Expander graphs and their applications

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Part I:
Expansion and the spectral gap

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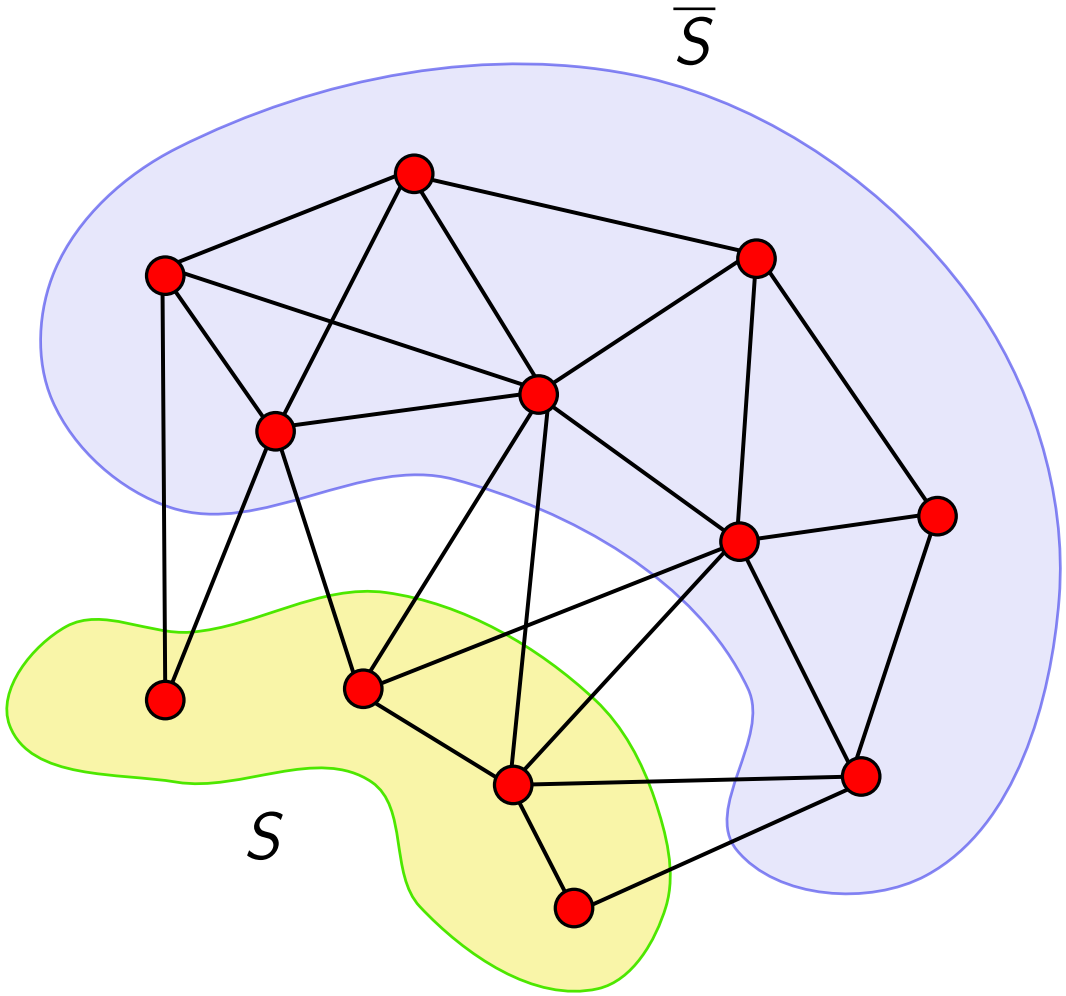
  - All vertices that can be reached within one step from S.

