Characterizations of monadic NIP/dependence

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November 15, 2023
LoGAlg

Overview

Definitions and examples

Definition

A class $\mathcal{C}$ of finite structures is *stable* if it does not encode the class of all finite linear orders.
$\mathcal{C}$ is *NIP* if it does not encode the class of all finite graphs.
$\mathcal{C}$ is *monadically stable/NIP* if it remains stable/NIP under arbitrary vertex colorings.

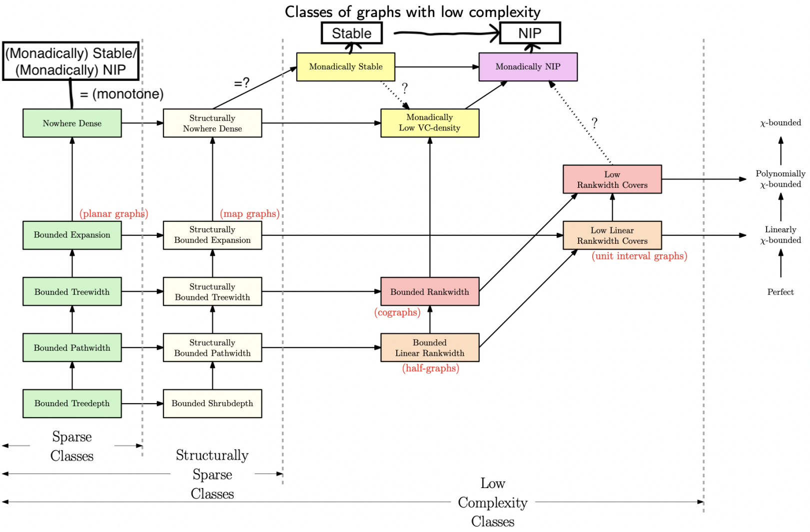
Theorem ([AA14], [PZ78])

Let $\mathcal{C}$ be a monotone graph class. Then $\mathcal{C}$ is nowhere dense $\iff$ $\mathcal{C}$ is (monadically) stable $\iff$ $\mathcal{C}$ is (monadically) NIP.

Theorem ([BGdM$^+$21])

Let $\mathcal{C}$ be a hereditary class of ordered graphs. Then $\mathcal{C}$ has bounded twin-width $\iff$ $\mathcal{C}$ is (monadically) NIP.

A MAP [NODMRS21]



Motivation

- Meta-conjecture: Monadic stability/NIP provide the "right" generalizations of nowhere denseness to hereditary classes.

Conjecture

Let $\mathcal{C}$ be a hereditary class of relational structures. Then first-order model checking is FPT on $\mathcal{C}$ $\iff$ $\mathcal{C}$ is monadically NIP.

- Baldwin-Shelah [BS85] gave model-theoretic characterizations of infinite monadically stable structures, hinging on *forking dependence*.
- Understanding forking dependence in monadically stable graph classes, and finitizing the Baldwin-Shelah theory, has been crucial to progress on the meta-conjecture.

Characterizations

- Building on results of Baldwin-Shelah on monadic stability [BS85] and Shelah on monadic NIP [She86].
- We characterize monadically NIP theories in the following ways.

1. The behavior of independence
2. A forbidden configuration
3. Decompositions of models
4. Type counting/width
5. The behavior of indiscernibles

- These are all characterizations of the theory T itself rather than its vertex colorings.
- An intuition is that models of monadically NIP theories are 1-dimensional, or alternatively are order-like (or tree-like).

INDEPENDENCE: THE F.S. DICHOTOMY

- Finite satisfiability gives a (possibly asymmetric) notion of independence in any theory.

Definition ([She86])

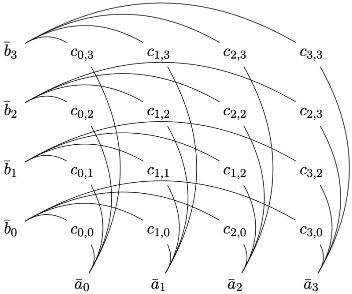
Let $A \downarrow^{fs}_M B$ mean that $tp(A/MB)$ is finitely satisfiable in M.
A theory T *has the f.s.-dichotomy* if given $A \downarrow^{fs}_M B$, then for any c, $cA \downarrow^{fs}_M B$ or $A \downarrow^{fs}_M Bc$.

- Intuitively, dependence reduces to singletons and is transitive on singletons.
- So f.s.-dependence induces a quasi-order ($a \prec b$ if $a \not\downarrow^{fs}_M b$).

Forbidden configuration: An infinite grid

Lemma (mostly [She86])

If T does not have the f.s. dichotomy, then some model of T codes an infinite grid (on tuples).



