FLIP-WIDTH

COPS AND ROBBER ON DENSE GRAPHS

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bounded treewidth

↓

excluding a minor

↓

excl. a topological minor

↓

bounded expansion

↓

nowhere dense

bounded treewidth
↓
excluding a minor
↓
excl. a topological minor
↓
bounded expansion
↓
nowhere dense

Theorem (Courcelle 1990)
Model checking Monadic Second Order logic is fpt
on every class of bounded treewidth

exhibit good
algorithmic,
combinatorial,
& logical behavior

Theorem (Grohe, Kreutzer, Siebertz, 2017)
Model checking First-Order logic is fpt
on every nowhere dense class.

Furthermore, for a monotone graph class C,
model checking FO is fpt ⇔ C is nowhere dense

BEYOND SPARSE

Example: treewidth $\to$ cliquewith/rankwidth

- retain many good properties of treewidth
- applicable to dense graphs

Theorem (Courcelle, Rotics, Makowsky 2000+Oum,Seymour 2006)
Model checking MSO is fpt on classes of bounded rankwidth

BEYOND SPARSITY

Project: extend from monotone classes to hereditary classes

monotone	*hereditary*
bounded treewidth	bounded cliquewidth
excluding a minor	?
excl. a top. minor	?
bounded expansion	?
nowhere dense	?

Quest (Grohe): which hereditary classes have fpt model checking?

TWIN-WIDTH

monotone		*hereditary*
bounded treewidth	→	bounded cliquewidth
excluding a minor	→	?
bd. sparse twin-width	→	bounded twin-width
bounded expansion	→	?
nowhere dense	→	?

class of graphs with max. degree *d*

GRAND UNIFICATION

Common generalization of Sparsity theory and Twin-width

bounded treewidth	bounded cliquewidth
excluding a minor	?
bd. sparse twin-width	bounded twin-width
bounded expansion	?
nowhere dense	?

WANTED

Analogues of the fundamental parameters
studied in Sparsity Theory:

degeneracy

generalized coloring numbers:
star-chromatic number, weak coloring numbers

CONTRIBUTION

- *flip-width* parameters

  - dense analogues of the fundamental parameters studied in sparsity theory

  - include degeneracy, twin-width, and clique-width as a *special cases*.

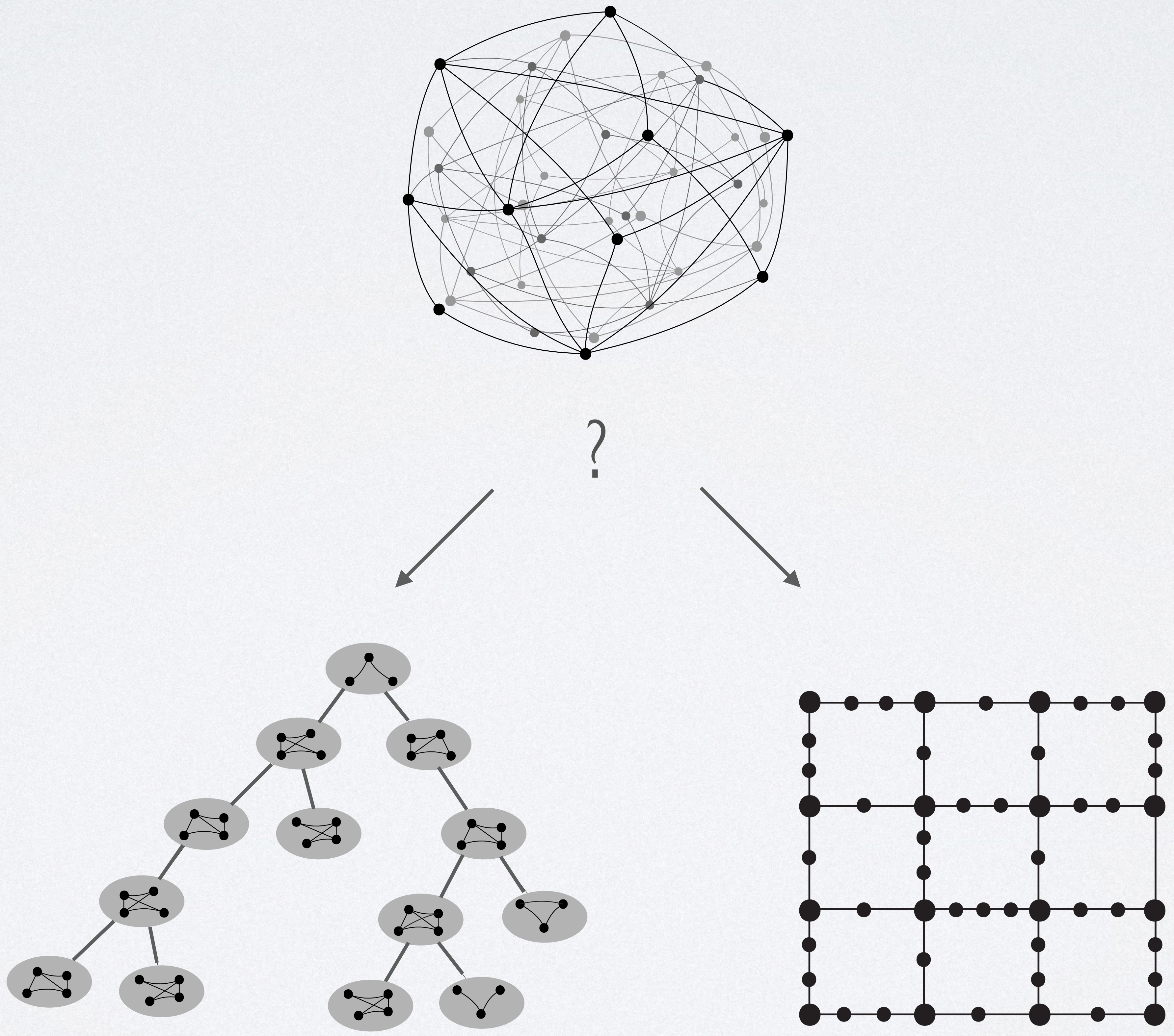
- notion of classes of *bounded flip-width* and *almost bounded flip-width*

GRAND UNIFICATION?

bounded treewidth	→	bounded cliquewidth
↓		↓
excluding a minor	→	?
↓		↓
bd. sparse twin-width	→	bounded twin-width
↓		↓
bounded expansion	→	**bounded flip-width**
↓		↓
nowhere dense	→	**almost bd. flip-width**

HOW TO DEFINE GRAPH PARAMETERS

- through decompositions
- through obstructions

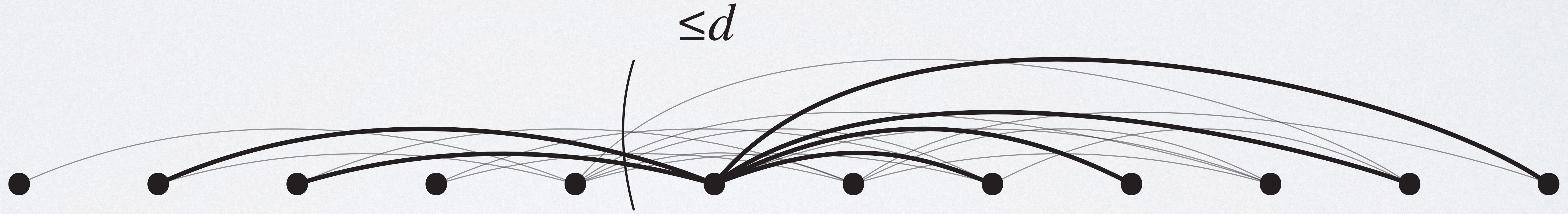


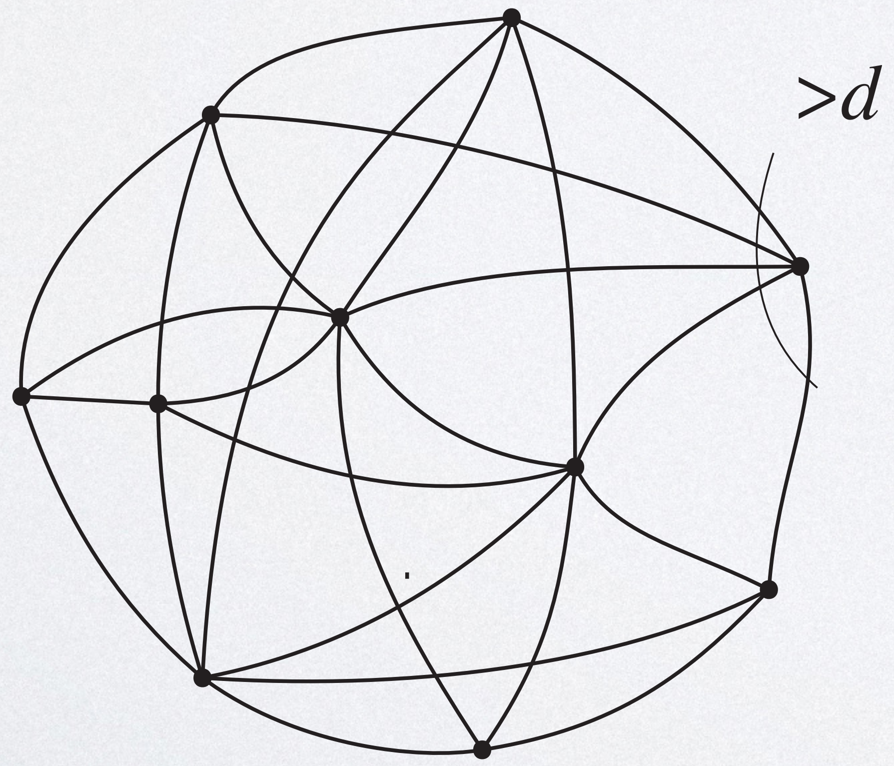
DEGENERACY

A graph is d-degenerate
if every subgraph has a vertex of degree $\leq d$

Or: there is an ordering such that every vertex has $\leq d$ neighbors before it

dichotomy:





HOW TO DEFINE GRAPH PARAMETERS

- through decompositions
- through obstructions
- through games

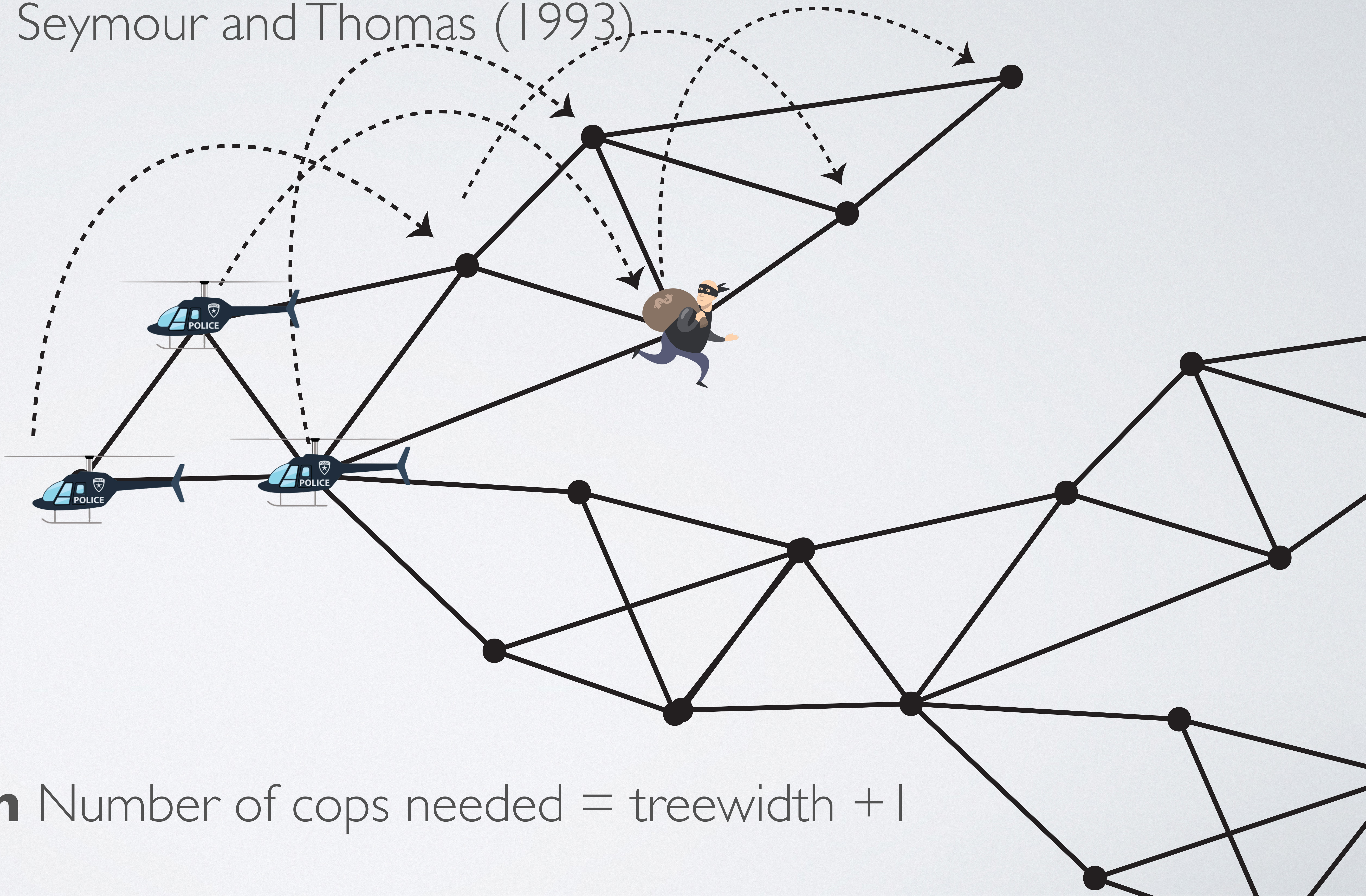
COPS AND ROBBER

Seymour and Thomas (1993)

