Elementary first-order model checking for sparse graphs

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16.XI.2023
2nd Workshop on Logic, Graphs, and Algorithms

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FPT: solvable in time $f(|\varphi|, \mathcal{C}) \cdot |G|^c$,
for some function f and $c \geq 1$.

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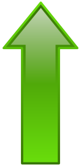

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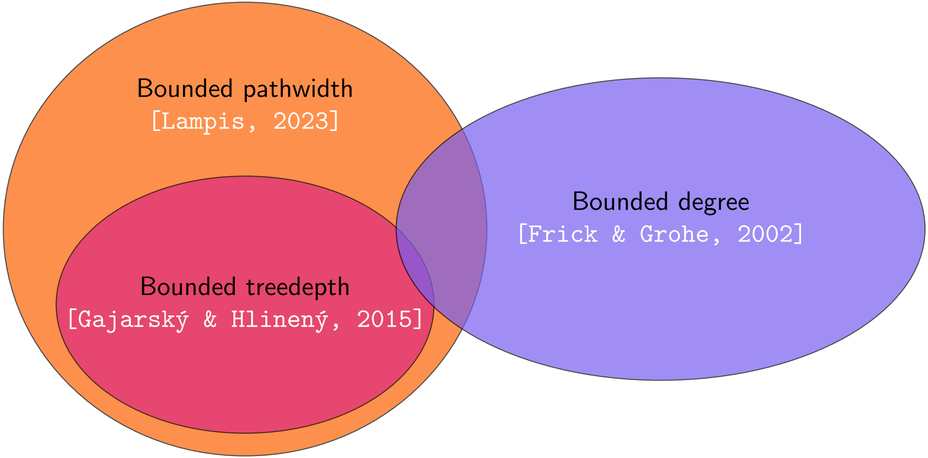

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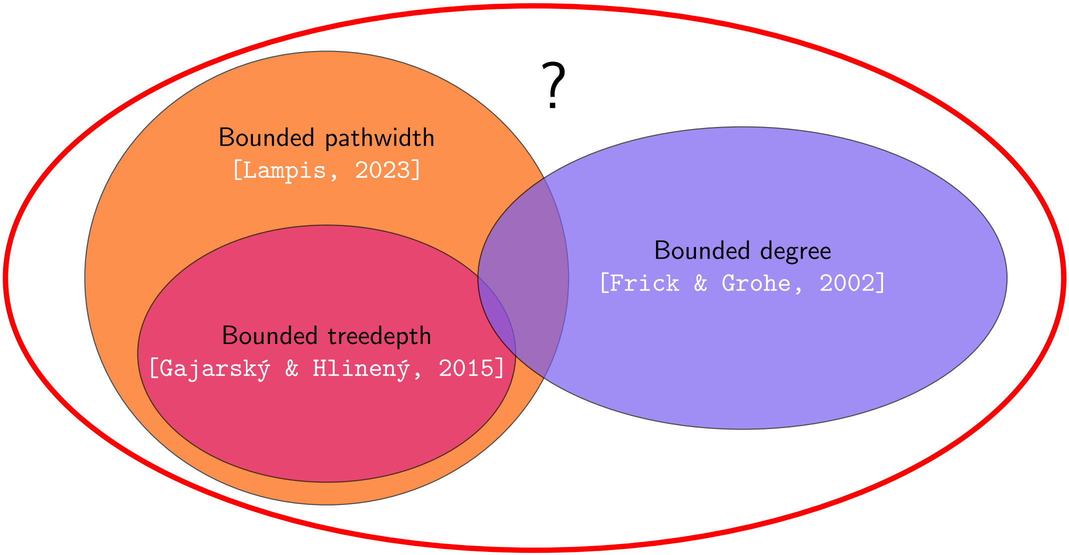

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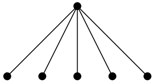 has depth 2.

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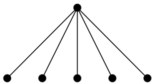

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The *tree rank* of a graph class $\mathcal{C}$ is defined as

$$\max\{d \in \mathbb{N} \mid \exists r \in \mathbb{N} : \mathcal{T}_d \subseteq \mathrm{TopMinors}_r(\mathcal{C})\}.$$

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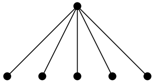

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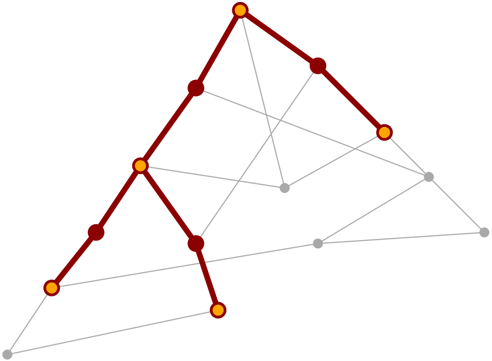

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- The class $\mathcal{C}$ of graphs of pathwidth d has tree rank exactly $d + 1$.