Expanding via edges and to vertices



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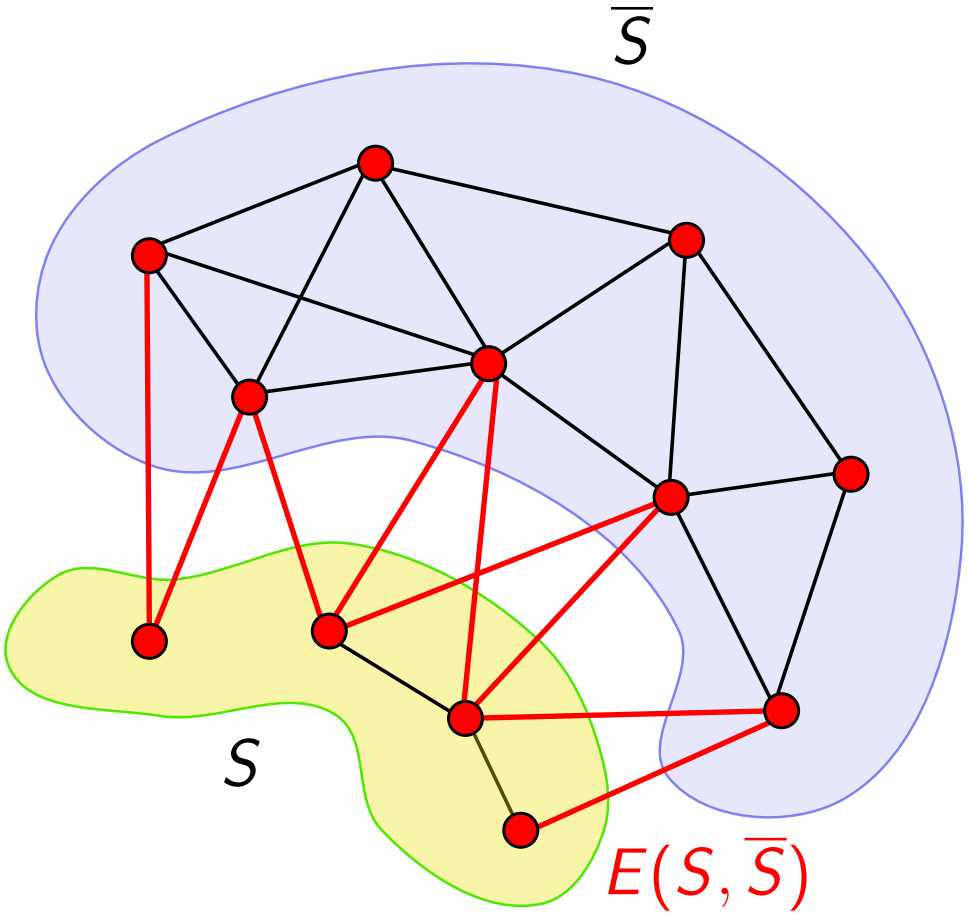

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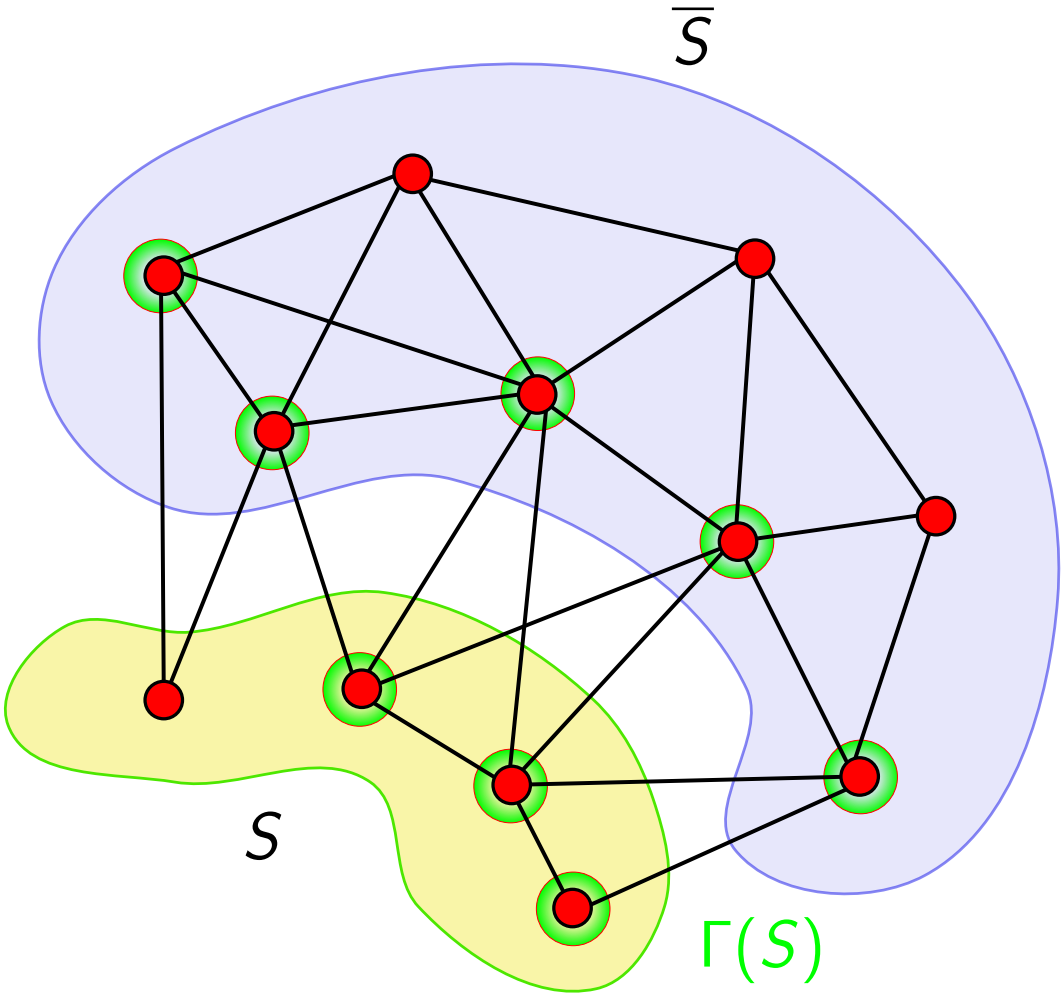

