Boolean CSPs and Directed flow-augmentation

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Min Sat(Γ)

CSPs (Constraint Satisfaction Problems)

- A CSP problem defined by fixing a domain D and a constraint language Γ over D.
- An instance of $\mathrm{CSP}(\Gamma)$ is given as a set of constraints (X, R) over Γ.

Constraint language Γ over a domain D

- a set of relations R over D, each relation $R \subseteq D^r$ for finite r (*arity*).

A constraint (X, R) over a constraint language Γ

- $X = (x_1, \ldots, x_r)$ is an r-tuple of variables (*scope* of the constraint)
- $R \in \Gamma$
- *satisfied* by an assignment $\alpha : X \to D$ if $(\alpha(x_1), \ldots, \alpha(x_r)) \in R$

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CSP(Γ), EXAMPLES

	CSP	3-COLORING	2-SAT
Problem	domain D	$\{1,2,3\}$	$\{0,1\}$
	constraint language Γ, $R \subseteq D^r$ for each $R \in \Gamma$ of arity r	$\{\neq\}$	$\{\{0,1\}^2 \setminus (a,b) : a,b \in \{0,1\}\}$
Instance	variables V	vertices	variables
	constraints (X,R), $R \in \Gamma$, $X \in V^r$	$\{(u,v),\neq)\}_{uv \in E(G)}$	clauses

We focus on boolean constraint languages, where CSP(Γ) is now called SAT(Γ).

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We focus on boolean constraint languages, where $\mathrm{CSP}(\Gamma)$ is now called $\mathrm{SAT}(\Gamma)$.

A natural variant of Sat(Γ): Min Sat(Γ)

Min Sat(Γ)

Input: A formula (i.e. a set of constraints) $\mathcal{F}$ over Γ, where the domain of Γ is boolean, a non-negative integer k.

Question: is there a set of at most k constraints $Z \subseteq \mathcal{F}$ such that $\mathcal{F} - Z$ is satisfiable?

st-Min-Cut as Min Sat(Γ)

Alternative formulation of st-Min-Cut: Given $G = (V, E)$ with $s, t \in V$ and integer k,

- Variables V.
- $(s = 1)$ and $(t = 0)$ as *crisp* constraints (i.e. $k + 1$ copies).
- For every edge $e = (s, v) \in E$, a constraint $(v = 1)$.
- For every edge $e = (u, t) \in E$, a constraint $(u = 0)$.
- For every other edge $e = (u, v) \in E$, a constraint $(u = v)$.
- Find a boolean assignment $\phi : V \to \{0, 1\}$ such that all but at most k constraints are satisfied.

This is a formula of Min Sat(Γ) over $\Gamma = \{0, 1, =\}$.

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k-Clique as Min Sat(Γ)

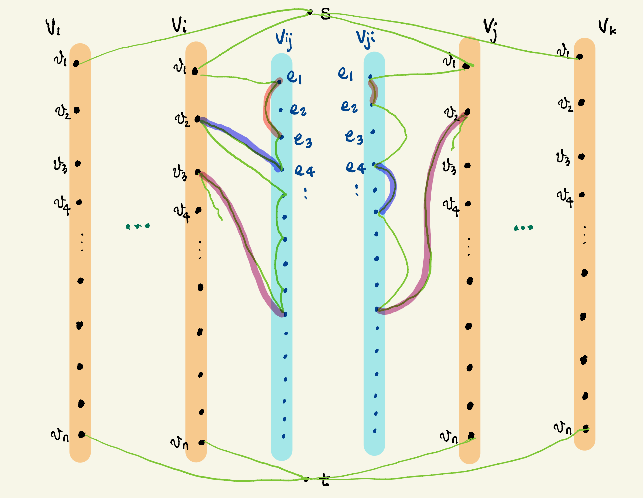
k-Multicolored Clique (equivalent to k-Clique):

- Input: $G = (V_1 \uplus V_2 \uplus \cdots V_k, E)$ with each V_i stable, integer k.
- Task: find a k-clique K (report No if none exists).

Alternative formulation as Min Sat(Γ) with $\Gamma = \{=, 0, 1, \text{double equality}\}$
(Marx and Razgon 2009)

k-Clique as Min Sat(Γ)

Does the following instance of Min Sat(Γ), where $\Gamma = \{=, 0, 1, \text{DOUBLE EQUALITY}\}$, admit a boolean assignment violating at most $\binom{k}{2}$ constraints?



