisfying

$$\overline{\varepsilon} = \mathrm{id} \qquad \varepsilon \circ \overline{f} = f \qquad \overline{g \circ \overline{f}} = \overline{g} \circ \overline{f}$$

Coalgebras and tree-width

$\alpha\colon A \to \mathbb{P}_k(A)$ is a **coalgebra** if

$$\begin{array}{ccc} A & & \\ {\scriptstyle\alpha}\downarrow & \searrow^{\mathsf{id}} & \\ \mathbb{P}_k(A) & \xrightarrow{\varepsilon_A} & A \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow{\alpha} & \mathbb{P}_k(A) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\overline{\mathsf{id}}} \\ \mathbb{P}_k(A) & \xrightarrow{\overline{\alpha\,\varepsilon_A}} & \mathbb{P}_k^2(A) \end{array}$$

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$$

Lemma

For fixed A, there is a one-to-one correspondence between:

- *coalgebras* $A \to \mathbb{P}_k(A)$
- *compatible* k*-pebble forest covers* $(A, \leq, p)$ *where* $(\leq) \subseteq A \times A$ *and* $p\colon A \to \{1, \ldots, k\}$

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Lemma

For fixed A, there is a one-to-one correspondence between:

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Theorem (Abramsky, Dawar, Wang 2017)

A has tree-width $< k$ *iff exists coalgebra* $A \to \mathbb{P}_k(A)$.

Capturing logics

- $A \equiv_{\exists^+\mathcal{L}^k} B$ iff exist $\mathbb{P}_k(A) \to B$ and $\mathbb{P}_k(B) \to A$

(Equality has to be added to the signature for the last one.)

Capturing logics

- $A \equiv_{\exists^+\mathcal{L}^k} B$ iff exist $\mathbb{P}_k(A) \to B$ and $\mathbb{P}_k(B) \to A$

- $A \equiv_{\mathcal{C}^k} B$ iff exist $f\colon \mathbb{P}_k(A) \to B$ and $g\colon \mathbb{P}_k(B) \to A$
  s.t. $\overline{g} \circ \overline{f} = \mathrm{id}$ and $\overline{f} \circ \overline{g} = \mathrm{id}$.

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- $A \equiv_{\mathcal{C}^k} B$ iff exist $f\colon \mathbb{P}_k(A) \to B$ and $g\colon \mathbb{P}_k(B) \to A$ s.t. $\overline{g} \circ \overline{f} = \mathrm{id}$ and $\overline{f} \circ \overline{g} = \mathrm{id}$.

- $A \equiv_{\mathcal{L}^k} B$ iff there is a bisimulation between (coalgebras)

$$(\mathbb{P}_k(A), \sqsubseteq, p) \text{ and } (\mathbb{P}_k(B), \sqsubseteq, p)$$

where

$$\overline{x} \sqsubseteq \overline{y} \iff \overline{x} \text{ is a prefix of } \overline{y}$$

$$p\,(\,[(p_1, a_1),\ \ldots,\ (p_n, a_n)]\,) = p_n$$

(Equality has to be added to the signature for the last one.)

Other game comonads*

Comonad	Combinatorial property	logic fragment
$\mathbb{E}_k$	tree-depth	qrank $\leq k$ fragment
$\mathbb{P}_k$	tree-width	k-variable fragment
$\mathbb{M}_k$	sync. tree depth	modal depth $\leq k$
$\mathbb{PR}_k$	path-width	restricted conjunction k-variable
$\mathbb{He}_k$	k-ary generalised t.-w.	generalised quantifier k-variable extension
$\mathbb{G}_k$	guarded tree decomp	quantifier-guarded
$\mathbb{LG}_k$	hypertree-width	k-conjunct guarded
$\mathbb{Hy}_k$	generated tree-depth	hybrid modal depth
$\mathbb{B}_k$	generated tree-depth	bounded quantifiers
...	...	...

(*) Since 2018, due to Abramsky, Marsden, Shah, Dawar, etc.