Edge and vertex expansion

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Edge expansion of a graph G is defined as

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- How to certify that a graph has a good expansion?

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- **Note**: $A(G)$ is a symmetric real matrix.

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- Multiset $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is called the **spectrum** of A (denoted $\sigma(A)$).

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- Let $\mathbf{1} = (1, 1, \ldots, 1)$. Then $A\mathbf{1} = d \cdot \mathbf{1}$ since the graph is d-regular.
- We have one eigenvector: $\frac{1}{\sqrt{n}} \cdot \mathbf{1}$, with eigenvalue d.

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- Ergo $|\lambda_i| \leq d$ for all i, and $\lambda_1 = d$.

Spectral gap

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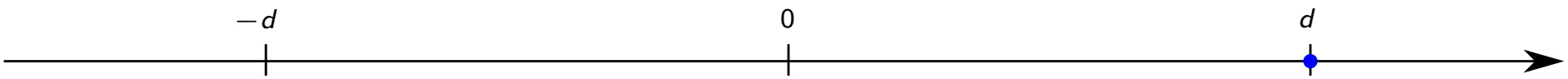

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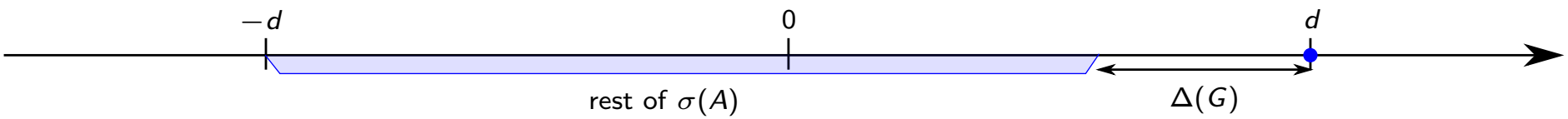

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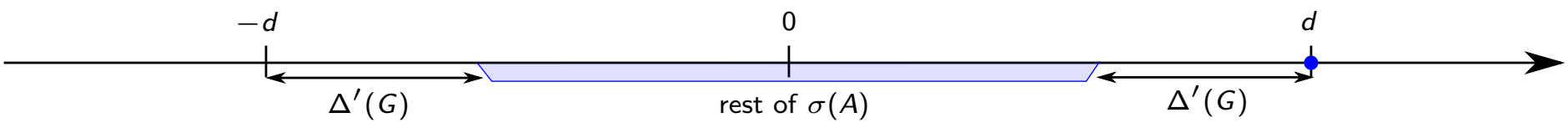

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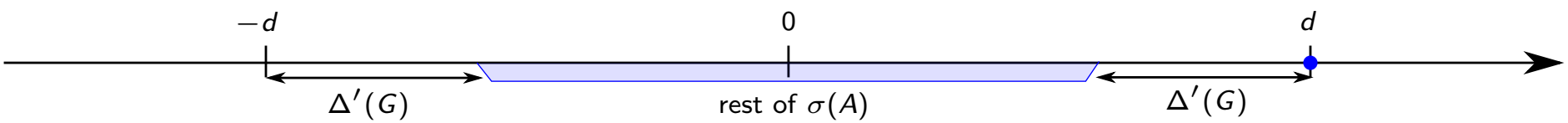
- G is an (n, d, α)-**expander** if $|V(G)| = n$, G is d-regular, and $|\lambda_i| \leq \alpha \cdot d$ for all $i = 2, 3, \ldots, n$.