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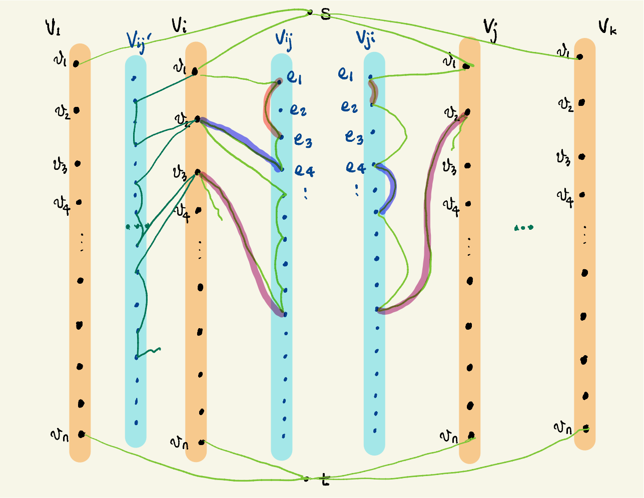

Toward FPT dichotomy for Min Sat(Γ)

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FPT dichotomy for Min Sat(Γ)

K. Kratsch, Pilipczuk, Wahlström 2021,22,23

Theorem

Let Γ be a finite boolean constraint language. Then parameterize by the number of unsatisfied constraints, one of the following holds.

1. Weighted Min Sat(Γ) *is FPT.*
2. Min Sat(Γ) *is FPT, but* Weighted Min Sat(Γ) *is W[1]-hard.*
3. Min Sat(Γ) *is W[1]-hard.*

The hard gist of the tractable cases critically relay on the flow-augmentation technique.

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Example of a tractable case: ℓ-Chain Sat

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- Input: a formula Φ as a set of constraints of the following form.
  - $(x_1 \rightarrow x_2) \wedge (x_2 \rightarrow x_3) \wedge \cdots (x_{\ell-1} \rightarrow x_\ell)$,
  - or unary clauses, i.e. (x) or $(\neg x)$

  each constraint C has weight $\omega(C)$. Two integers k and W.
- Task: find a truth assignment $V(\Phi) \rightarrow \{1, 0\}$ violating at most k constraints of weight at most W.

$\Phi = \bigwedge_{i=1}^{4}(x_i) \wedge \bigwedge_{i=1}^{4}(\neg y_i) \wedge \bigwedge_{i=1}^{4} C_i$,

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FPT in $k + \ell$? Was THE bottleneck cut problem.
Tractable case in both bijunctive and IHS-B languages.
$\ell = 1, 2$ is st-Min-Cut.

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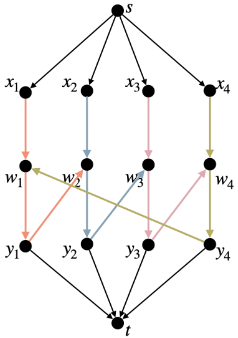

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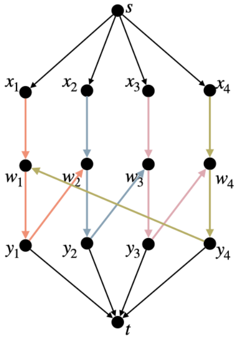

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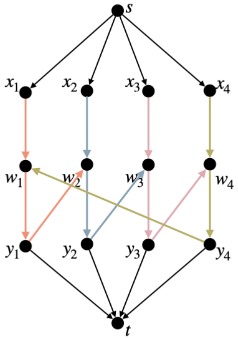

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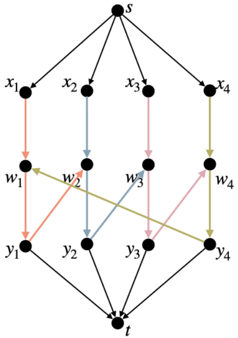

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Flow augmentation

Flow Augmentation Theorem (simple version)

There exists a polynomial-time algorithm that, given

- a directed graph G with $s, t \in V(G)$ and an integer k,

returns

- a set $A \subseteq V(G) \times V(G)$

such that for every minimal st-cut Z of size at most k, with probability $2^{-\mathcal{O}(k^4 \log k)}$

- Z is an st-cut of minimum cardinality in $G + A$.