k cops

a robber

moves along edges
at *infinite speed*



Theorem Number of cops needed = treewidth + 1

SPEED LIMIT

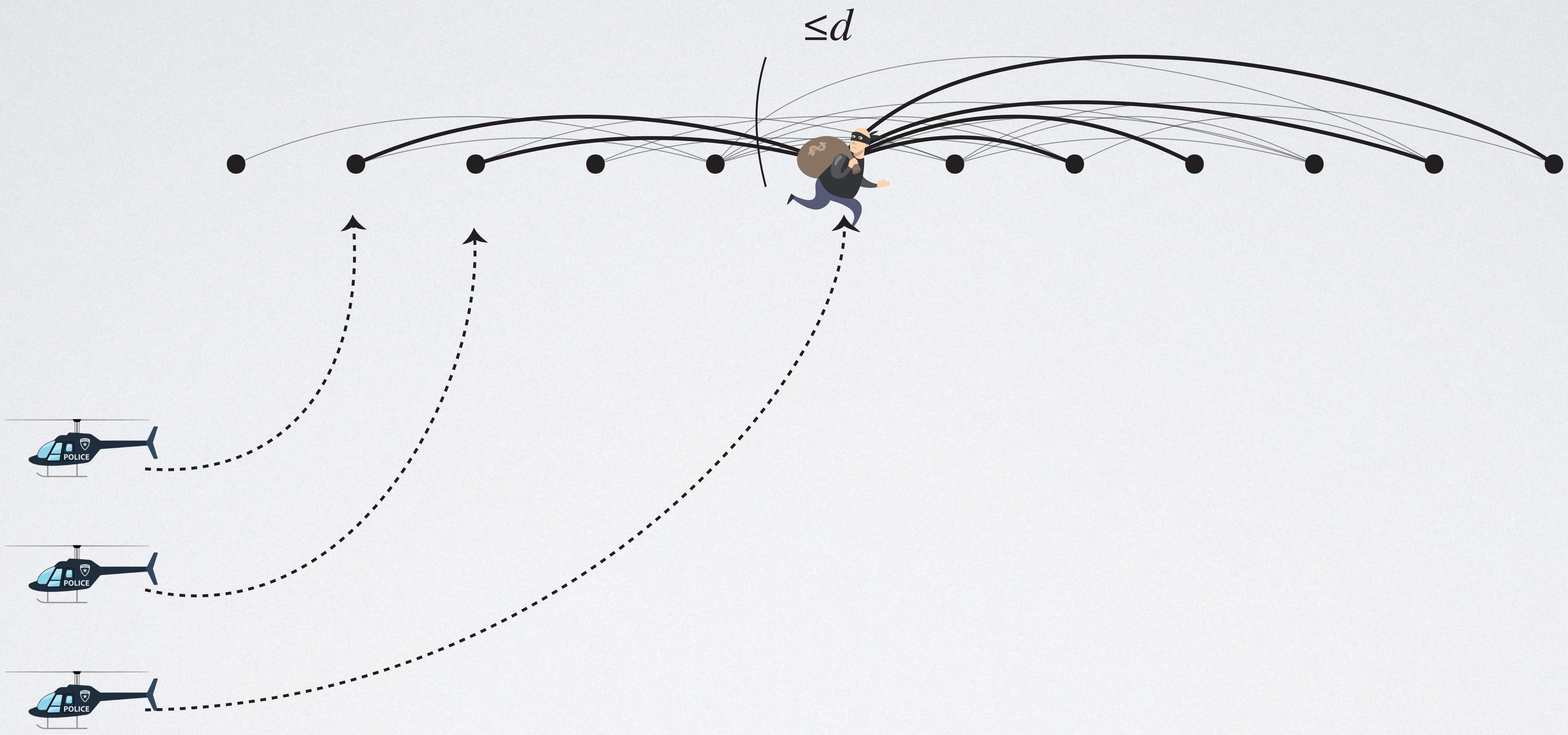
robber moves at speed r

$\mathrm{copwidth}_r(G)$:= number of cops needed to capture robber

Fact. $\text{copwidth}_1(G) = \text{degeneracy}(G)+1$

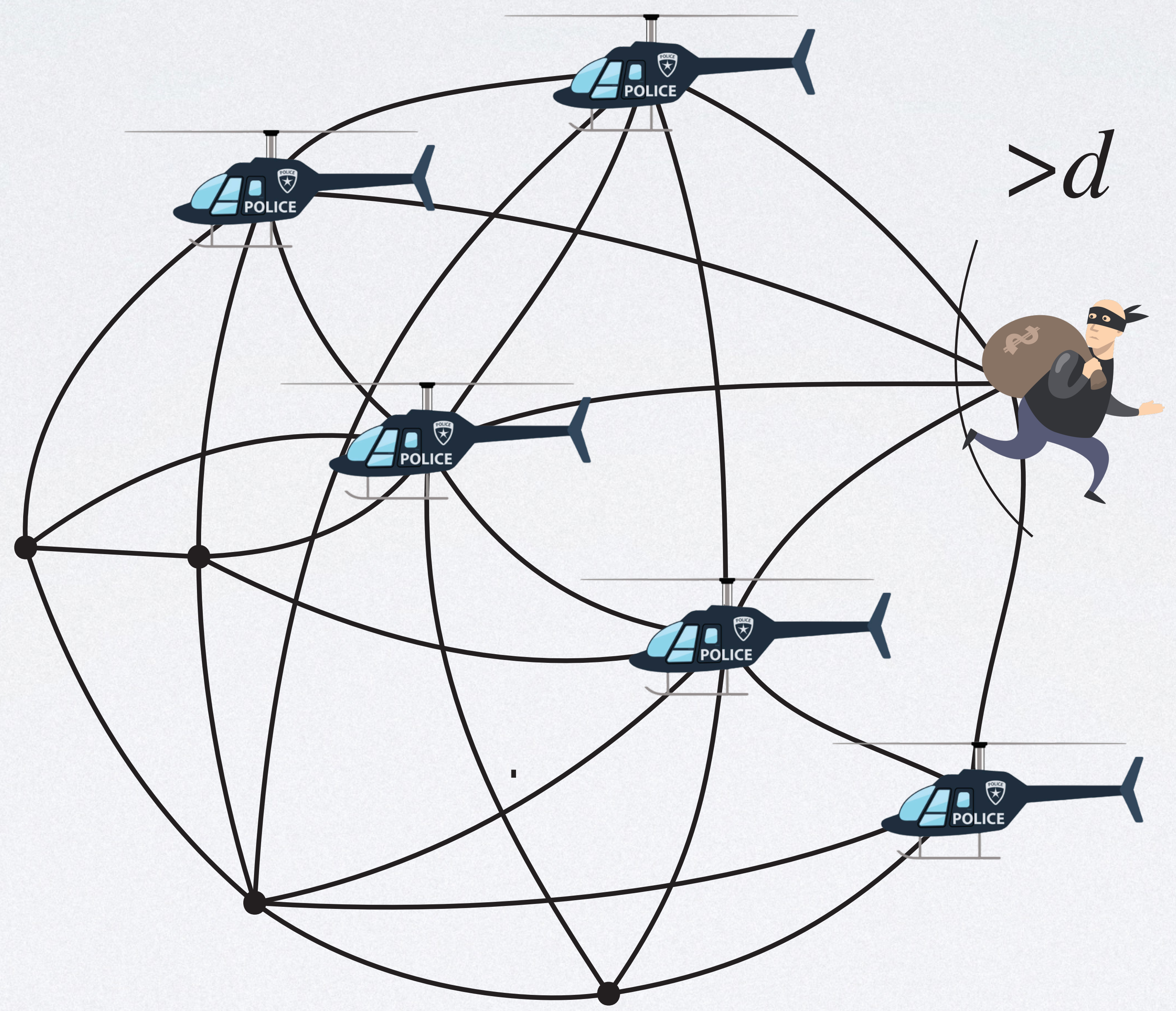
[≈Richerby & Thilikos 2008]

$\mathrm{copwidth}_1(G) \leq \mathrm{degeneracy}(G)+1$



copwidth$_1$(G) ≥ degeneracy(G)+1

G is not d-degenerate → d+1 cops do not suffice



Theorem $\text{copwidth}_1(G) = \text{degeneracy}(G)+1$

$$\text{copwidth}_\infty(G) = \text{treewidth}(G)+1$$

$$\text{copwidth}_r(G) \approx \Theta(r)\text{-weak coloring number}(G)$$

Theorem. Let C be a graph class. Then:

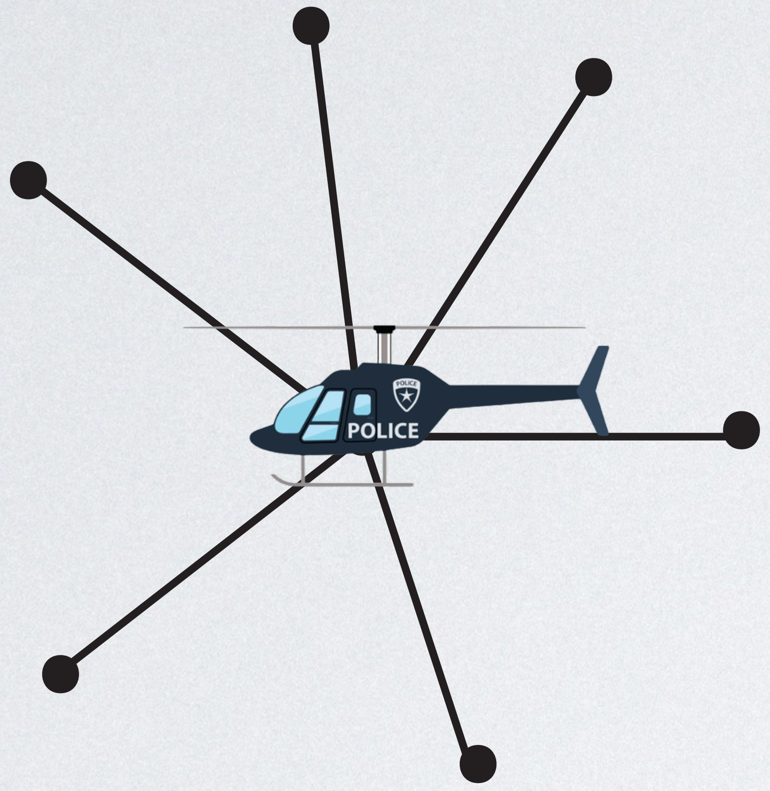
C has bounded expansion

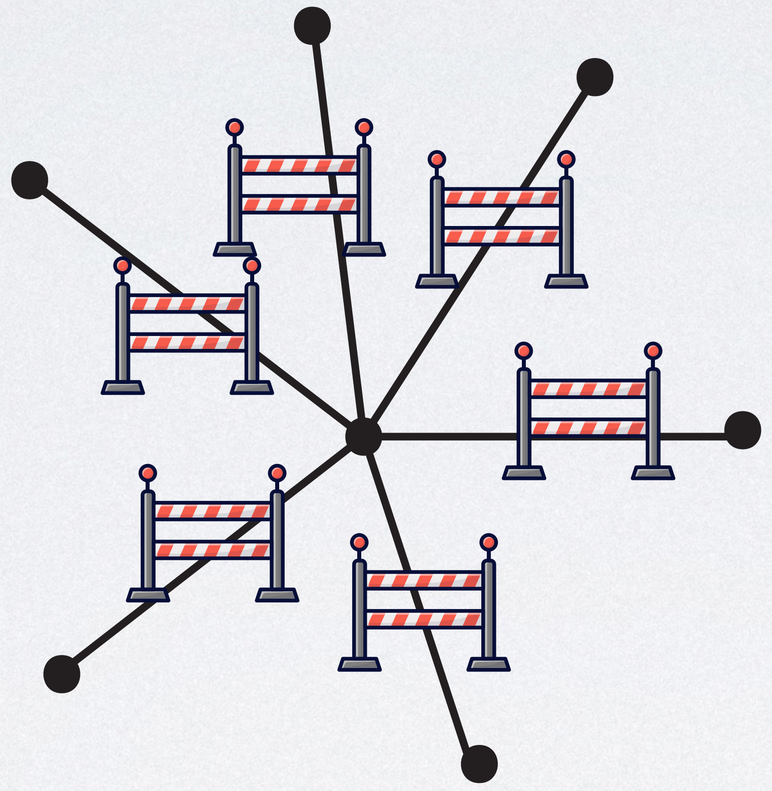
$$\Updownarrow$$

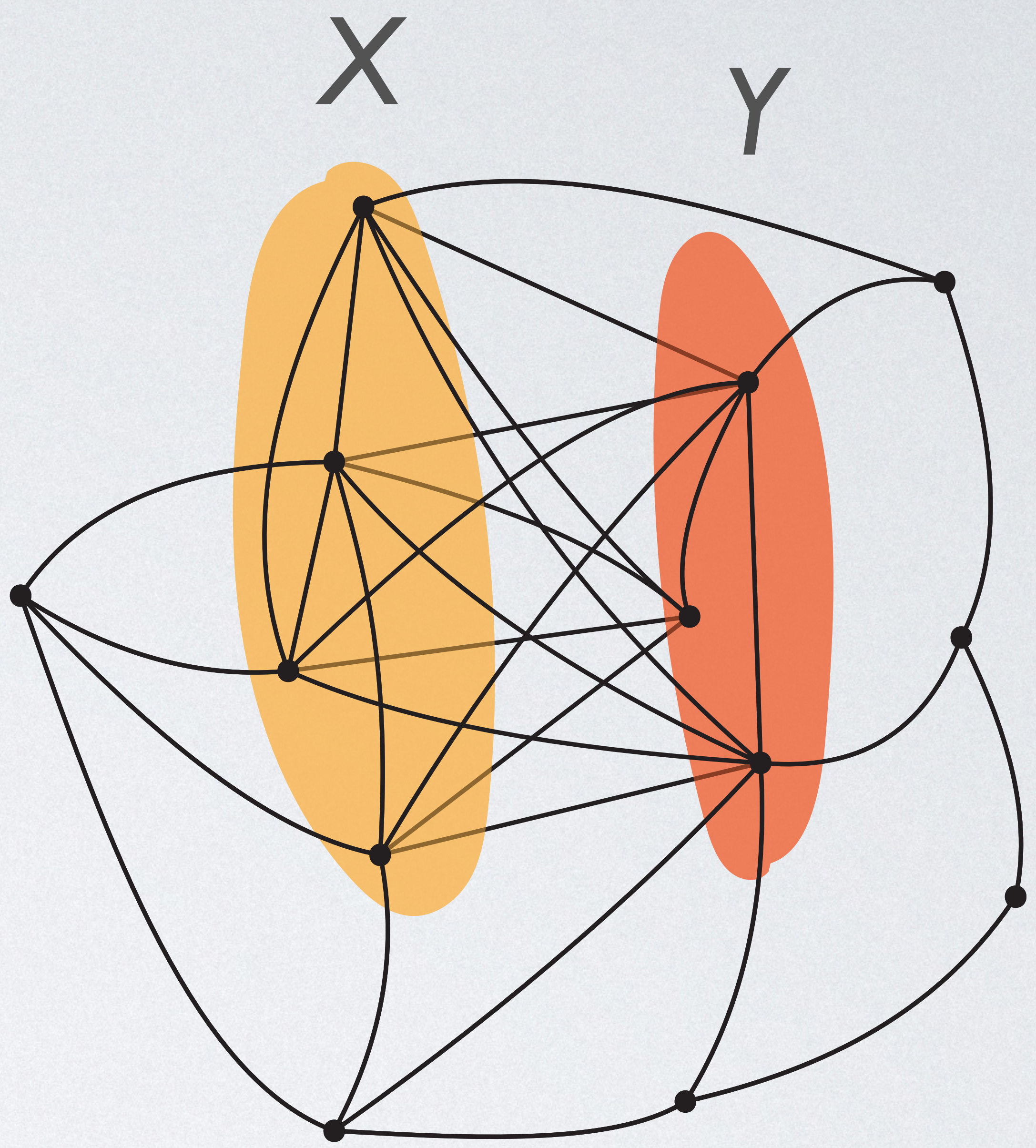
for all $r \in \mathbf{N}$, $\text{copwidth}_r(C) < \infty$