Cheeger's inequalities

$$\begin{array}{ccccc} \Delta(G)/2 & \leq & h^E(G) & \leq & \sqrt{2d\Delta(G)} \\ \frac{d^2-(d-\Delta'(G))^2}{d^2+(d-\Delta'(G))^2} & \leq & h^V(G) & \leq & d\sqrt{2}\sqrt{d^2-(d-\Delta'(G))^2} \end{array}$$

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- Spectral properties are much easier to control.

Part II:
Random walks on expanders

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  - Hence $p^t = \tilde{A}p^{t-1} = \tilde{A}^t p^0$, where $\tilde{A} = \frac{1}{d} \cdot A$.

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- If G is an (n, d, α)-expander, then $\tilde{\lambda}_1 = 1$ and $|\tilde{\lambda}_i| \leq \alpha$ for $i = 2, 3, \ldots, n$.

Iterating $\tilde{A}$

- Let $p^0 = (b_1, b_2, \ldots, b_n)$ in basis $(e_1, e_2, \ldots, e_n)$.
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- **Note**: $b_1 e_1 = \frac{1}{n} \cdot \mathbf{1}$, so p^t converges to uniform distribution exponentially fast.

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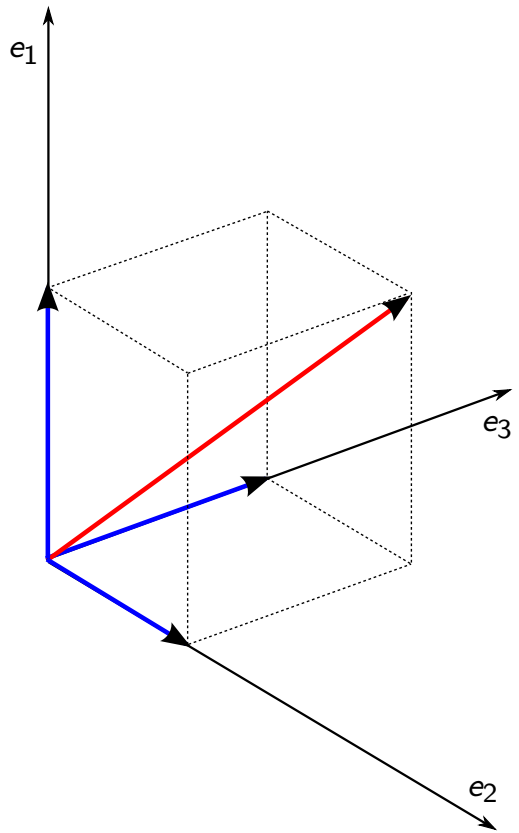

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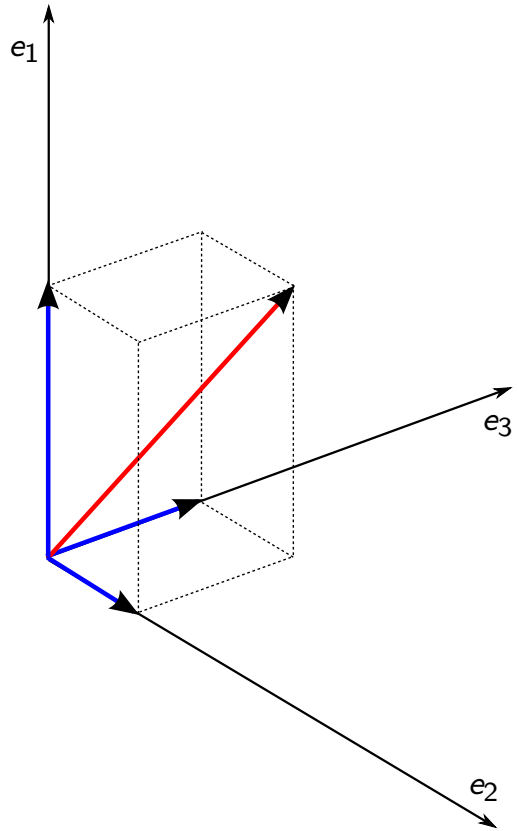