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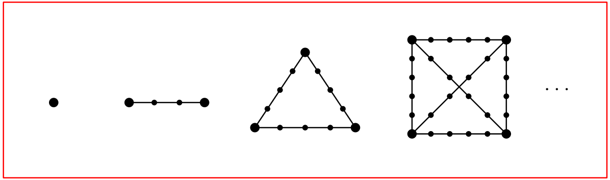

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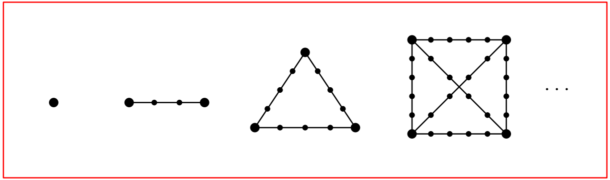

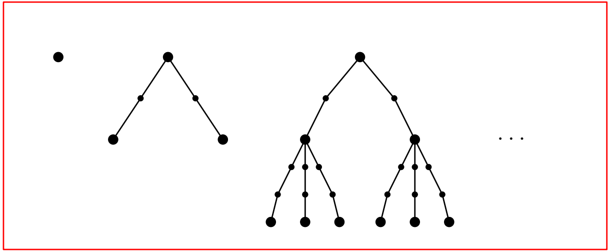

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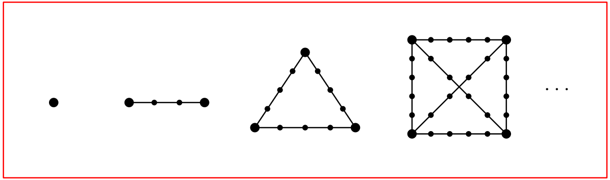
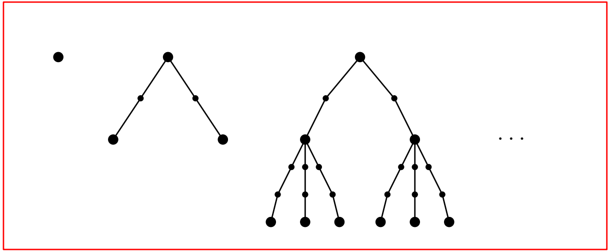
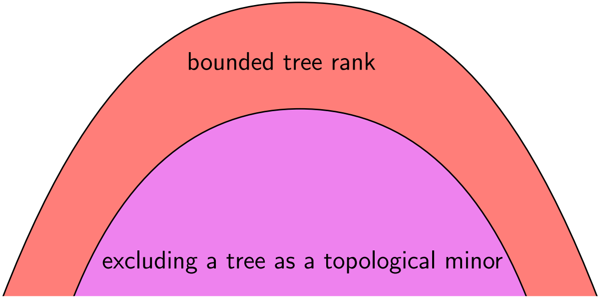