$$\sup_{G \in C} \text{copwidth}_r(G)$$

GOING DENSE

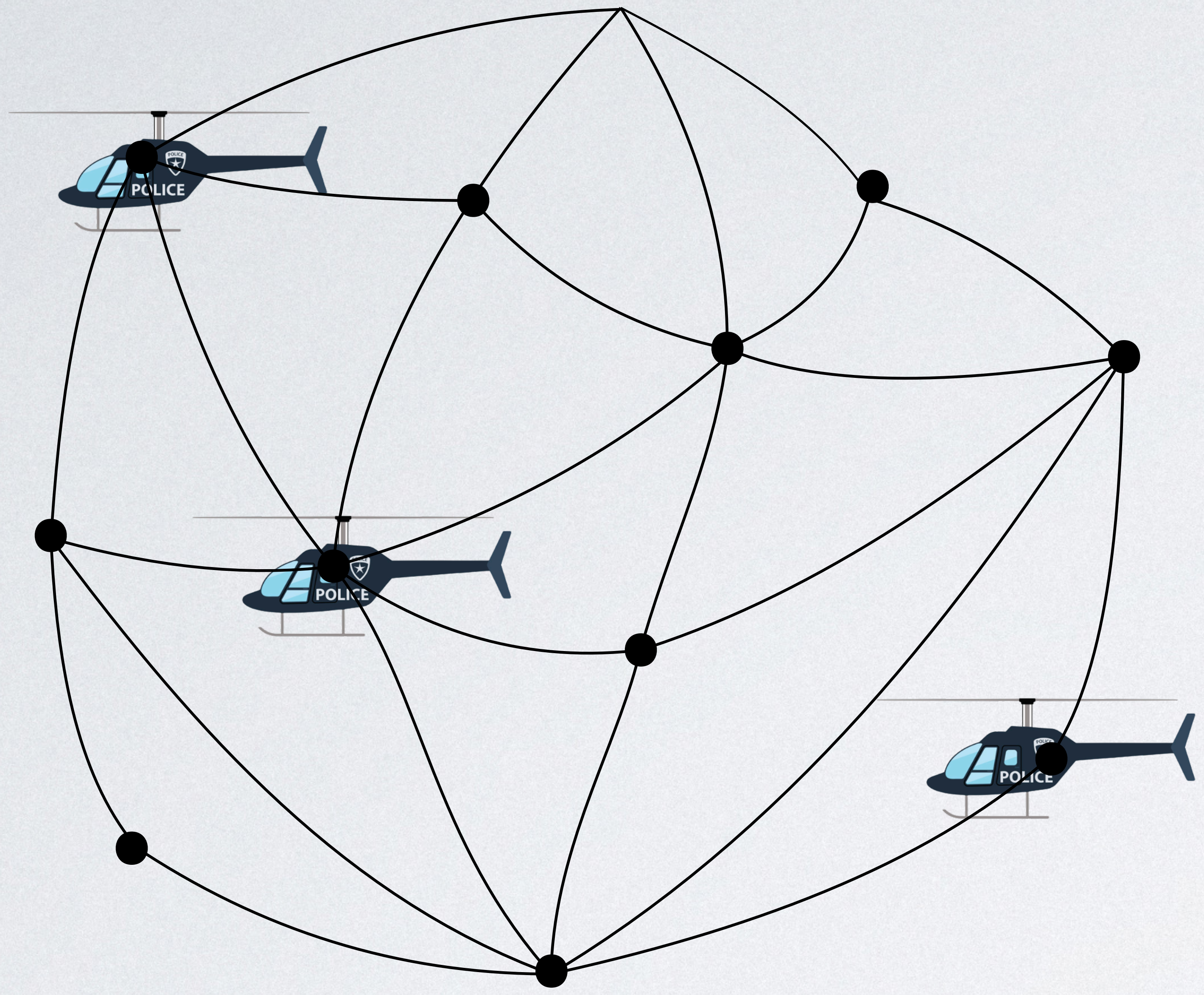






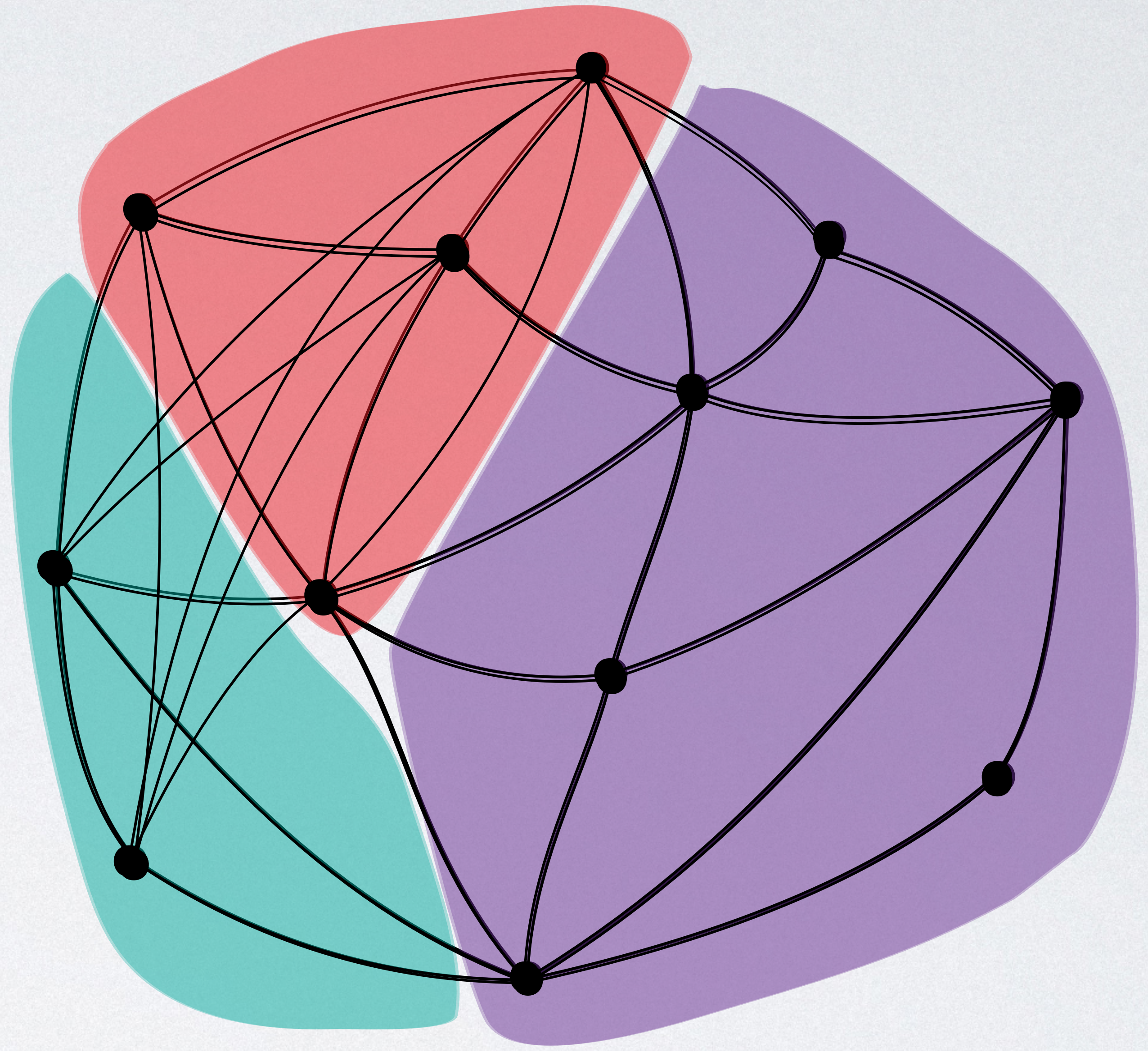
blocking a vertex = isolating a vertex → ***flip X and Y in G***

GOING DENSE



blocking k vertices

→



***k*-flip of *G*:**

partition $V(G) = A_1 \cup \ldots \cup A_k$

For each pair A_iA_j

flip or not

FLIPPER GAME

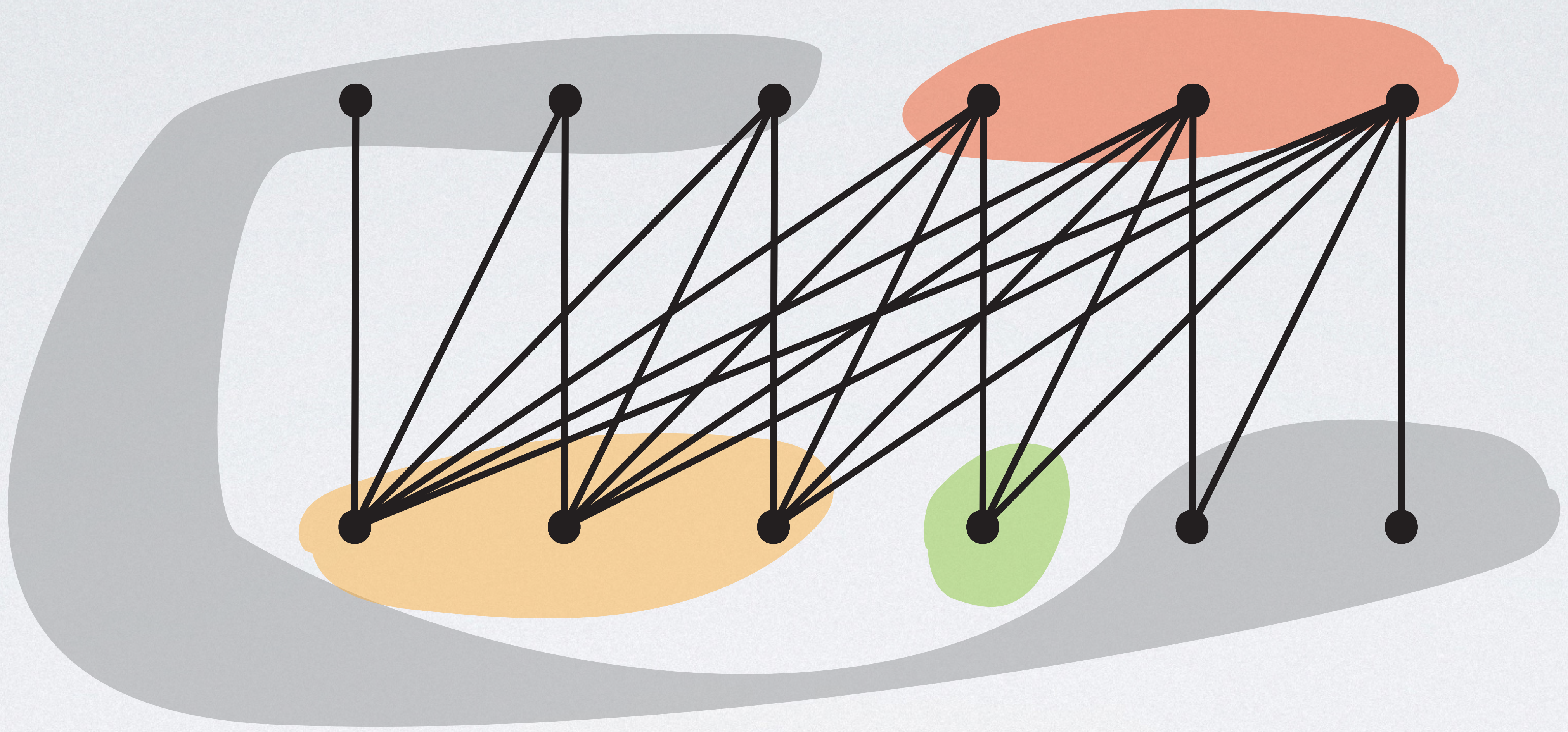
With radius r and flip power k

In each round:

- Flipper announces *next* k-flip G_k of G
- Then runner runs at speed r in *previous* k-flip G_{k-1} of G
- Runner looses if new position is isolated in G_k