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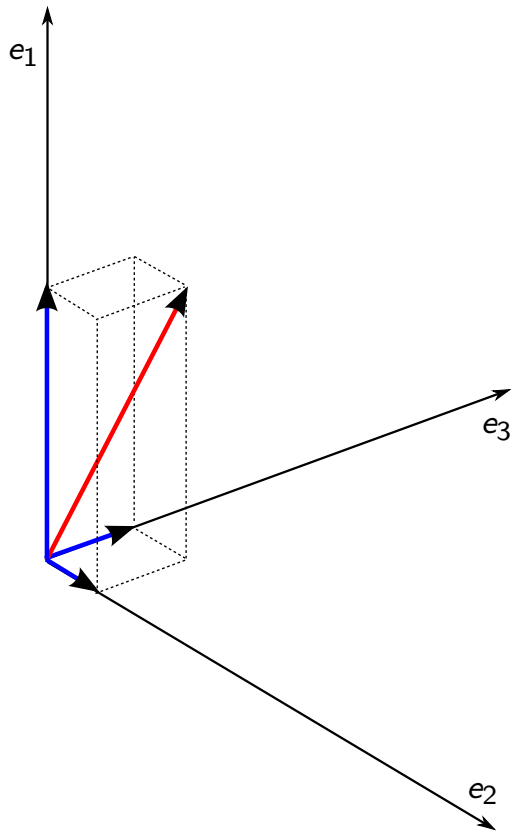

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  - We say that random walks are *rapidly mixing*.
- **Corollary**: an (n, d, α)-expander has diameter $\mathcal{O}(\log n)$.

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- Sample one element u, and perform a random walk of length t from u to collect t samples.

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Random sampling of expander walks

Assume that G is a (n, d, α)-expander for some constants d, α, and $B \subseteq V(G)$ is a vertex subset of size βn. Then the probability that a random walk of length t from a vertex chosen uniformly at random is entirely contained in B, is at most $(\beta + \alpha)^t$.

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- Sample a random walk of length t from a randomly chosen vertex: $\mathbb{P}(\text{all bad}) \leq (\beta + \alpha)^t$, but for constant d we use only $\log n + \mathcal{O}(t \log d) = \log n + \mathcal{O}(t)$ random bits.

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- **Strategy 1**: $\log n$ independent repetitions, $r \log n$ random bits.

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  - **Note**: with poly(n) overhead we can iterate through these $\mathcal{O}(\log n)$ additional bits, so we can reduce the error probability to $\frac{1}{n}$ with poly(n, r) overhead **without** increasing the number of random bits used.

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- We now sketch a construction of an infinite family of constant-degree expanders for $\alpha_0 = \frac{1}{2}$ (without traversing).

Part III:
Construction of an expander

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  2. **Zig-zag product** with H: vertex count increases, $\alpha(\cdot)$ increases only slightly, but the degree becomes again constant.

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- **Note**: G^t has still n vertices, but is d^t-regular.

Squaring the graph: spectrum

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- **Corollary**: $\alpha(G^t) = \alpha(G)^t$.
- If $\alpha(G) \leq \frac{1}{2}$ then $\alpha(G^2) \leq \frac{1}{4}$.

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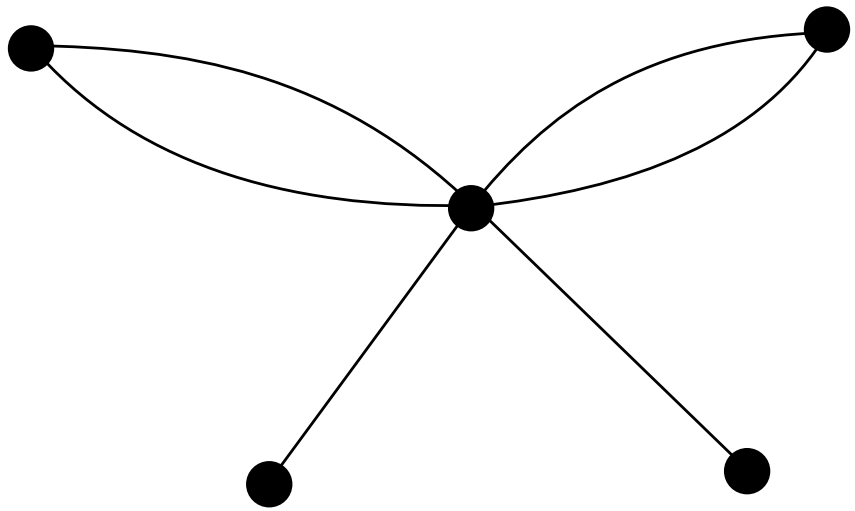

