Kernelization hardness of connectivity problems

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Source

Kernelization hardness of connectivity problems in d-degenerate graphs

WG 2010, joint work with Marek Cygan, Marcin Pilipczuk and Jakub Onufry Wojtaszczyk

Setting

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  ST does not admit a polynomial kernel.
  - Reduction from Set Cover, for which the proof is difficult.
- **Goal:** Develop a simple technique for proving hardness of domination–style connectivity problems

d–degenerate graphs

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- However, d-degenerate graphs do not have „topological" structure.
  - Given a graph G obtain G' by inserting a vertex inside each edge. Then G' is 2–degenerate.

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- Dominating Set has a polynomial kernel in $K_{d+1,d+1}$–free graphs (a superclass) [Philip et al.].
- A natural question: does CDS have a polynomial kernel as well? [WorKer'09]

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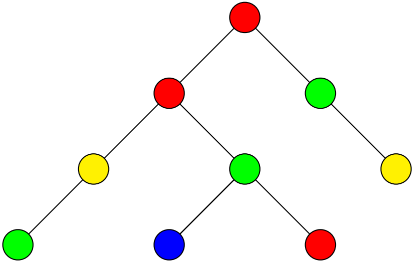
- Special case of Group Steiner Tree — $2^k|G|^{O(1)}$ FPT algorithm.

Colourful Graph Motif – example



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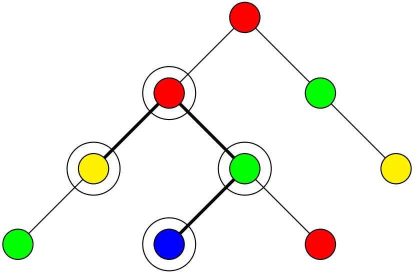

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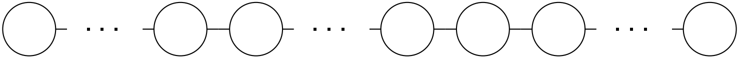
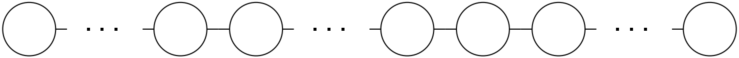


Colourful Graph Motif – NP hardness

$$\alpha \vee \beta \vee \gamma$$



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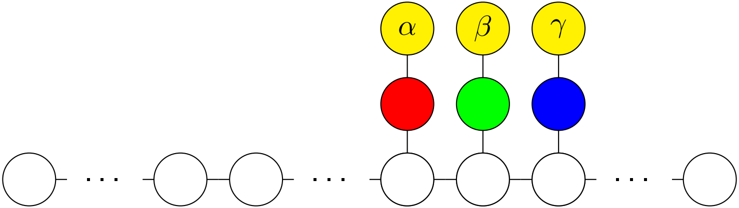

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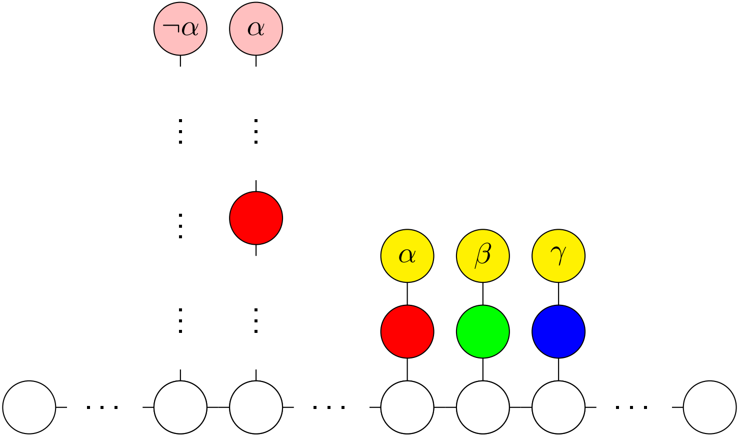

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- Composition algorithm: disjoint sum.
- But **resuing** the colours — the parameter do not increase!
- Thus Colourful Graph Motif does not admit polynomial kernel in sets of disjoint combs (unless $\Sigma_3 = PH$).
  - **Remark:** There is a simple reduction to a case of one comb (technical).

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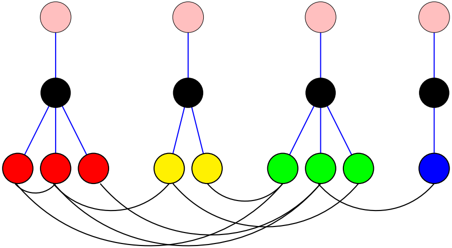

Connected Dominating Set

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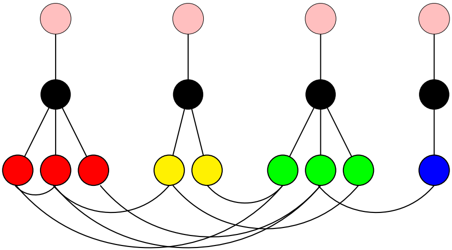

- Ask for Connected Dominating Set with $2k = 8$ nodes.

Connected Dominating Set

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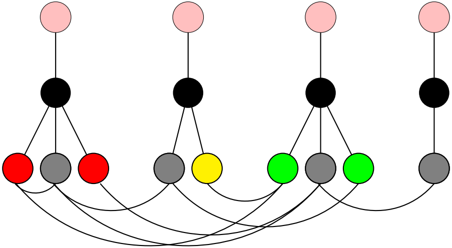

- Ask for Connected Dominating Set with $2k = 8$ nodes.
- Need to use black nodes and one node of each colour.

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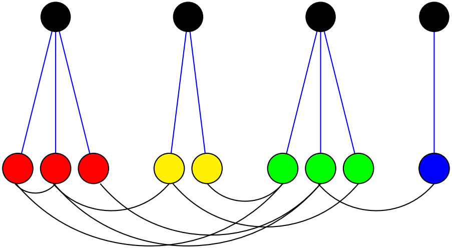

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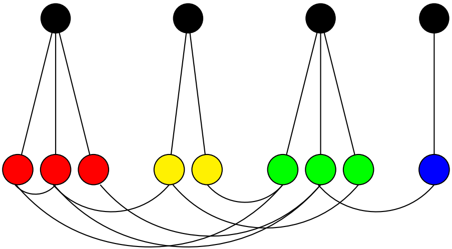

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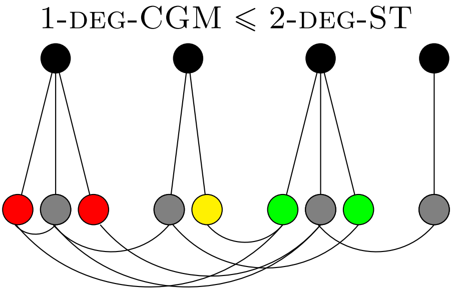

- Black nodes are terminals.
- Ask for Steiner Tree with $k = 4$ Steiner nodes.

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- COLOURFUL GRAPH MOTIF is a good way of encapsulating connectivity requirements even for special classes of graphs.
- d-degeneracy does not help in the existence of polynomial kernel.
- Misra et al. very recently used COLOURFUL GRAPH MOTIF to resolve existence of polynomial kernels for CDS restricted to the classes of graphs of bounded girth.
- Unfortunately, graphs obtained in the reductions still can be far from being planar (or having any topological properties).

Thank You for your attention

Questions?