Hitting forbidden subgraphs in graphs of bounded treewidth

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The problem

H-Subgraph Hitting

Input: A graph G and an integer k

Question: Does there exist a subset of vertices X with $|X| \leq k$ such that $G - X$ does not contain H as a subgraph?

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  - k can be arbitrarily large; we do not consider it as a parameter.

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- **Our question**: what is the right exponent at the $\mathrm{poly}(t)$ factor?

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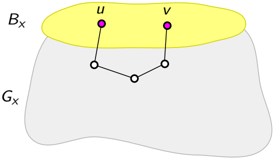
- (P., MFCS 2011) For $\ell \geq 5$, there is no algorithm with running time $2^{o(t^2)} \cdot n^{\mathcal{O}(1)}$ for this problem, unless ETH fails.
- *"[...] these problems admit a simple $2^{\mathcal{O}(t^2)} n^{\mathcal{O}(1)}$ dynamic programming algorithm, where t is the width of a given tree decomposition. In the state, one remembers for every pair of vertices of bag B_x, whether in G_x they can be connected via paths of length $1, 2, \ldots, \ell - 1$ disjoint with the solution."*



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- In the updates, the information about the folios should be sufficient to ensure that no C_ℓ is created.

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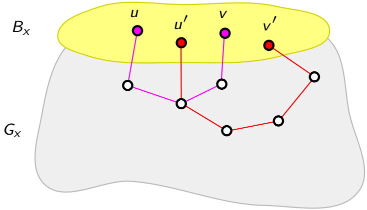

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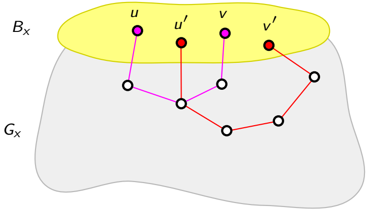

- We also need to store information about intersecting paths.
- Starts to be a mess.

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    - For hitting cycles: $2^{\mathcal{O}(t^2 \log t)} \cdot n$ algorithm.
  - However, there are cases when the complexity of the standard variant is provably *lower* than the colorful one.
  - The exponent is governed by sizes of minimal separators in H.

The colorful problem

Colorful H-Subgraph Hitting

Input: A graph G with vertices partitioned into classes $\{V_u \colon u \in V(H)\}$ and an integer k

Question: Is there a subset of vertices X with $|X| \leq k$ such that $G - X$ does not contain a colorful H-subgraph?

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- **Remark**: no problem with intersecting parts in the DP!

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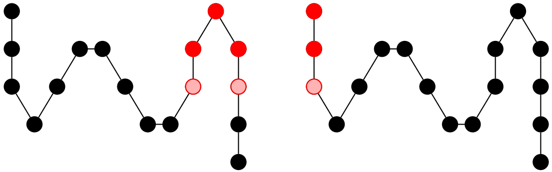
$$\mu^\star(G) = \max\{|N(X)| : G[X] \text{ is connected and } N[X] \neq V(H)\}.$$

... on a picture



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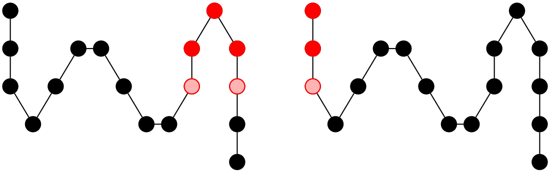

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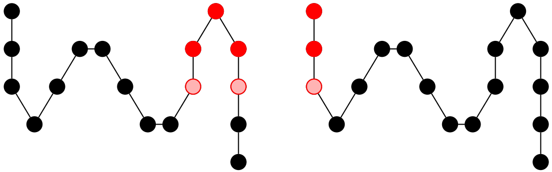

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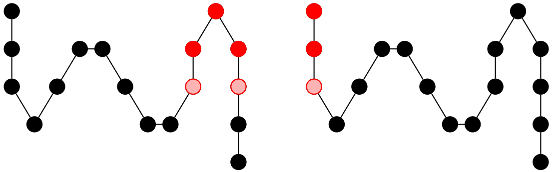