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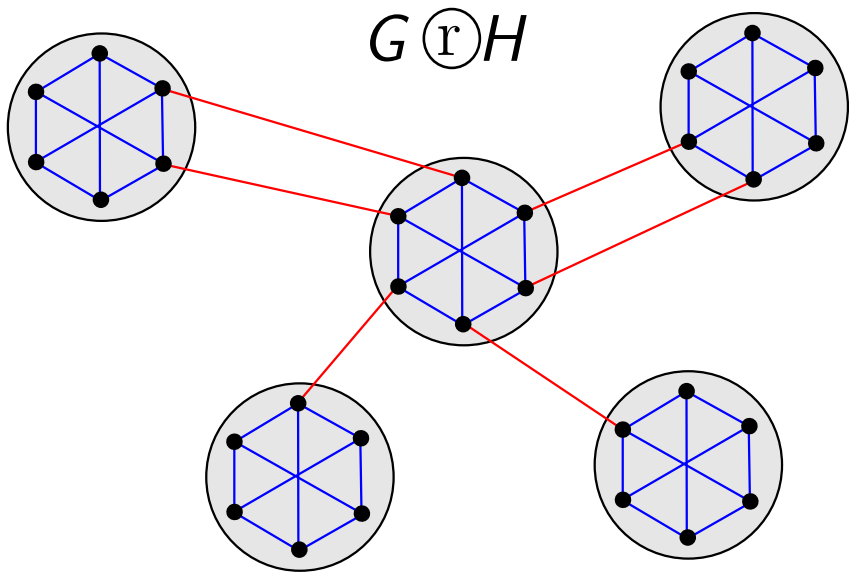

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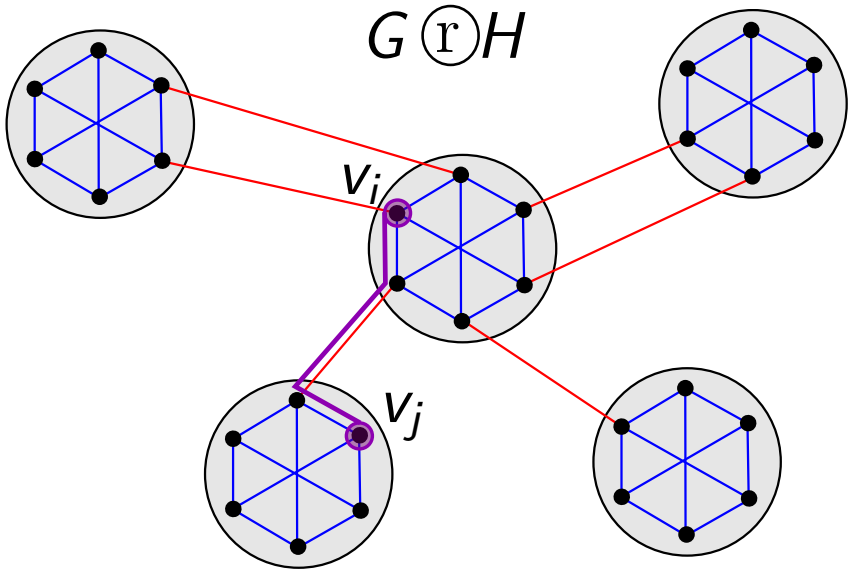

- $H - G - H$ walks in $G \textcircled{r} H \to$ edges in $G \textcircled{z} H$

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- **Fact**: If $\beta = \alpha(H)$, then $\alpha(G \textcircled{z} H) \leq \alpha(G) + \beta + \beta^2$.
- **Gain**: By applying the zig-zag product, we get constant degree at a small price in the spectral gap.

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- Then $\alpha(G_{i+1}) \leq \frac{1}{4} + \frac{1}{100} + \frac{1}{100^2} < \frac{1}{2}$, G_{i+1} is d^2-regular and has d^4-times more vertices.

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- Many other connections and applications:
  - compressed sensing, metric embeddings, correction codes, theory of random matrices, representation theory, harmonic analysis, PDEs, geometric group theory...

Further reading

- Hoory, Linial, Widgerson,
  Expander graphs and their applications, a survey.
- Lecture notes from course *Selected topics of graph theory* by Marcin Pilipczuk, Jakub Onufry Wojtaszczyk and students, editions 2010 and 2013, University of Warsaw
  (in Polish, notes about $\mathbf{L} = \mathbf{SL}$ in English).
- Script from summer school *Hardness of Approximation* by Cygan, Pilipczuk, P, Wojtaszczyk, 2010
  (in Polish).