Every tree as a topological minor and tree rank 2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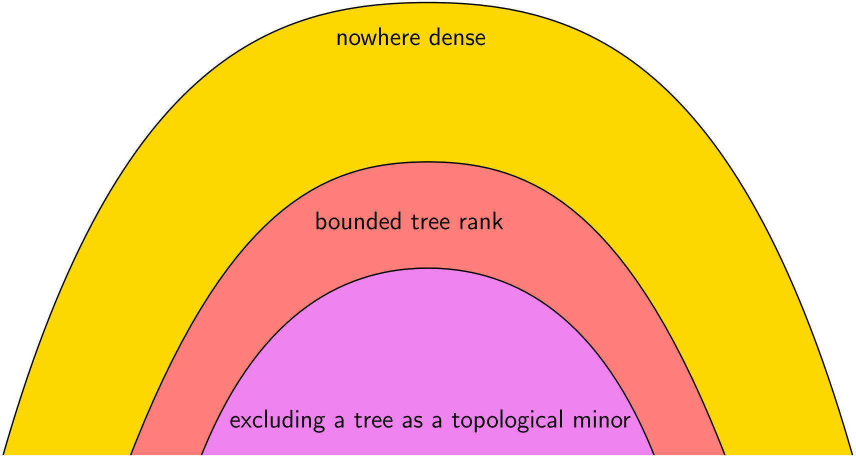
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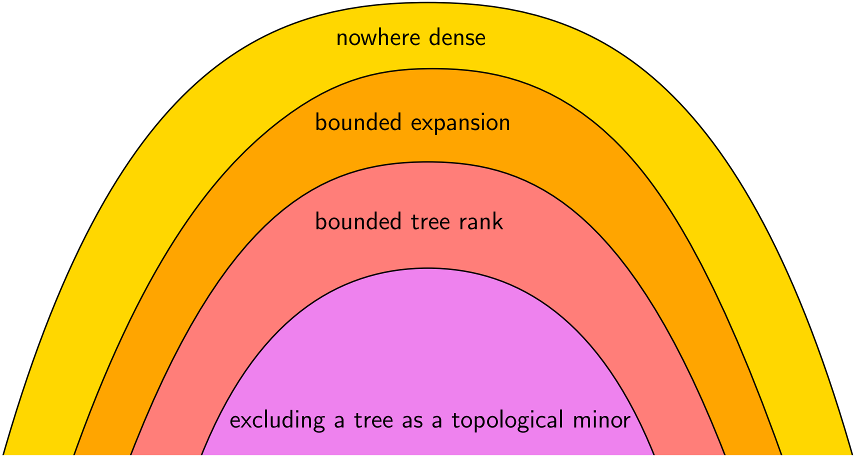
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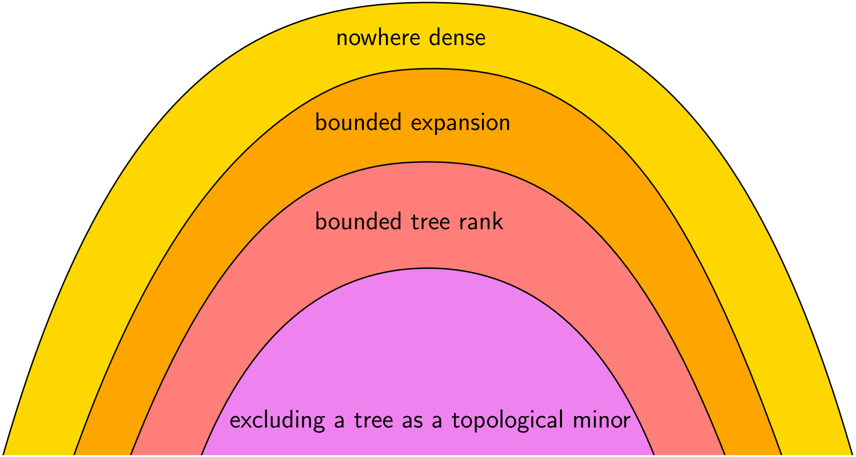

Fact: A graph of minimum degree δ contains every tree on δ vertices as a subgraph.

bounded tree rank $\implies$ bounded degeneracy

$T_k^d :=$ tree of depth d and branching k.

Tree rank of $\mathcal{C}$:
the least number $d \in \mathbb{N}$ such that
for every $r \in \mathbb{N}$ there is $k \in \mathbb{N}$ s.t. **no graph** in $\mathcal{C}$ contains T_k^{d+1} as an r-shallow topological minor.

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Theorem [Gajarský, Pilipczuk, Sokołowski, *S.*, Toruńczyk, 2023]

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Almost complete characterization of elementarily-FPT FO model checking on sparse classes.

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Lemma

Let $\mathcal{C}$ be a graph class of tree rank d.
Every formula φ is equivalent on $\mathcal{C}$ to a formula ψ of alternation rank $3d$.

Also, if $\mathcal{C}$ has *elementary* tree rank d, then $|\psi|$ is elementary in $|\varphi|$.

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Theorem [Gajarský, Pilipczuk, Sokołowski, *S.*, Toruńczyk, 2023]

Let $\mathcal{C}$ be a monotone graph class. The following are equivalent:

- $\mathcal{C}$ has bounded tree rank
- $\exists k \in \mathbb{N}$ such that for every formula φ, there is an equivalent (on $\mathcal{C}$) formula ψ of alternation rank k.

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Let $d \in \mathbb{N}$. The following conditions are equivalent:

(1) $\mathcal{C}$ has (elementary) tree rank d,

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(1) $\implies$ (2) *small # of "candidate roots" for* $T^i_{k_i}$ *as an* r*-shallow topological minor in* $B^r_G(v)$.

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FO model checking algorithm on bounded expansion classes [Dvořák, Král, & Thomas, 2014]

This algorithm is elementarily-FPT for sentences of *constant* alternation rank.

Conclusion

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Almost complete characterization of elementarily-FPT FO model checking on sparse classes.

Elementary query enumeration, query answering, or counting answers to queries?

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Conjecture:

A hereditary graph class $\mathcal{C}$ has elementarily-FPT model checking if and only if it has bounded rank.

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Let $\mathcal{C}$ be a hereditary graph class.
$\mathcal{C}$ has bounded rank $\iff$ $\exists k \in \mathbb{N}$ such that every φ is equivalent on $\mathcal{C}$ to a ψ of alternation rank k.