Categorical versions of (F)MT theorems

1. Lovász homomorphism counting **[Dawar, TJ, Reggio 2021] [Reggio 2022]**

   uses: comonadicity of forgetful functors + finite rank comonad

2. Feferman–Vaught–Mostowski theorems **[TJ, Marsden, Shah 2023]**

   generalises: tensor products and bilinearity (linear algebra) for monads

3. Courcelle **[TJ, Marsden, Shah 2022+]**

4. van Benthem-Rosen **[Abramsky, Marsden 2022] [Abramsky, Reggio 2023]**

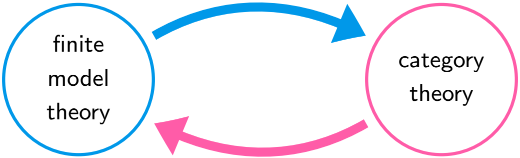
5. equi-rank HPT **[Abramsky, Reggio 2022+]**

   uses: model saturation $\sim$ small object argument (algebraic topology)

6. Hudges' word construction **[Reggio, Riba 2023+]**

7. Gaifman/Hanf Locality **[TJ tba]**

Homomorphism Counting & Interchange of Ideas



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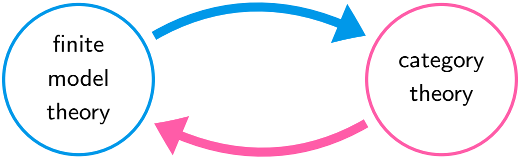

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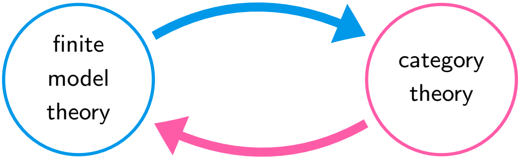

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2. $\mathbb{P}_k$, new description of tw and its new td refinement

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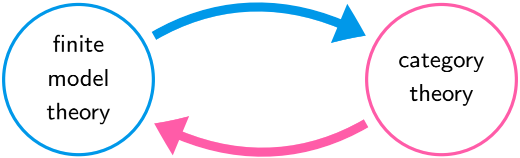

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[Dvořák 2010], **[Grohe 2020]**, . . .

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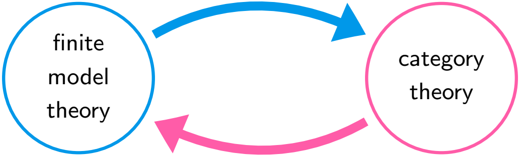

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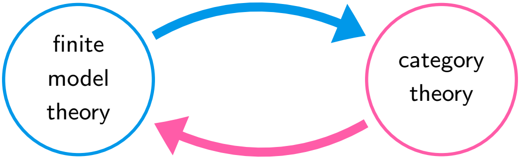

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5. unexpected corollary: equality elimination for $\mathcal{C}^k$

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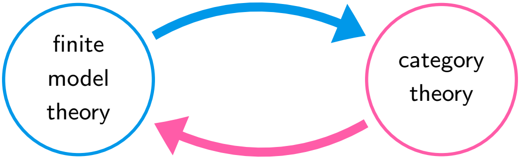

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6. **[Lichter, Pago, Seppelt 2024]** negative answer to **[Conghaile, Dawar 2021]**

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Feferman–Vaught–Mostowski theorems ($\approx$ composition methods)

Theorem (TJ, Marsden, Shah 2023)

Assume $\mathbb{C}$ *classifies* $\mathcal{L}$ *and* Ψ *is n-ary and functorial. If* Ψ *has a "Kleisli law"* $\mathbb{C}(\Psi(A_1, \ldots, A_n)) \to \Psi(\mathbb{C}A_1, \ldots, \mathbb{C}A_n)$,

$$A_i \equiv_{\exists^+\mathcal{L}} B_i, \quad \textit{for } i = 1, \ldots, n,$$

implies $$\Psi(A_1, \ldots, A_n) \equiv_{\exists^+\mathcal{L}} \Psi(B_1, \ldots, B_n).$$

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Feferman–Vaught–Mostowski theorems ($\approx$ composition methods)

Theorem (TJ, Marsden, Shah 2023)

Assume $\mathbb{C}$ *classifies* $\mathcal{L}$ *and* Ψ *is n-ary and functorial. If* Ψ *has a "***smooth** *Kleisli law"* $\mathbb{C}(\Psi(A_1, \ldots, A_n)) \to \Psi(\mathbb{C}A_1, \ldots, \mathbb{C}A_n)$,

$$A_i \equiv_{\mathcal{L}} B_i, \quad \text{for } i = 1, \ldots, n,$$

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implies $$\Psi(A_1, \ldots, A_n) \equiv_{\#\mathcal{L}} \Psi(B_1, \ldots, B_n).$$