$\text{flipwidth}_r(G)$:= minimum k needed to capture runner

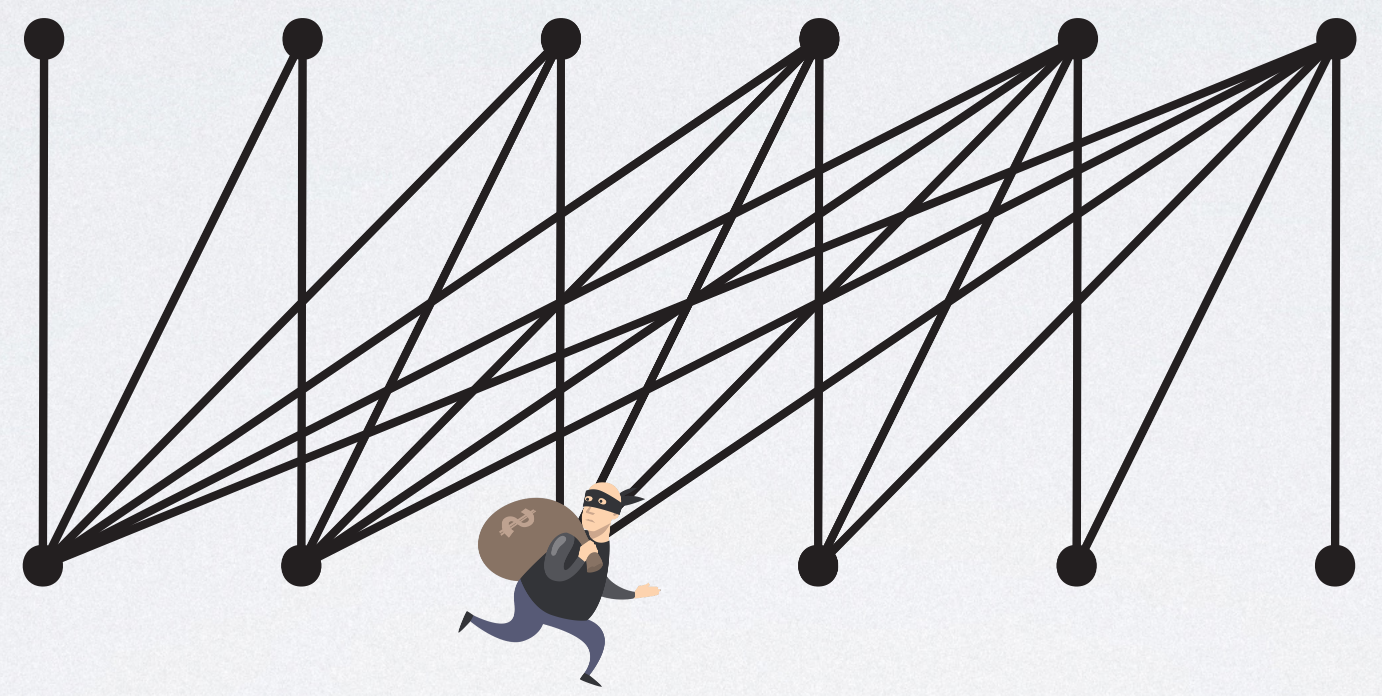
EXAMPLE



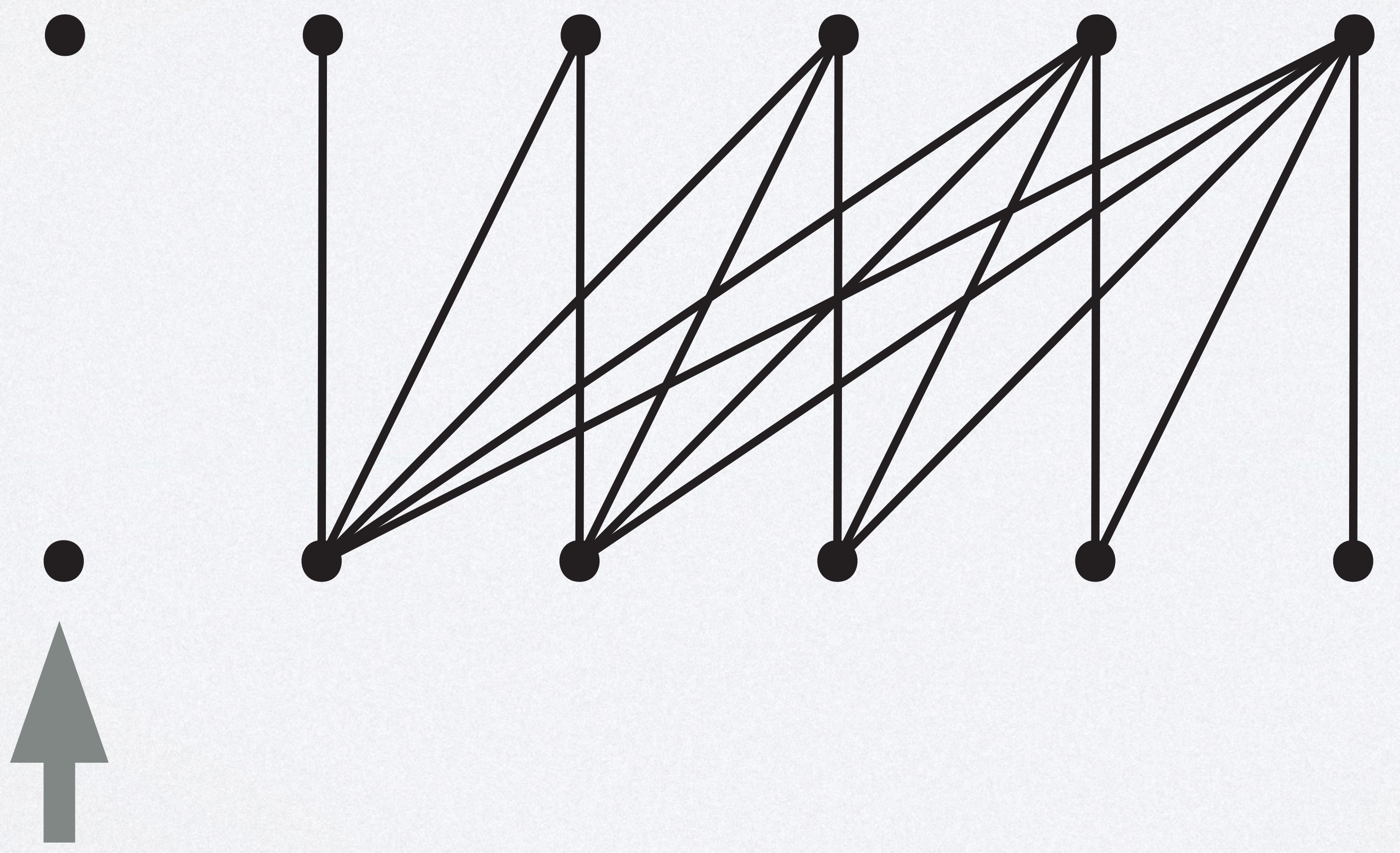
4-flip

EXAMPLE

speed $r = \infty$, flip power k=4

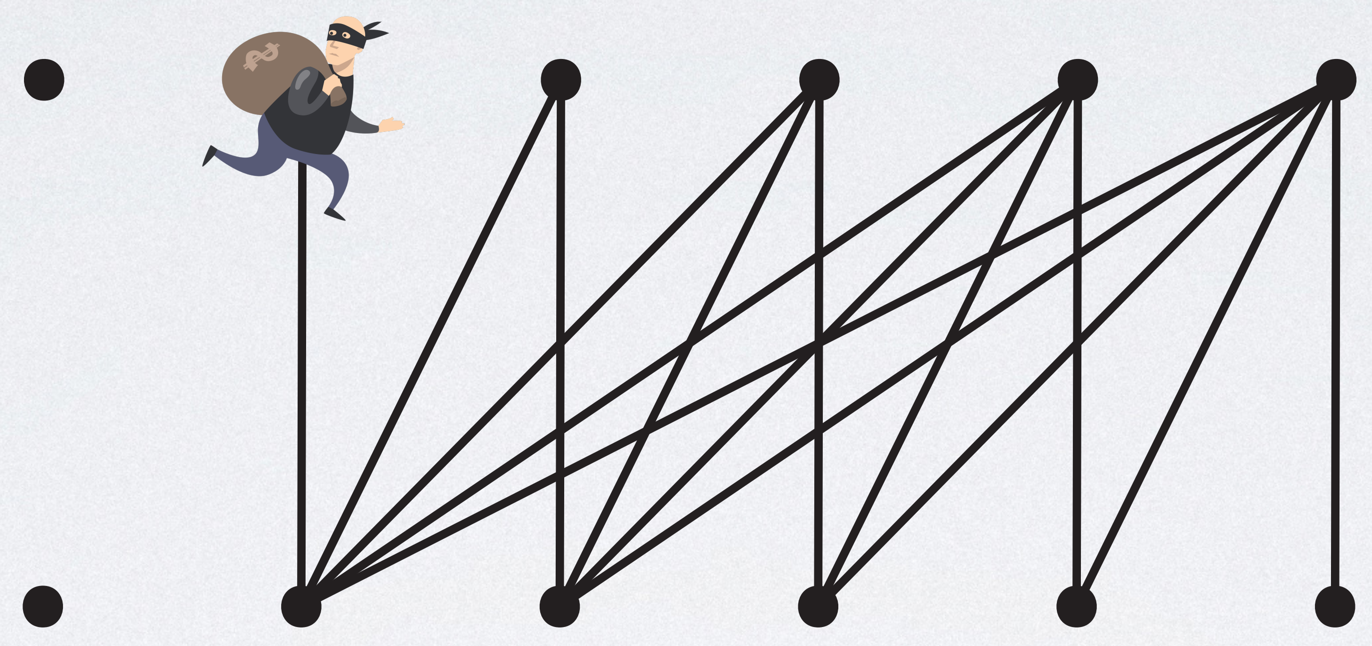


next flip:

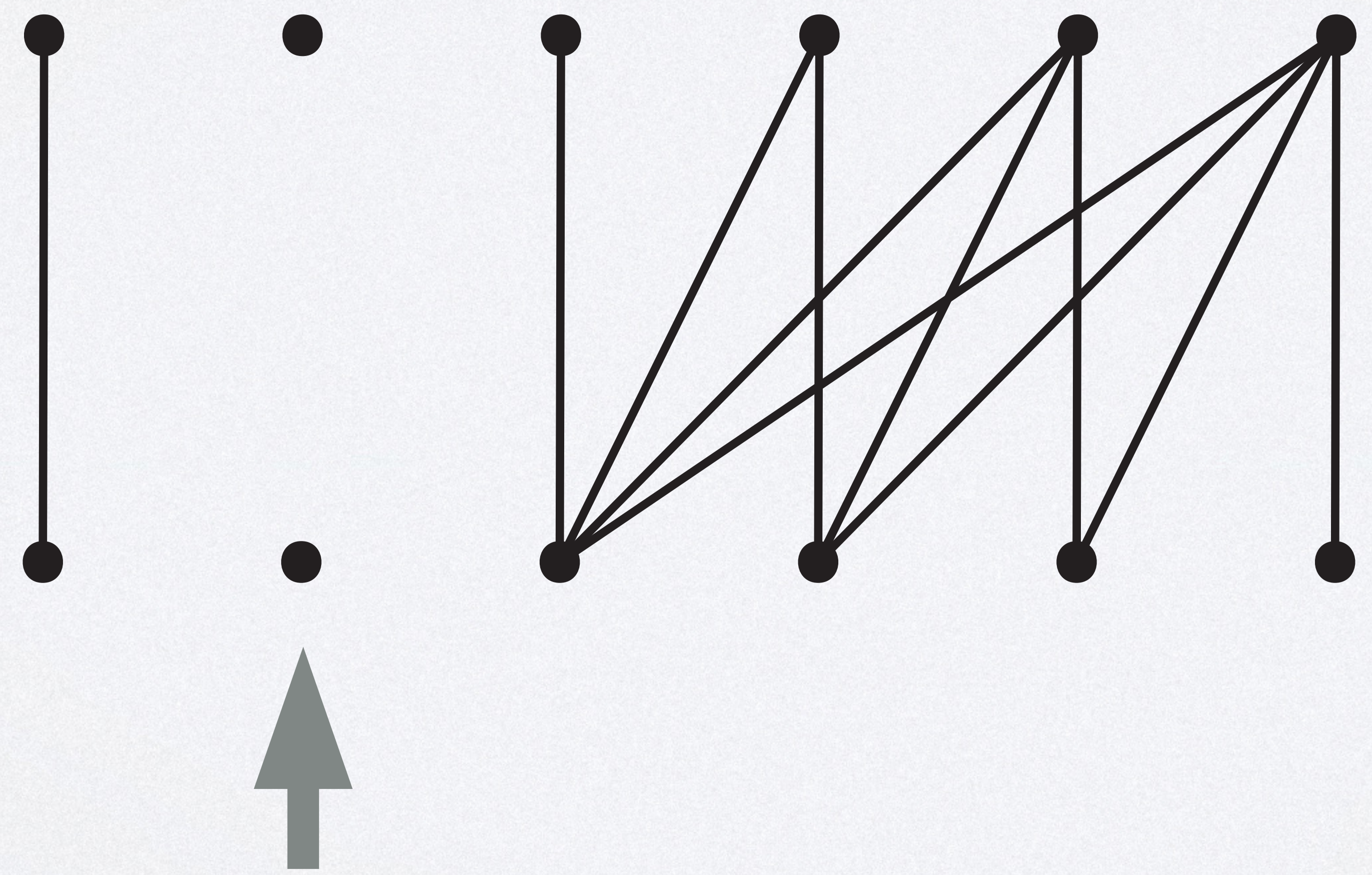


EXAMPLE

speed $r = \infty$, flip power $k=4$



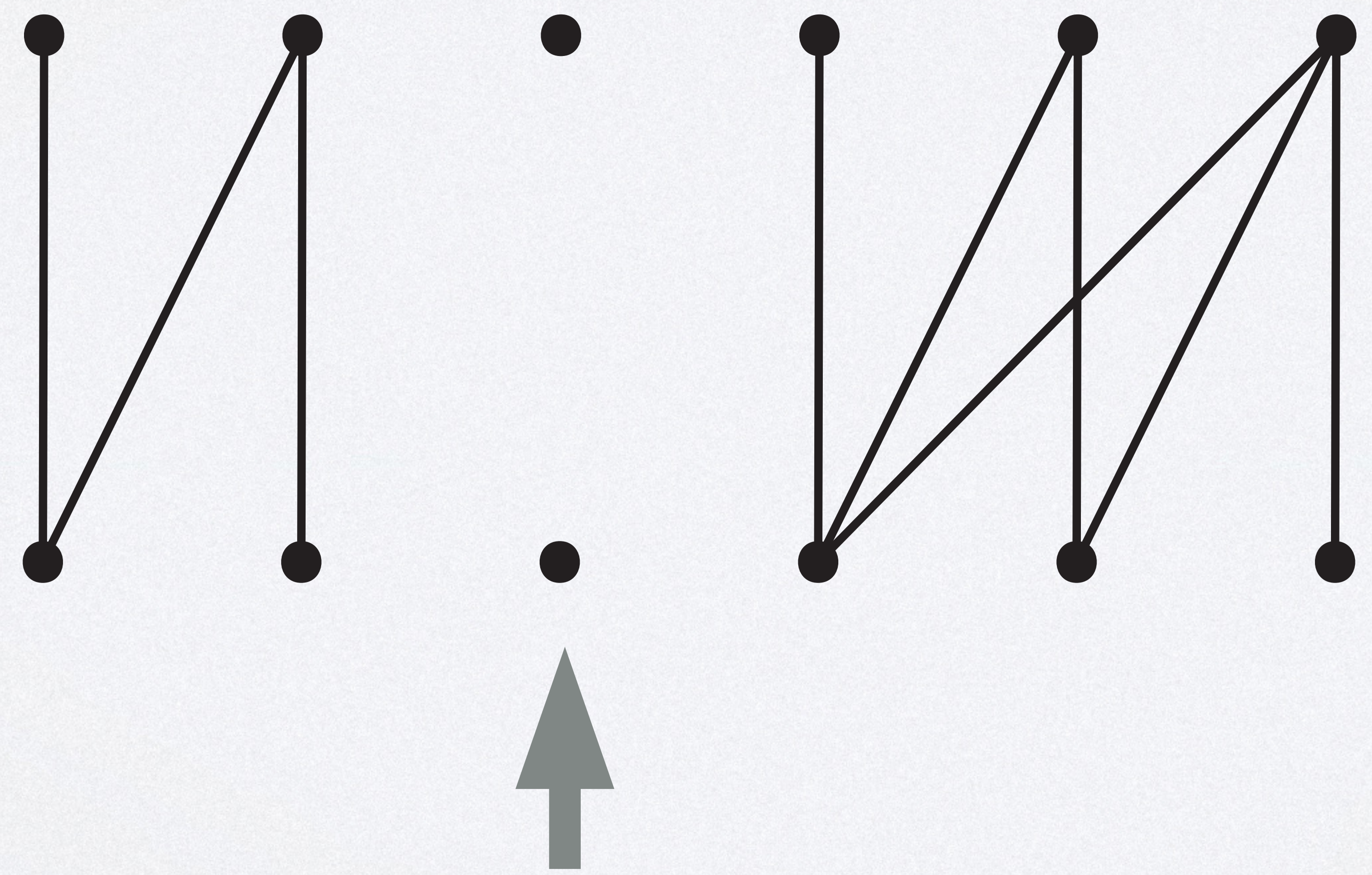
next flip:



EXAMPLE

speed $r = \infty$, flip power k=4

next flip:



EXAMPLE

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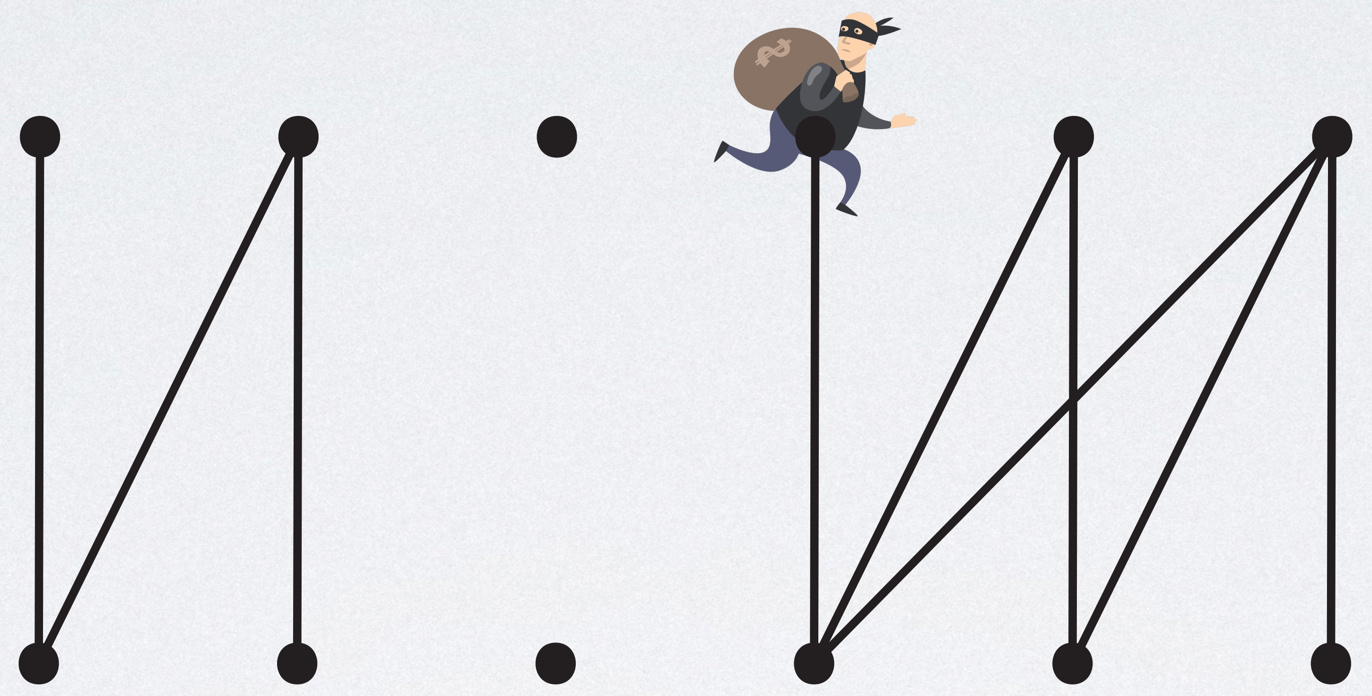
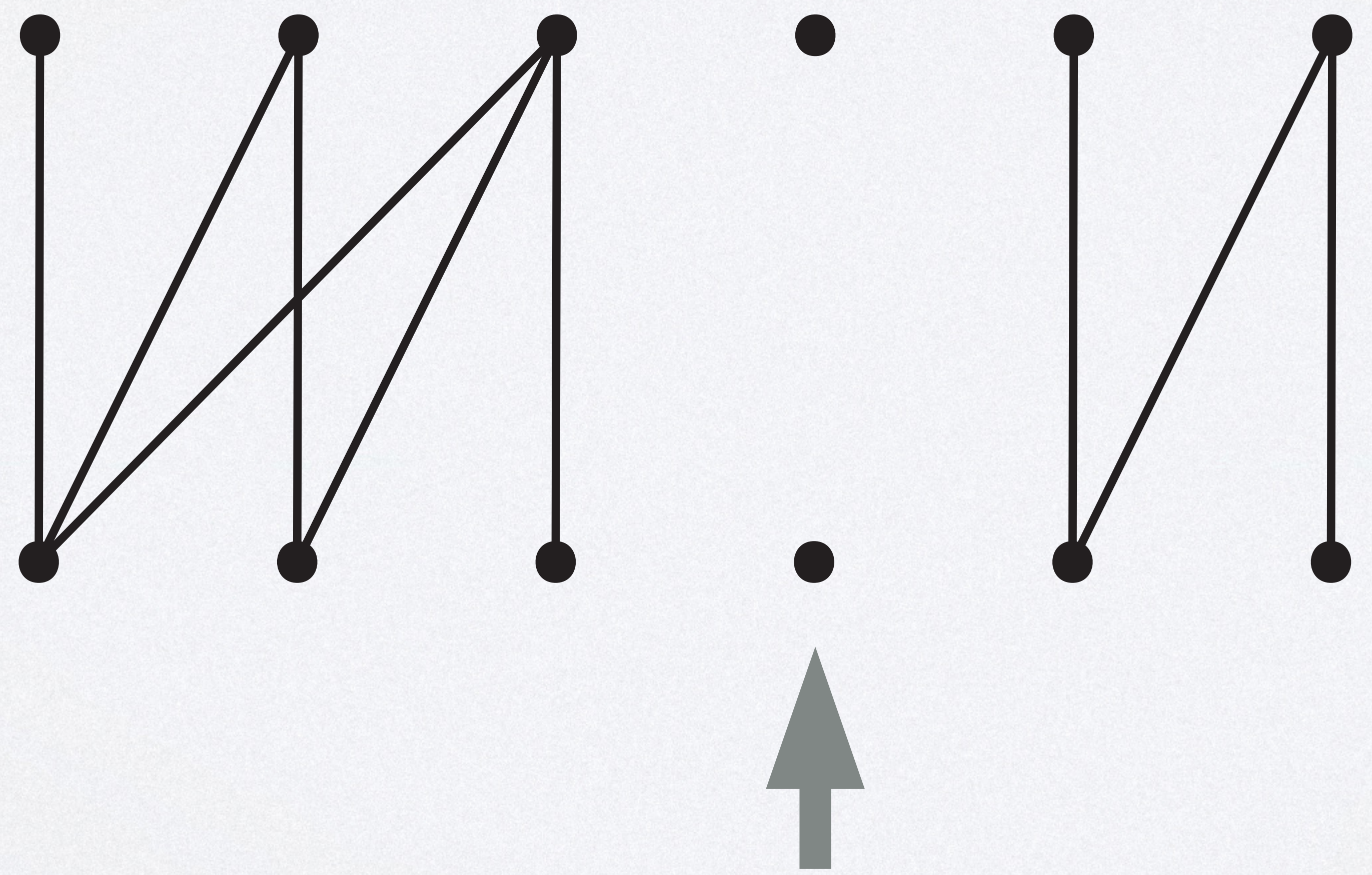