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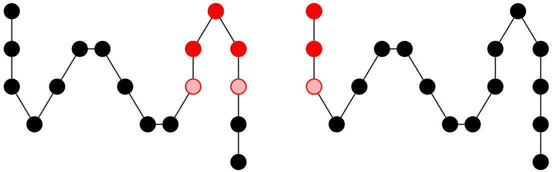

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- $\mu(H) \leq \mu^\star(H)$.
- **Note**: μ and $\mu^\star$ are undefined for a clique.

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- To get a $2^{\mathcal{O}(t^{\mu(H)})} \cdot n$-time algorithm, we use *prediction*.
- Find the smallest solution in the subgraph that is consistent with a predicted final behaviour in the bag for the whole graph.

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- The total number of literals is at most $3m$.

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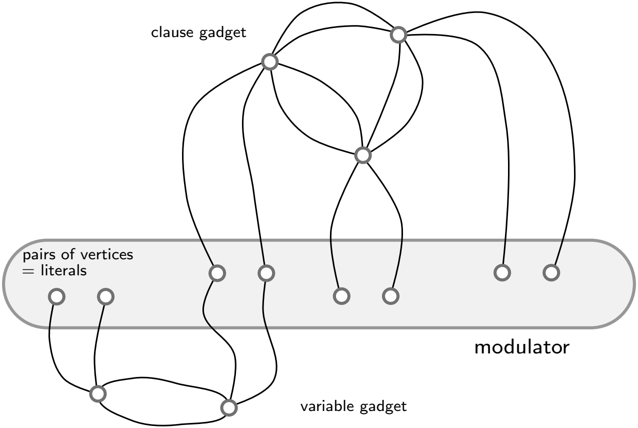

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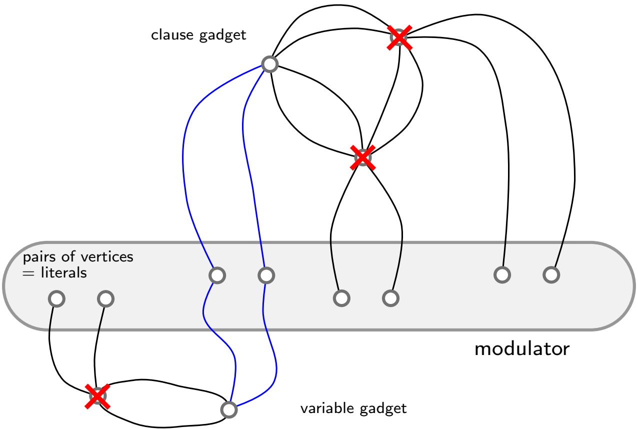

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- Colors help us not to create unwanted copies.

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  - **Note**: H' can be disconnected in this definition.

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- Many technicalities omitted.

Difference between variants

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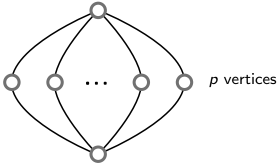

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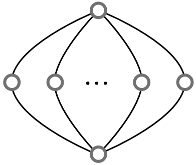

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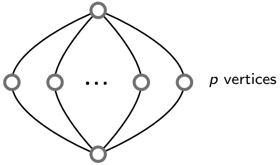

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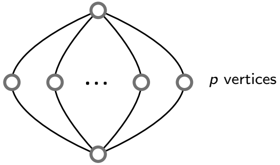

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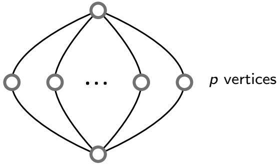

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- If one attaches triangles to the top and bottom, then a $2^{o(t^p)} \cdot n^{\mathcal{O}(1)}$ lower bound in the standard variant.
- Seems like the answer depends on the symmetries of H.

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  - It is possible for paths.
- Minor hitting?
- **Thanks for attention!**