Many applications, e.g. **[Karamlou, Shah 2023+]**:

$$A \equiv_{\mathcal{L}} B \implies Q_d(A) \equiv_{\mathcal{L}} Q_d(B)$$

where

$\mathcal{L} =$ guarded $\mathcal{C}_{\infty,k}$, for structures with comeasurable relations

$Q_d(A) =$ projector-valued measurements on A from **[Abramsky, Barbosa, de Silva, Zapata 2017]**, for non-local quantum strategies

Conclusion

- 4 years of work for 2 professors, 3 postdocs, 3 PhDs
- many category theorists learning FMT and vice versa
  (including ESSLLI school, ACT Adjoint School)
- great interchange of ideas between the communities!

Open problems

1. comonads for twin-width, flip-width, clique-width, ...
   ([Abramsky, TJ, Paine 2022] there is uninteresting comonad for any Δ s.t. $A, B \in \Delta$ iff $A + B \in \Delta$)
2. comonads for fixpoint fragments and non-symmetric games
3. presentations of comonads
4. classification of nowhere dense/monadically stable comonads

Thank you!

References, i

Origins of the theory:

1. Anuj Dawar, Simone Severini, and Octavio Zapata. Pebble games and cospectral graphs. Electronic Notes in Discrete Mathematics 61 (2017): 323-329.
2. Samson Abramsky, Anuj Dawar, and Pengming Wang. The pebbling comonad in finite model theory. LICS 2017.

Theory basics and a survey of main achievements:

3. Samson Abramsky, Nihil Shah, Relating structure and power: Comonadic semantics for computational resources. CMCS 2018 & Journal of Logic and Computation, Vol. 31(6), 2021.
4. Samson Abramsky. Structure and Power: an emerging landscape. Fundamenta Informaticae 186(1-4), 2022.

References, ii

Papers introducing further new game comonads

5. Adam Ó Conghaile, Anuj Dawar. Game Comonads & Generalised Quantifiers, CSL 2021.
6. Samson Abramsky, Dan Marsden. Comonadic semantics for guarded fragments, LICS 2021.
7. Yoàv Montacute, Nihil Shah. The Pebble-Relation Comonad in Finite Model Theory, LICS 2022.
8. Samson Abramsky, Dan Marsden. Comonadic semantics for hybrid logic and bounded fragments, MFCS 2022.
9. Bartosz Bednarczyk, Mateusz Urbańczyk. Comonadic semantics for description logics games. Description Logics Workshop @ FLoC 2022.
10. Santiago Figueira, Gabriel Goren Roig. A Modal Logic with n-ary Relations Over Paths: Comonadic Semantics and Expressivity. arXiv:2307.09679

* (more to appear later)

References, iii

Categorical (F)MT theorems

11. Lovász-type homomorphism counting theorems:
    - i Anuj Dawar, Tomáš Jakl and Luca Reggio, Lovász-type theorems and game comonads, LICS 2021.
    - ii Luca Reggio. Polyadic Sets and Homomorphism Counting. Advances in Mathematics, Vol. 410, 2022.
    - iii Samson Abramsky, Tomáš Jakl, Thomas Paine. Discrete Density Comonads and Graph Parameters. CMCS 2022.
12. Tomáš Jakl, Dan Marsden, Nihil Shah. A categorical account of composition methods in logic. LICS 2023.
13. Tomáš Jakl, Dan Marsden, Nihil Shah. A game comonadic account of Courcelle and Feferman-Vaught-Mostowski theorems. arXiv:2205.05387
14. for van Benthem-Rosen theorem, see [8]
15. Samson Abramsky and Luca Reggio. Arboreal categories and equi-resource homomorphism preservation theorems. arXiv:2211.15808
16. Luca Reggio, Colin Riba. Finitely accessible arboreal adjunctions and Hintikka formulae. arXiv:2304.12709

References, iv

Synthetic finite model theory:

17. Samson Abramsky and Luca Reggio. Arboreal categories. An axiomatic theory of resources. ICALP 2021 & LMCS 2023.
18. Samson Abramsky, Yoàv Montacute, Nihil Shah. Linear Arboreal Categories. arXiv:2301.10088
19. Further see [15] and [16] for examples of usage of this abstract setting.