next 4-flip:



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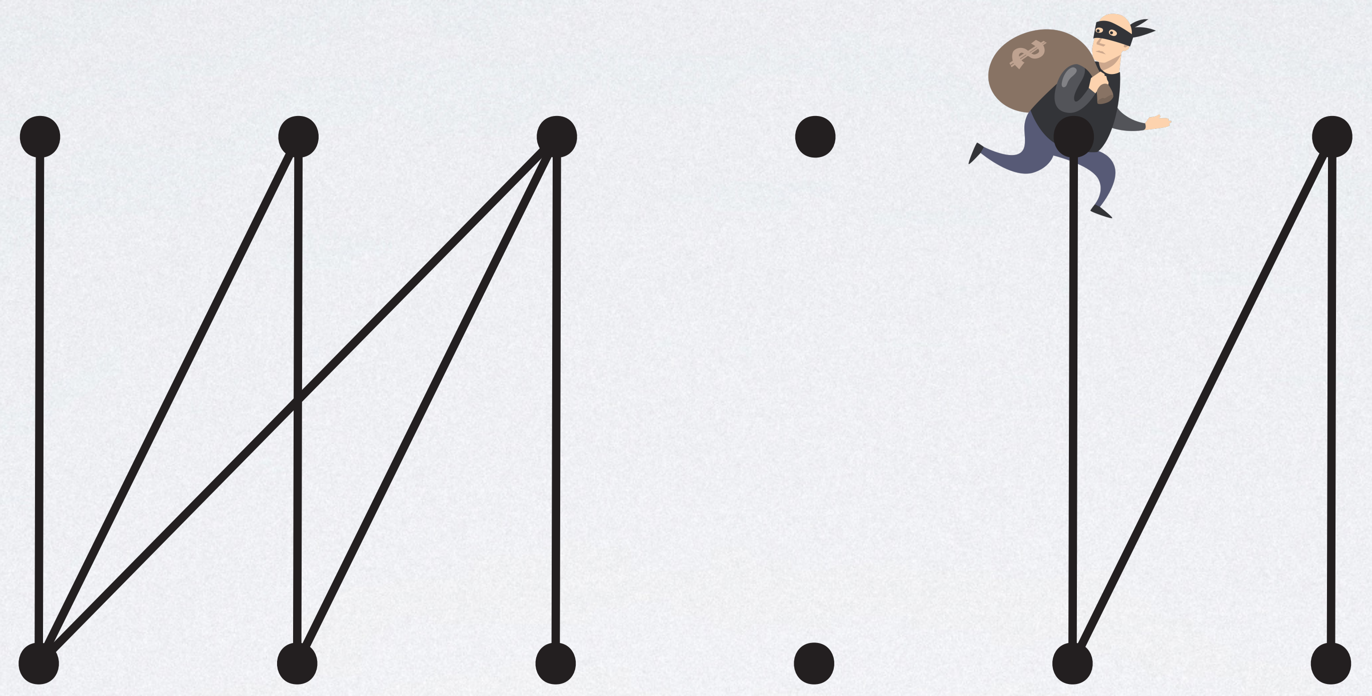
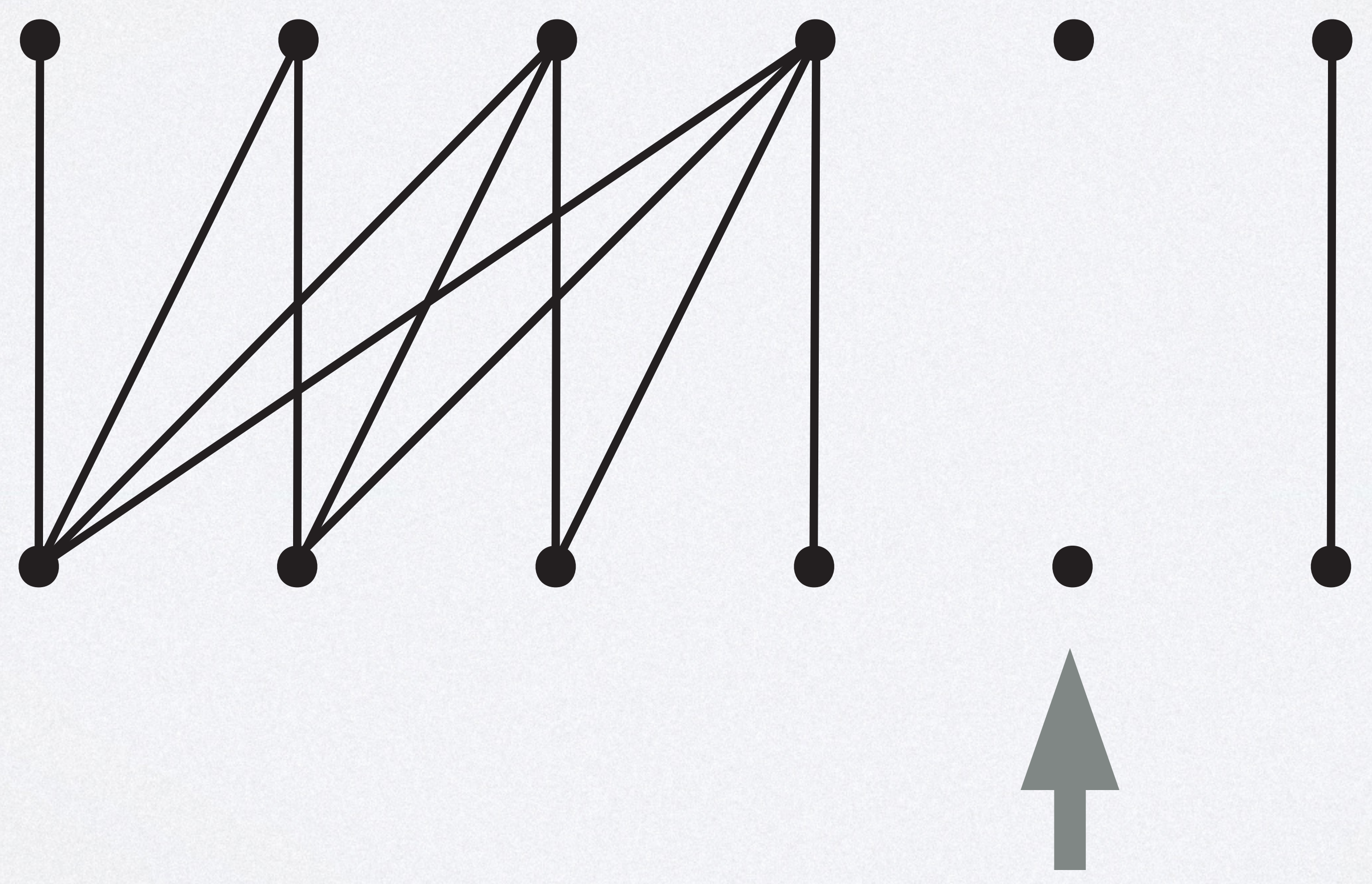
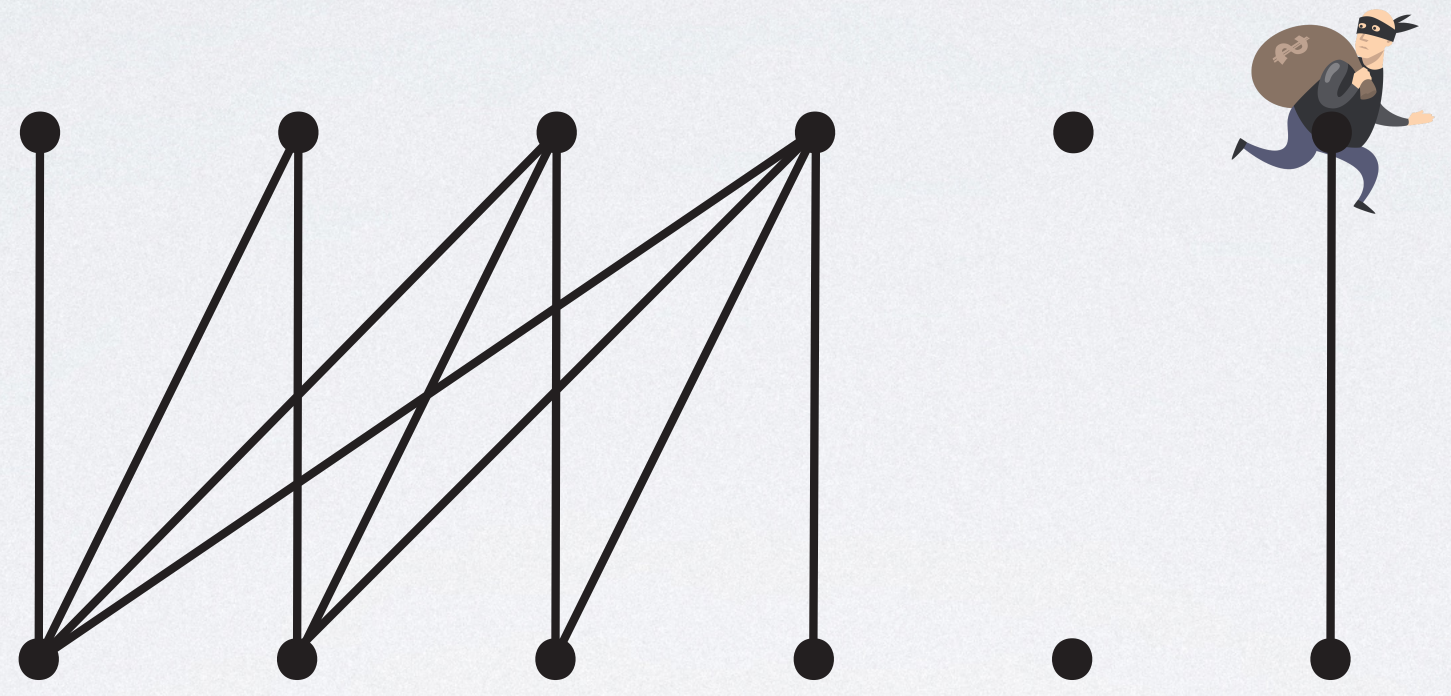


next 4-flip:

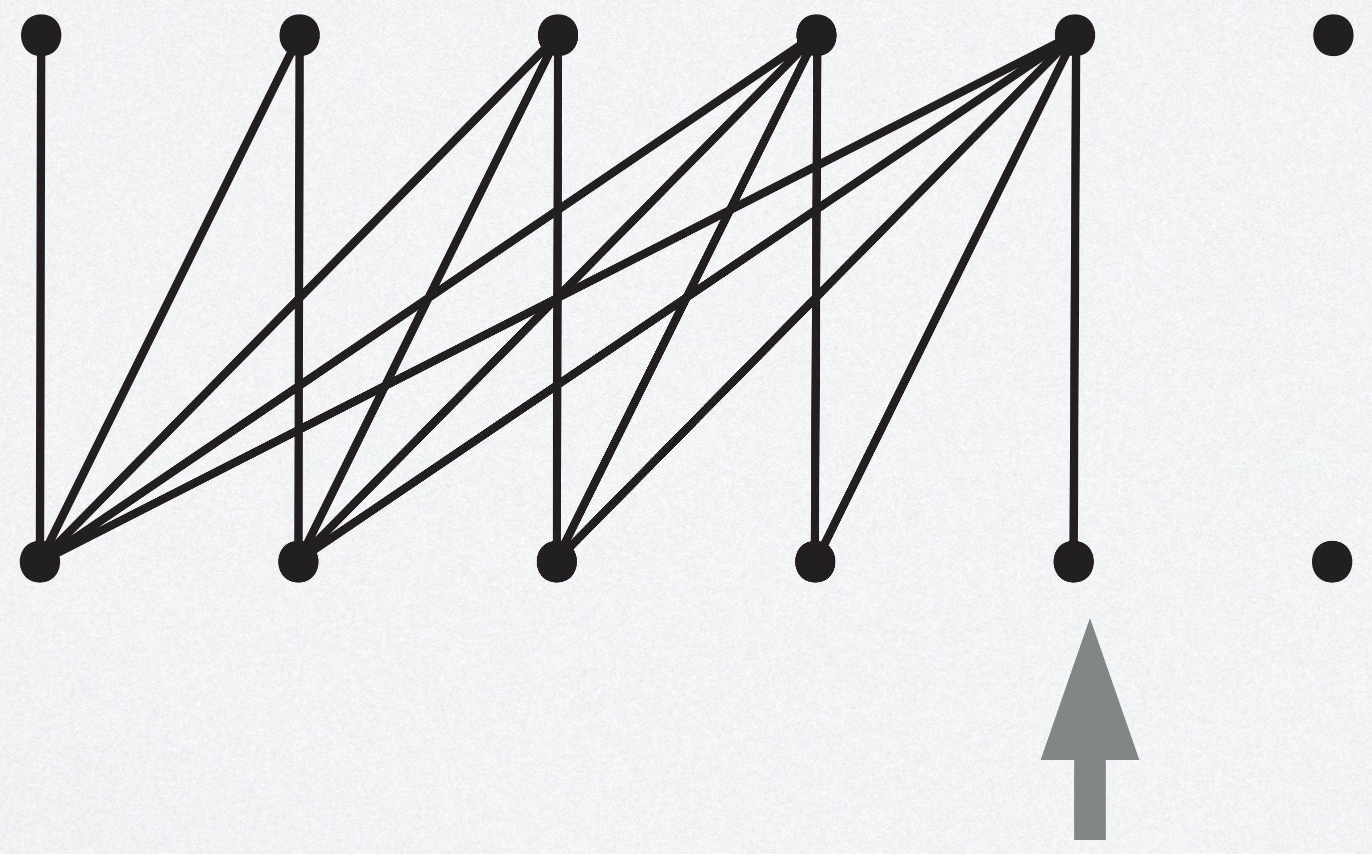


EXAMPLE

speed $r = \infty$, flip power k=4

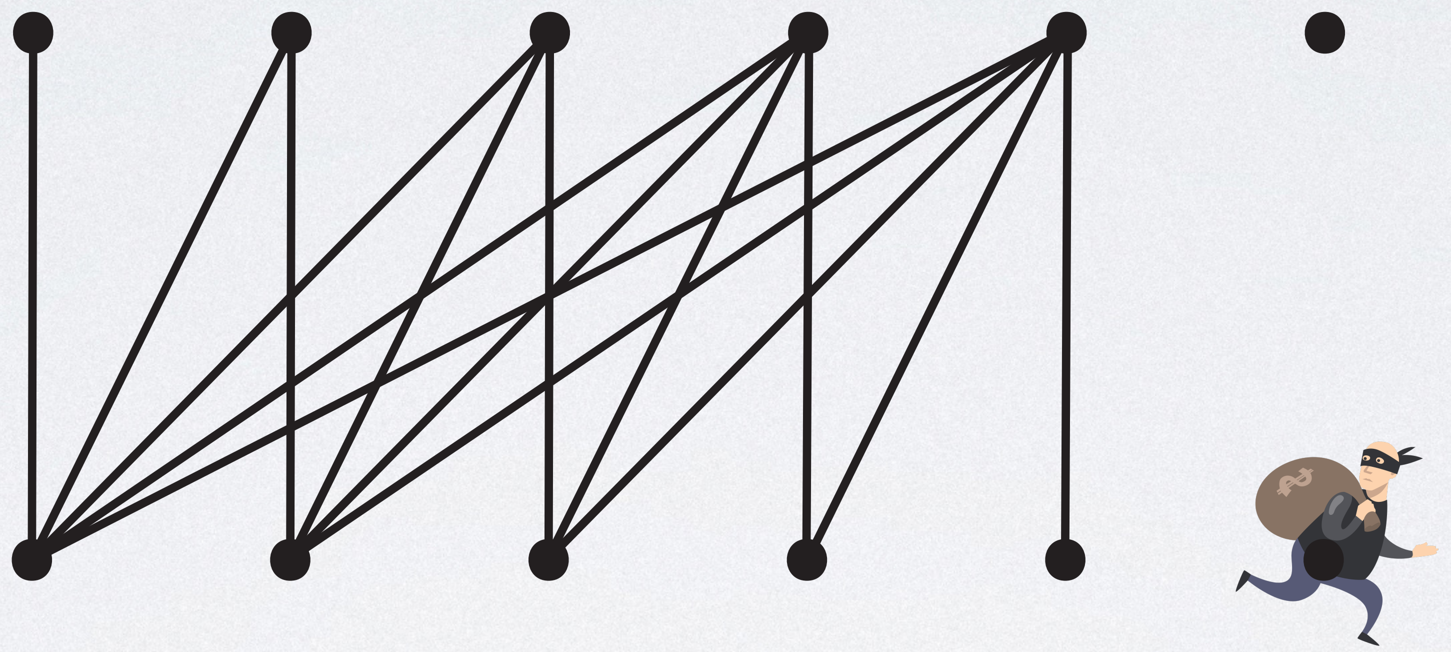


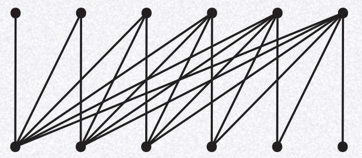
next 4-flip:



EXAMPLE

speed $r = \infty$, flip power k=4



flip-width()≤4

RADIUS ∞

Theorem $\text{flip-width}_\infty(G) \approx \text{clique-width}(G)$

characterization of clique-width via games

Corollary A class C has bounded clique-width

$\Leftrightarrow$

$\text{flip-width}_\infty(C)<\infty$

$$\text{rank-width}(G) \leq \text{flip-width}_\infty(G) \leq O(2^{\text{rank-width}(G)})$$

$\text{rank-width}(G) \leq k$

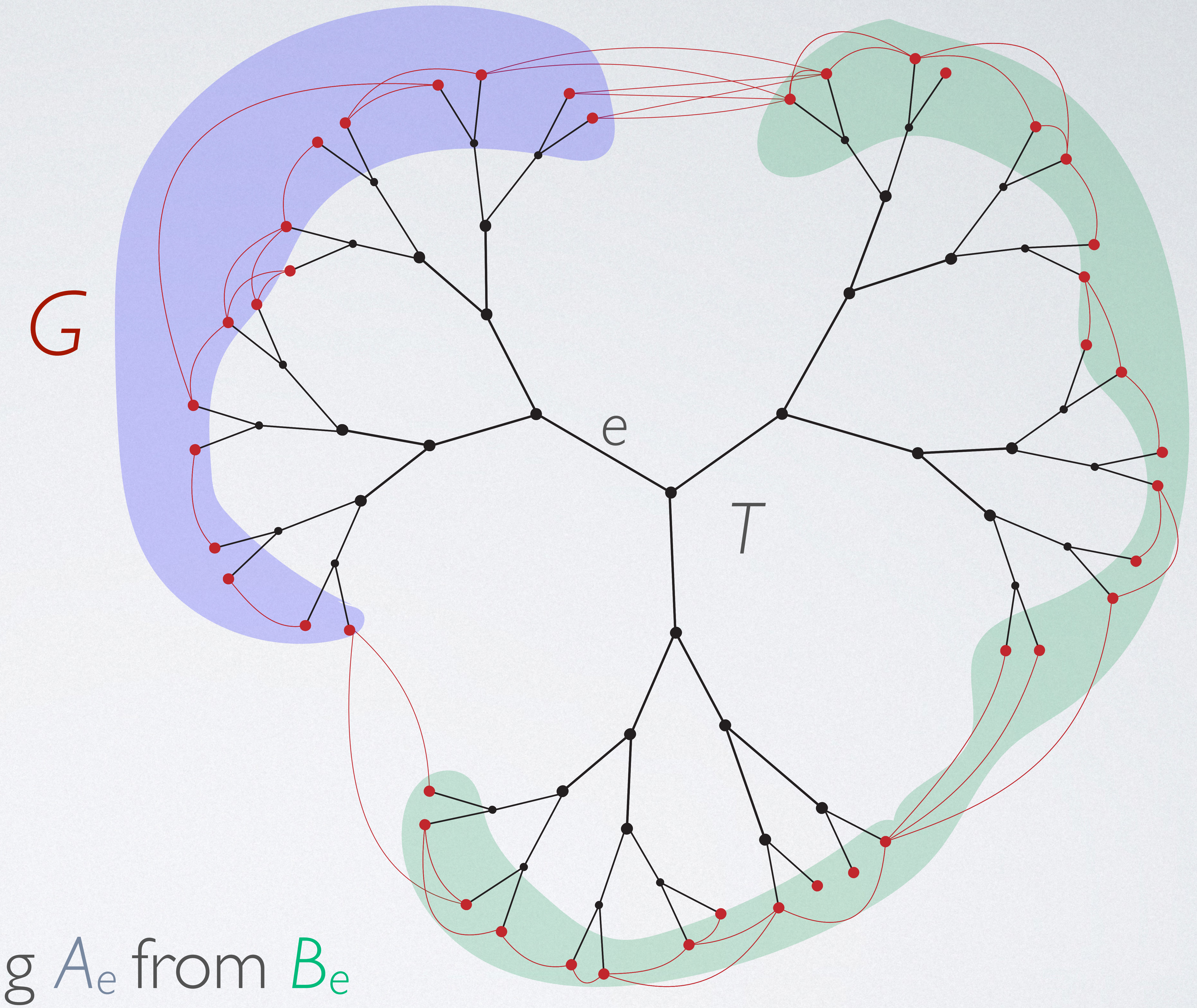
$\Downarrow$

exists cubic tree T:

- $V(G) = \text{Leaves}(T)$
- For every edge e of T

$V(G) = A_e \cup B_e$

$\text{Adj}_G[A_e, B_e]$ has rank $\leq k$



$\Rightarrow \exists\ O(2^k)$-flip G' of G separating A_e from B_e

$\text{flip-width}_\infty(G) \leq O(2^{\text{rank-width}(G)})$

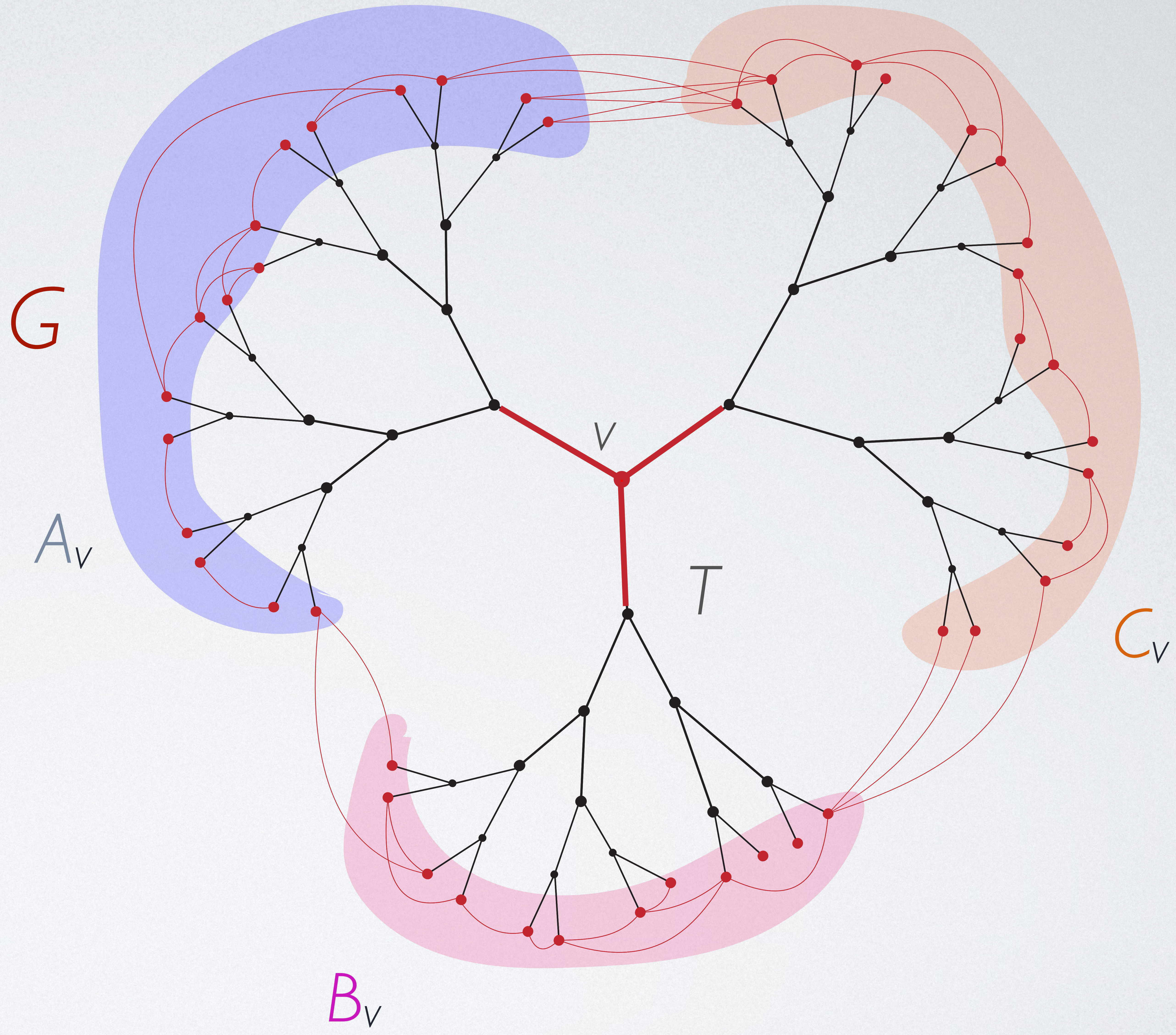
rank-width$(G) \leq k$

$\Downarrow$

For every node $v \in V(T)$

$\exists$ $O(2^k)$-flip G' of G

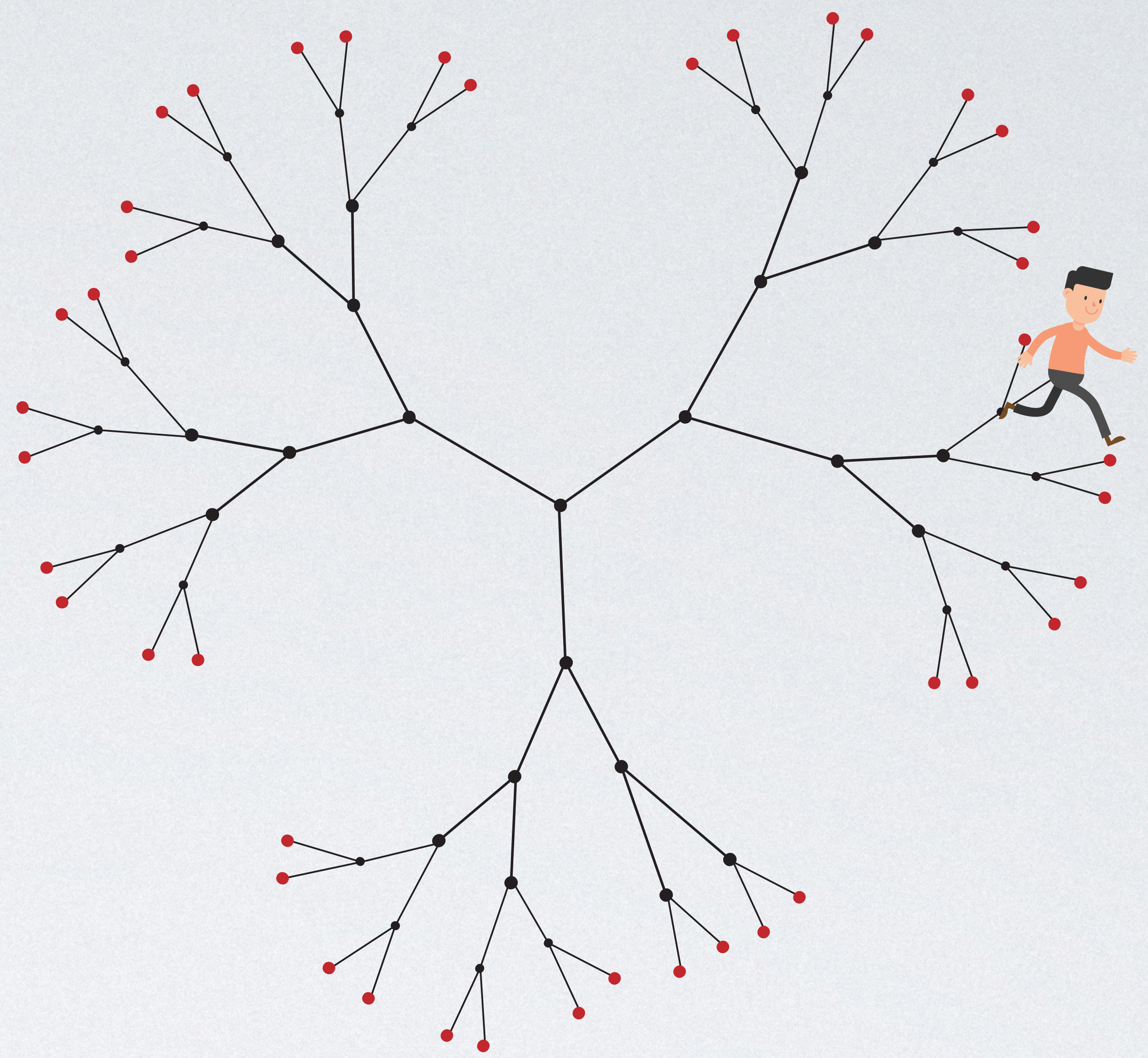
pairwise separating A_v, B_v, C_v



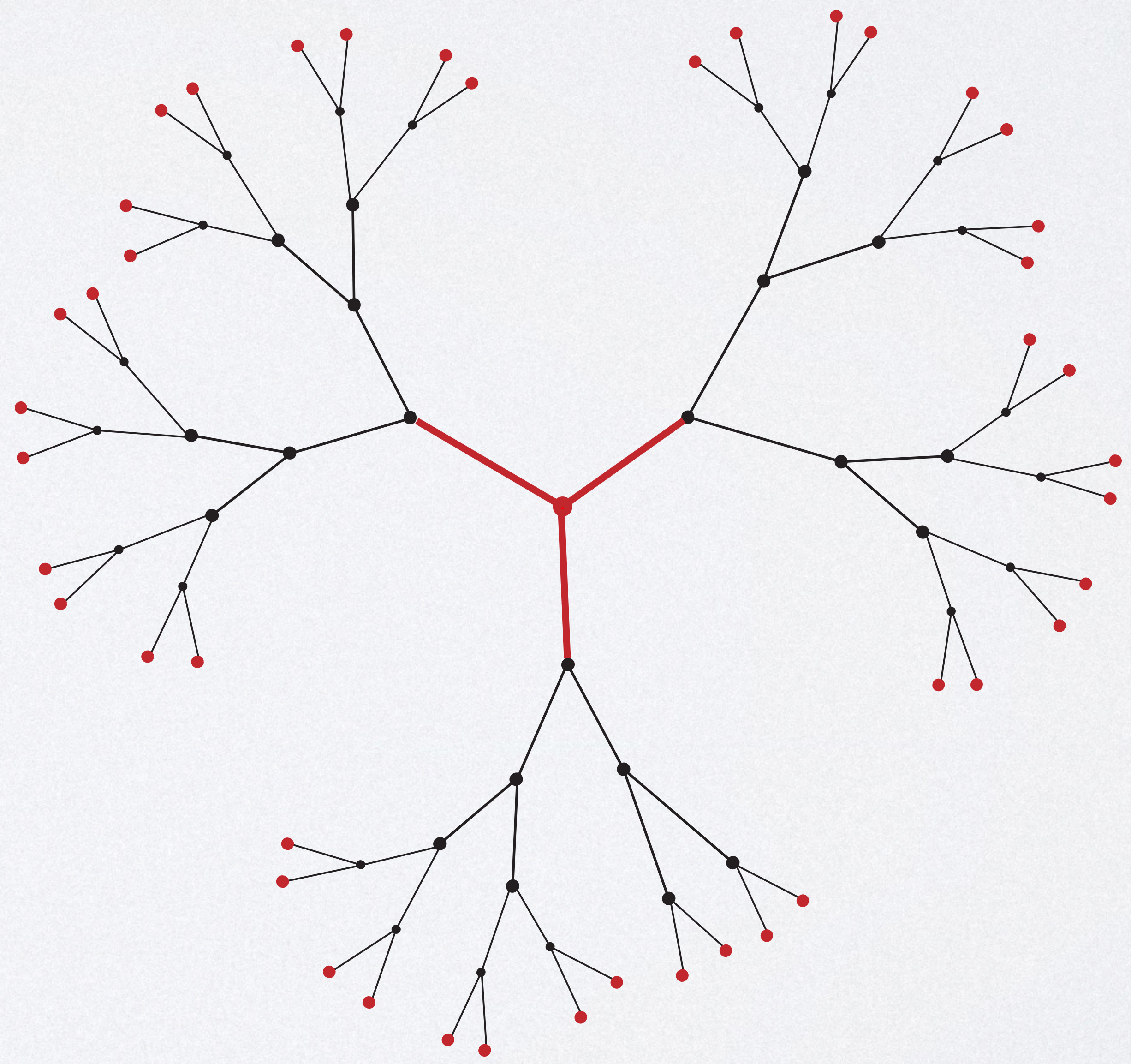
flip-width$_\infty(G) \leq O(2^{\text{rank-width}(G)})$

rank-width$(G) \leq k$

$\Downarrow$



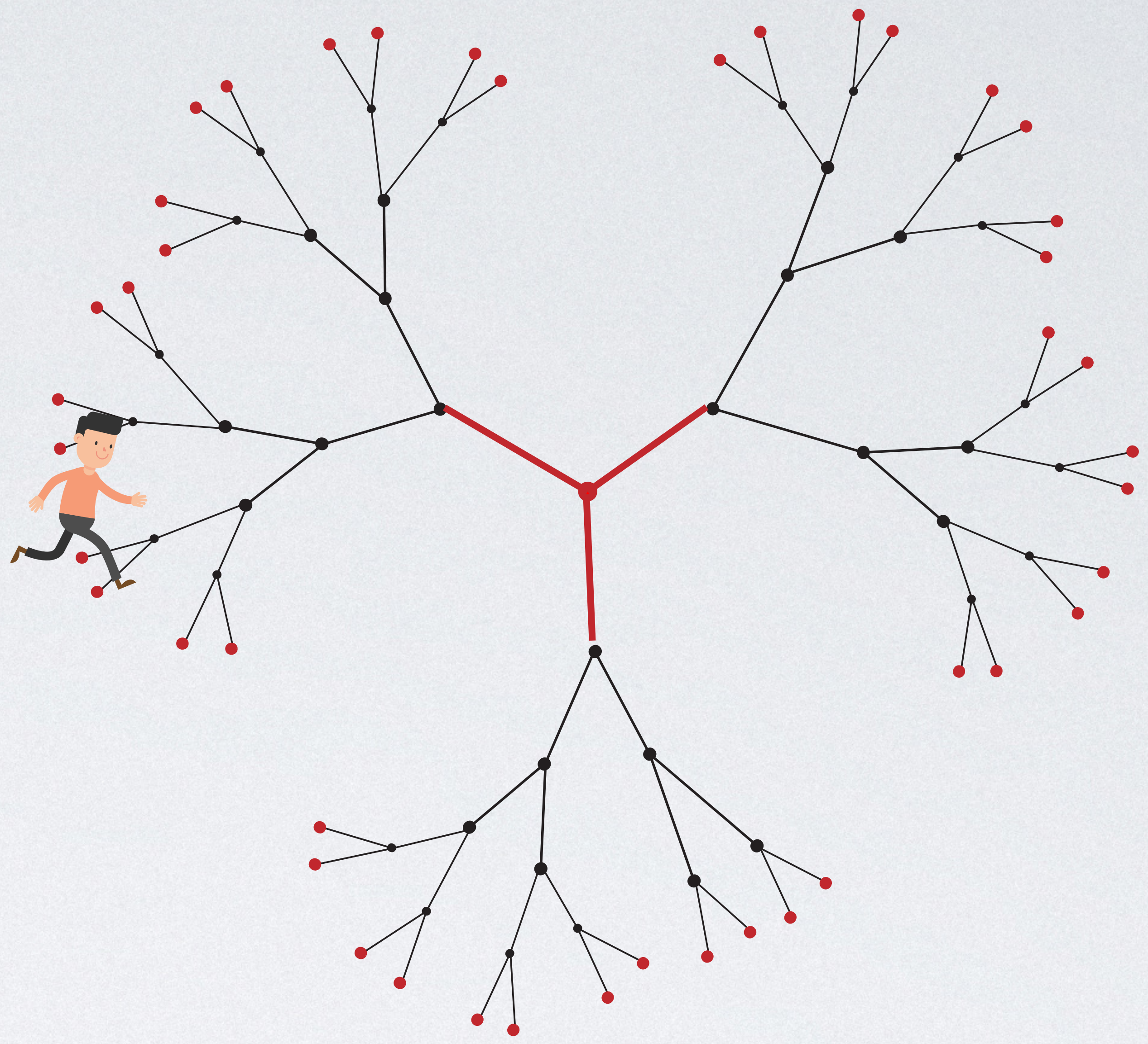
next flip:



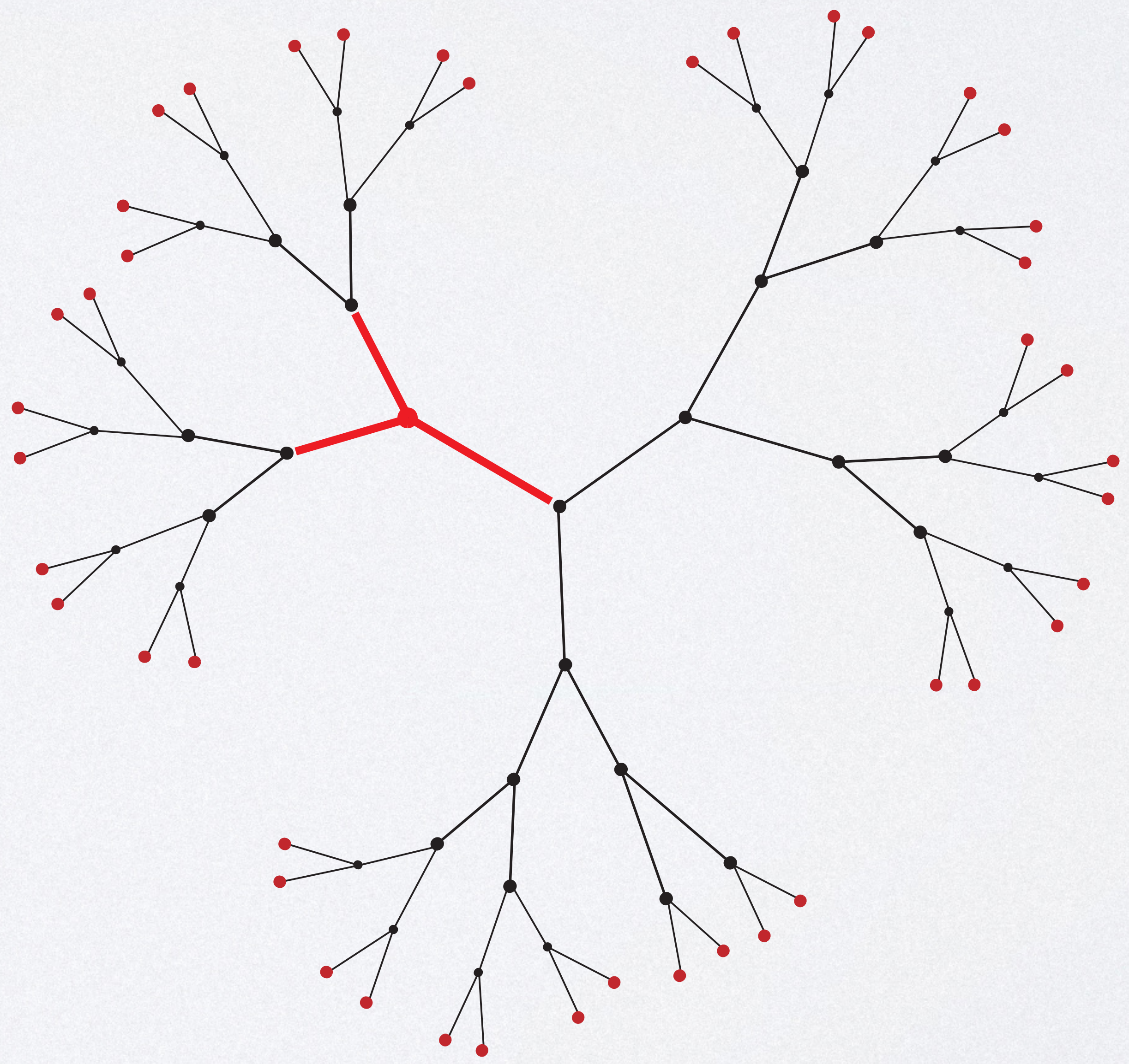
$\text{flip-width}_\infty(G) \leq O(2^{\text{rank-width}(G)})$

$\text{rank-width}(G) \leq k$

$\Downarrow$



next flip:

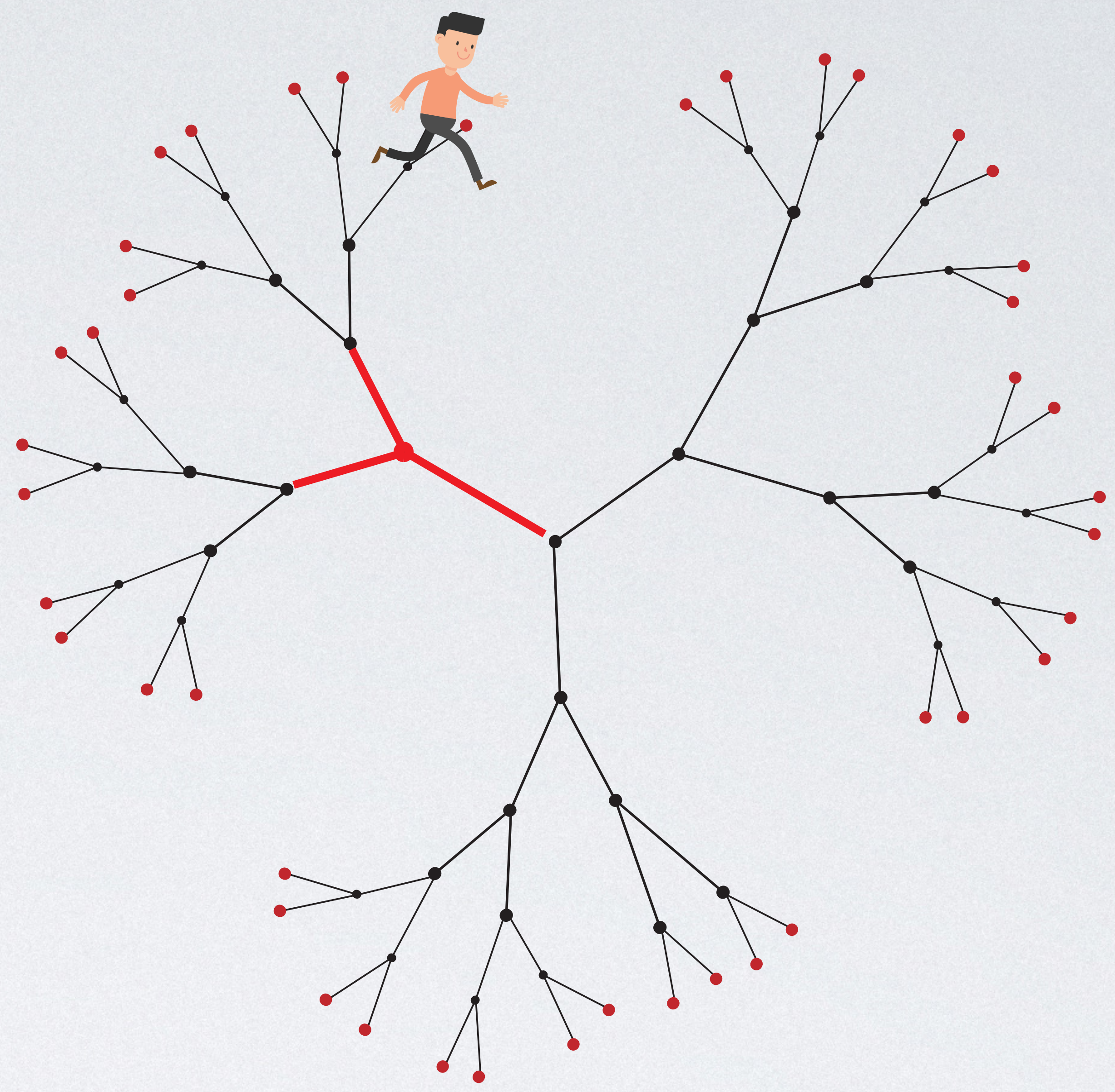


$\text{flip-width}_\infty(G) \leq O(2^{\text{rank-width}(G)})$

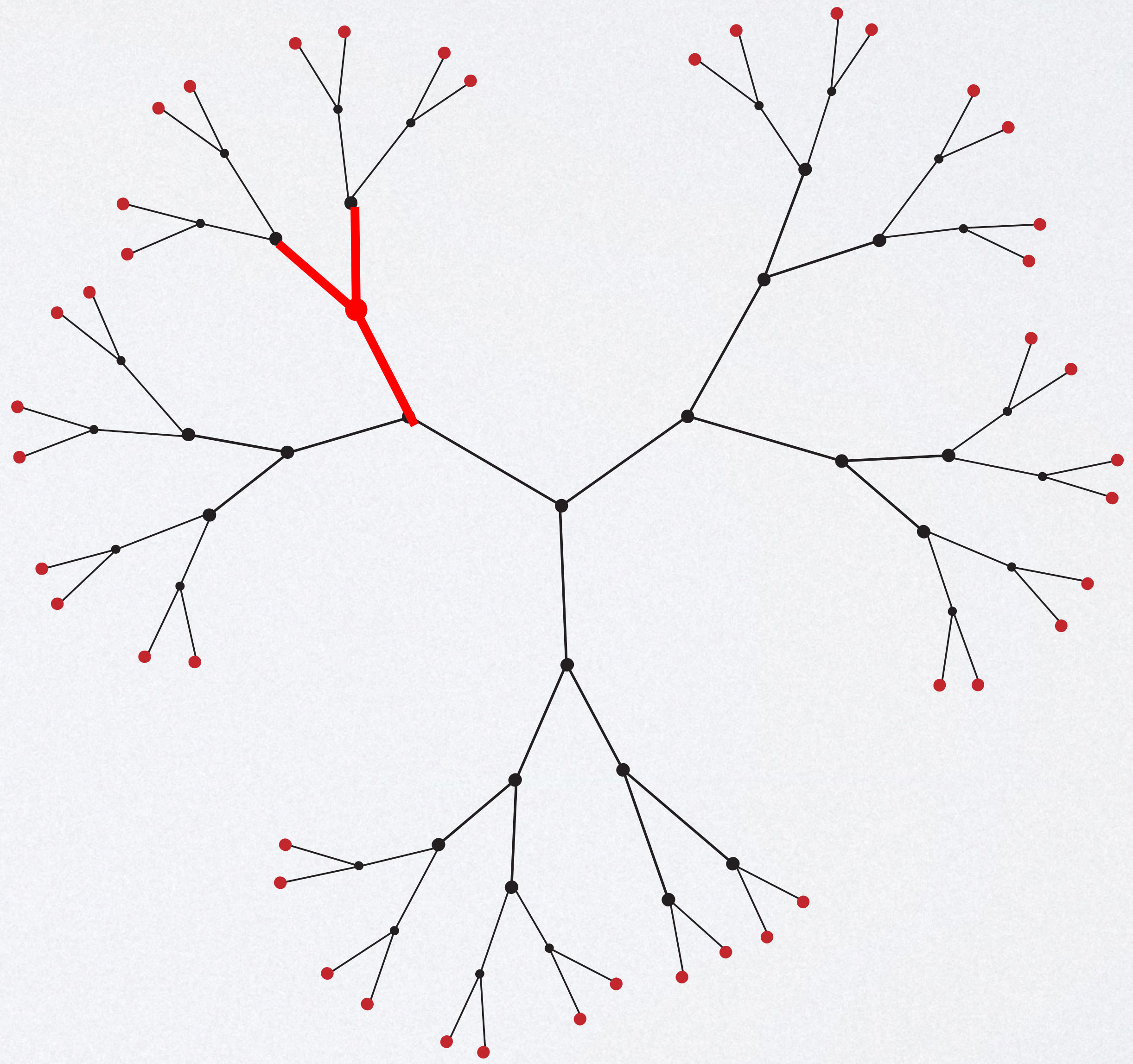
$\text{rank-width}(G) \leq k$

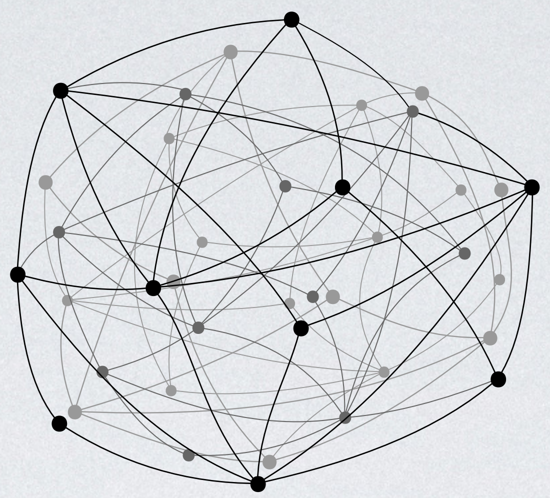
$\Downarrow$

$\text{flip-width}_\infty(G) \leq O(2^k)$



next flip:





?

small rankwidth

strategy for Flipper

small flip-width$_\infty$

tangle of large order

*

strategy for Runner

large flip-width$_\infty$

*Oum, private communication

BOUNDED FLIP-WIDTH

Definition A graph class C has *bounded flip-width* if
for all $r \in \mathbf{N}$, $\text{flip-width}_r(C) < \infty$

Bounded tree-width = bounded cop-width$_\infty$	→	Bounded clique-width = bounded flip-width$_\infty$
↓		↓
Bounded expansion = bounded cop-width$_r$ $\forall r \in \mathbf{N}$	→	Bounded flip-width = bounded cop-width$_r$ $\forall r \in \mathbf{N}$

BOUNDED FLIP-WIDTH

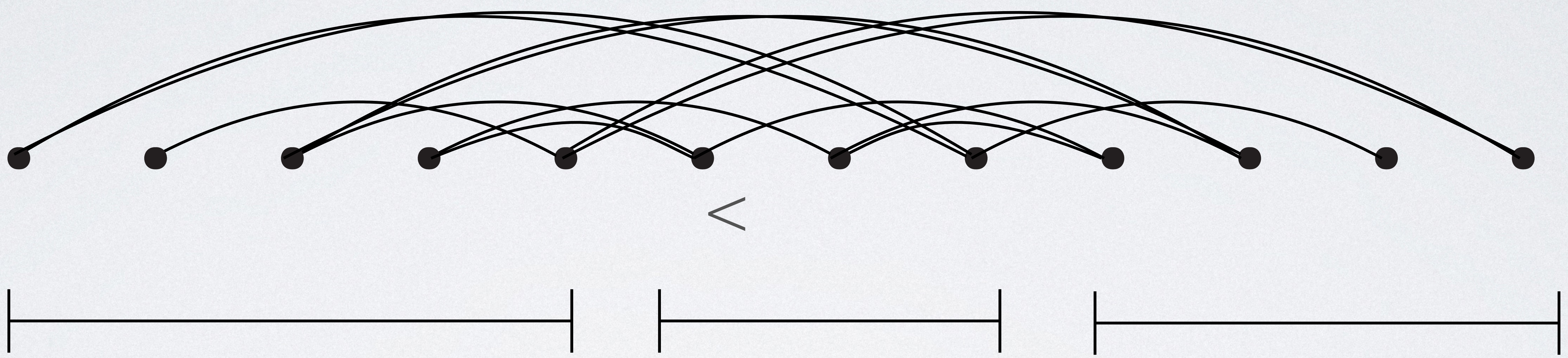
Examples:

- Classes of bounded expansion
- Classes of bounded clique-width
- Classes of bounded twin-width

bounded treewidth	→	bounded cliquewidth
excluding a minor	→	?
bd. sparse twin-width	→	bounded twin-width
bounded expansion	→	**bounded flip-width**
nowhere dense	→	monadically NIP

FLIP-WIDTH OF ORDERED GRAPHS

Variant of flip-width for ordered graphs $G=(V,E,<)$



Flipper performs k-flip on (V,E) and cuts $<$ into k intervals

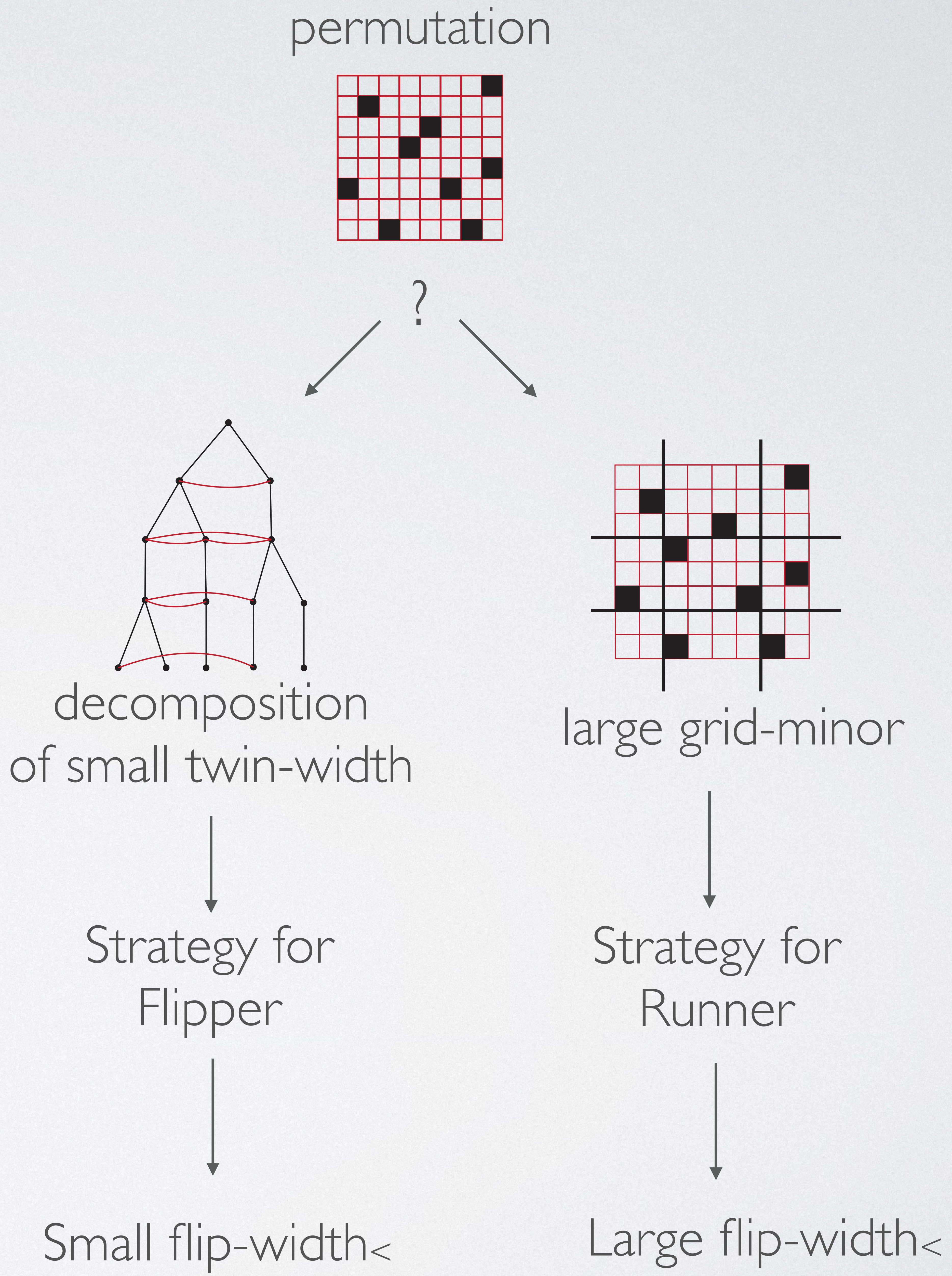
Runner moves along edges at speed 1 or within intervals at speed ∞

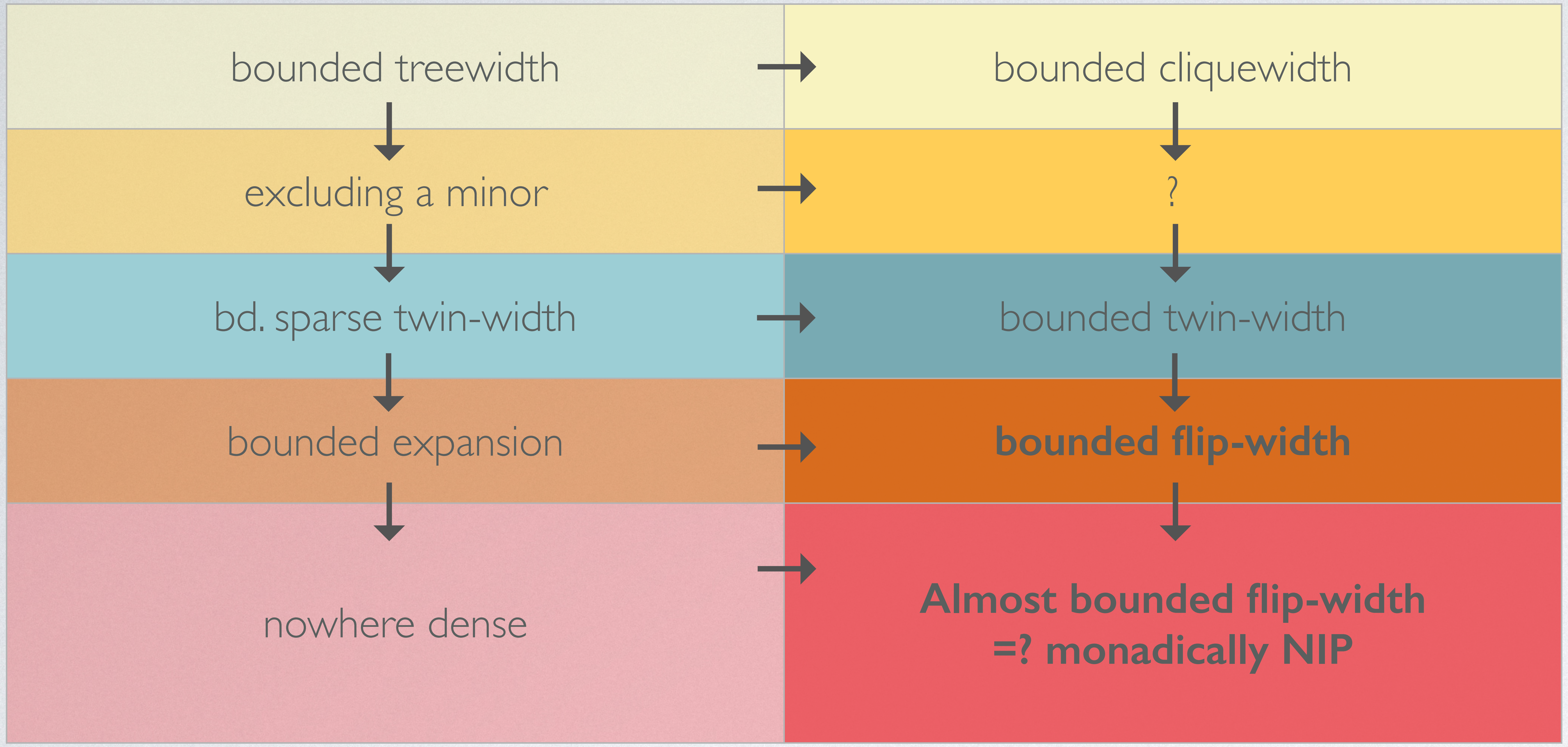
Theorem $\quad \text{flip-width}_<(G) \approx \text{twin-width}(G)$

Game characterization of twin-width

TWIN-WIDTH

- Klazar 2000, Marcus&Tardos 2004, Guillemot&Marx 2014 (dichotomy for permutations)
- Bonnet, Kim, Thomassé, Watrigant 2020 (twin-width)
- Bonnet, Giocanti, O. de Mendez, Thomassé, Simon, **T**. 2022 (dichotomy for ordered graphs)





Questions:

- FPT Model checking
- FPT approximation
- Decompositions
- Obstructions
- Dense variant of excluding a minor

THANK YOU !

Looking for students, postdocs – starting from 2024!

szymtor@uw.edu.pl