When a sought solution Z is a minimal st-cut, then flow-augmentation lifts st-mincut size to match $|Z|$ by adding (unbreakable) arcs in a way not messing the solution, with high enough probability.

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Algorithm for ℓ-Chain Sat

Weighted ℓ-Chain Sat

- Input: a directed graph $G = (V, E)$ with s, t, a collection $\mathcal{B}$ (bundles) of pairwise disjoint path of length at most b with weights $\omega : \mathcal{B} \to \mathbb{Z}_+$, integers k and W.
- Task: find a minimal st-cut $Z \subseteq \bigcup \mathcal{B}$ violating at most k bundles of weight at most W.

1. Invoke flow-augmentation; now Z is an st-mincut. Note $\lambda(s, t) \leq bk$

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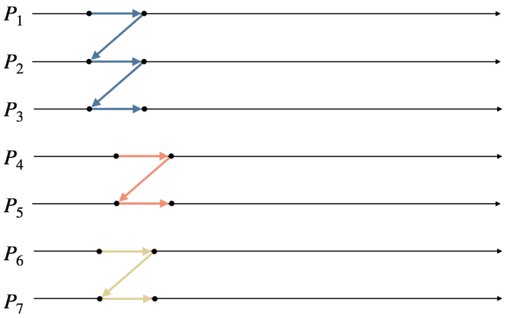

2. Guess how violated bundles overlay the flow paths.

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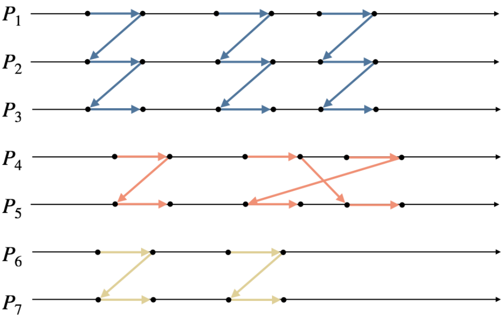

3. Consider (only) the bundles conforming the guess.

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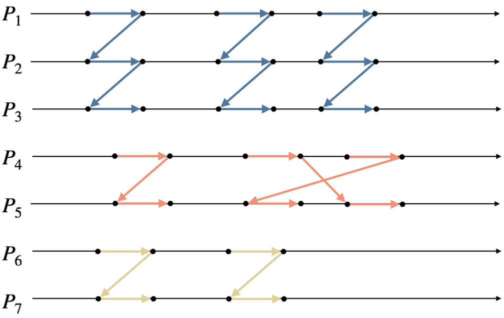

4. A bundle *crossed* by another bundle cannot be violated.

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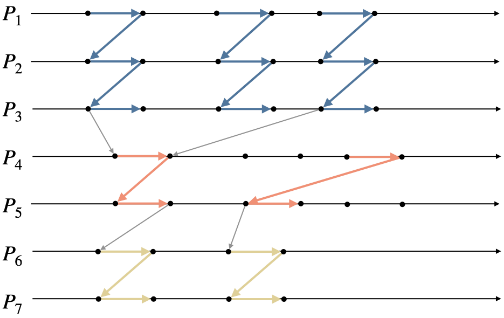

5. Bundles are *linearly ordered*.

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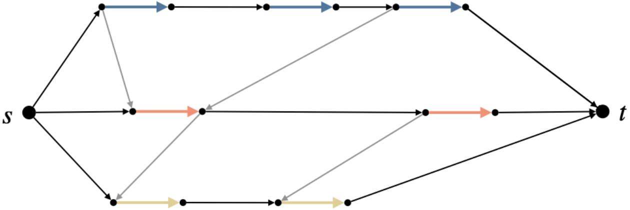

6. *Compress* the bundles, obtain (Weighted) st-MINCUT instance.

Concluding remarks

- $2K_2$-freeness allows one to use a similar argument in the more general cases.
- The presence of $2K_2$ leads to a reduction in the spirit of the previous one (k-Clique to Min Sat($\{=, 0, 1, \text{double equality}\}$).
- *Flow-augmentation* looks like the missing tool in directed graph separation problems.
- It closed some dichotomies and long-standing open problems.
- Key open problemArgh...it's closed recently by George Osipov and Marcin Pilipczuk: Symmetric Multicut.
  Directed graph G, unordered pairs of terminals $\mathcal{T}$, integer k. Delete k edges so that for every $st \in \mathcal{T}$, s and t are not in the same strong component.