- $M \models \phi(\bar{a}_i, \bar{b}_j, c_{k,\ell}) \iff (i,j) = (k,\ell)$
  (e.g. $\phi(x,y,z) := E(x,z) \wedge E(y,z)$)
- So monadically NIP $\Rightarrow$ f.s. dichotomy

DECOMPOSABILITY: M-F.S. SEQUENCES

Definition

Given a model M, $(A_i : i \in I)$ is an *M-f.s. sequence* if $A_i \downarrow^{fs}_M \{A_{<i}\}$.

A *linear decomposition* of N is a partition $N = \sqcup_i A_i$ and a model M (not necessarily in N) such that $(A_i : i \in I)$ is an M-f.s. sequence.

Lemma ([She86])

If T has the f.s. dichotomy, then any partial linear decomposition of $N \models T$ can be extended to a full linear decomposition of N.

- The f.s. dichotomy is exactly what is needed to extend one point at a time.
- Alternatively, use the quasi-order decomposition into f.s.-dependence classes.

Type-counting/width: "Blobby linear neighborhood-width"

Lemma

If T has the f.s. dichotomy, then the "blobby linear neighborhood-width" of every model is bounded by some λ (in fact, by $2^{\aleph_0}$).

If T is not monadically NIP, "blobby linear neighborhood-width" can be made arbitrarily large.

- So f.s. dichotomy $\Rightarrow$ monadically NIP
- Blumensath [Blu11] showed that T is monadically NIP $\iff$ the "rank-width" of models is cardinal-bounded.

Indiscernibles: dp-minimality and indiscernible triviality

Lemma

T has the f.s. dichotomy $\iff$ T is dp-minimal and indiscernible trivial.

Definition

T is *dp-minimal* if for every dense indiscernible $\mathcal{I}$, any new parameter a splits $\mathcal{I}$ into at most two mutually indiscernible sequences.
(Roughly, T does not have two independent dimensions.)

Definition

T is *indiscernible trivial* if whenever endpointless $\mathcal{I}$ is indiscernible over each $a \in A$, then $\mathcal{I}$ is indiscernible over A.

Main theorem

Theorem ([BL21])

The following are equivalent for a complete theory T.

1. *T is monadically NIP.*
2. *T has the f.s. dichotomy.*
3. *T does not code an infinite grid on tuples.*
4. *Partial linear decompositions of models of T extend to full linear decompositions.*
5. *Models of T have cardinal-bounded "blobby linear neighborhood-width".*
6. *T is dp-minimal and indiscernible trivial.*

Collapse of dividing lines

- Why do monadic stability/NIP appear as dividing lines in combinatorial problems about hereditary classes?

Theorem ([BL22])

Let $\mathcal{C}$ be a hereditary class in a relational language. Then $\mathcal{C}$ is stable $\iff$ $\mathcal{C}$ is monadically stable, and $\mathcal{C}$ is NIP $\iff$ $\mathcal{C}$ is monadically NIP.

- A sharp dichotomy: either $\mathcal{C}$ is tree-like (monadically NIP) or totally chaotic (not even NIP).
- Key point: If T is not monadically NIP, then T codes a "pre-grid" by an *existential* formula.

Infinite and finite

- What does the infinitary viewpoint bring?
- Dependence is easier to understand and manipulate.
- Allows for asymptotic analysis at the scale of cardinals.

- There are two sides to finitizing the theory.
- The non-structure results finitize directly by compactness.
- The structure results seem to require coming up with a "localized" notion of dependence and then re-developing the theory.

References I

- Hans Adler and Isolde Adler, *Interpreting nowhere dense graph classes as a classical notion of model theory*, European Journal of Combinatorics **36** (2014), 322–330.
- Édouard Bonnet, Ugo Giocanti, Patrice Ossona de Mendez, Pierre Simon, Stéphan Thomassé, and Szymon Toruńczyk, *Twin-width IV: low complexity matrices*, arXiv preprint arXiv:2102.03117 (2021).
- Samuel Braunfeld and Michael C Laskowski, *Characterizations of monadic NIP*, Transactions of the AMS, Series B **8** (2021), 948–970.
- ______, *Existential characterizations of monadic NIP*, arXiv preprint arXiv:2209.05120 (2022).

References II

- Achim Blumensath, *Simple monadic theories and partition width*, Mathematical Logic Quarterly **57** (2011), no. 4, 409–431.
- John T Baldwin and Saharon Shelah, *Second-order quantifiers and the complexity of theories.*, Notre Dame Journal of Formal Logic **26** (1985), no. 3, 229–303.
- Jaroslav Nešetřil, Patrice Ossona de Mendez, Roman Rabinovich, and Sebastian Siebertz, *Classes of graphs with low complexity: the case of classes with bounded linear rankwidth*, European Journal of Combinatorics **91** (2021), 103223.
- Klaus-Peter Podewski and Martin Ziegler, *Stable graphs*, Fund. Math **100** (1978), no. 2, 101–107.

References III

Saharon Shelah, *Monadic logic: Hanf numbers*, Around classification theory of models, Springer, 1986, pp. 203–223.