Subexponential parameterized algorithm for computing the cutwidth of a semi-complete digraph

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Cutwidth of an undirected graph

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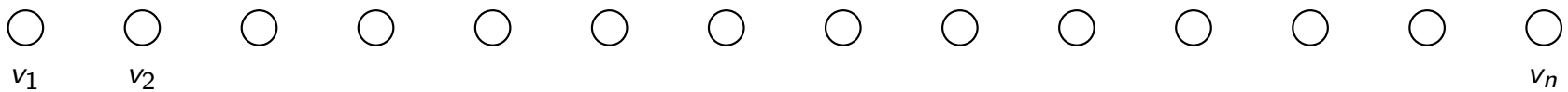

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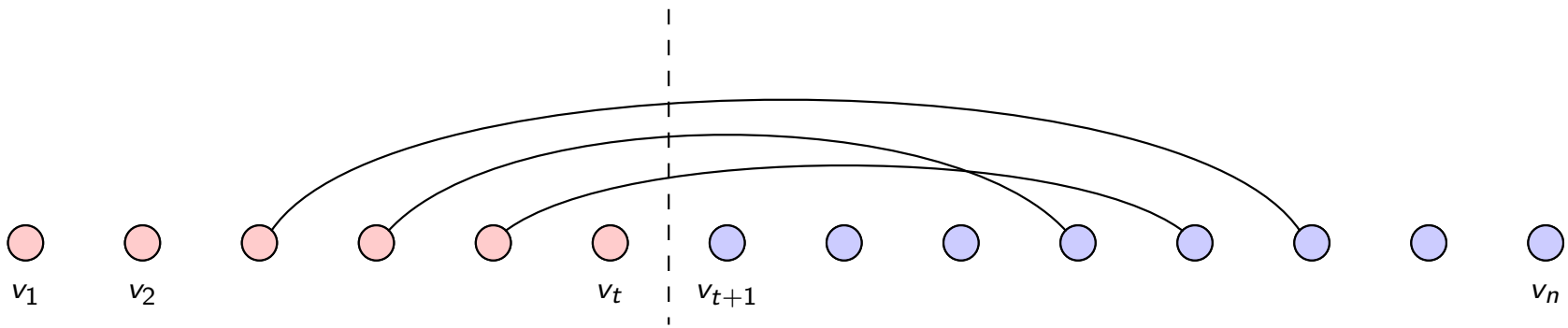

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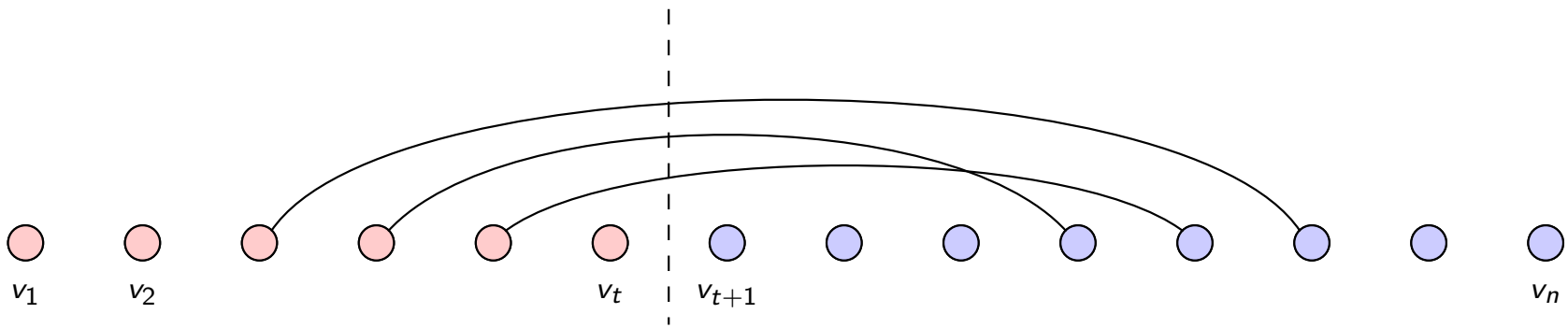

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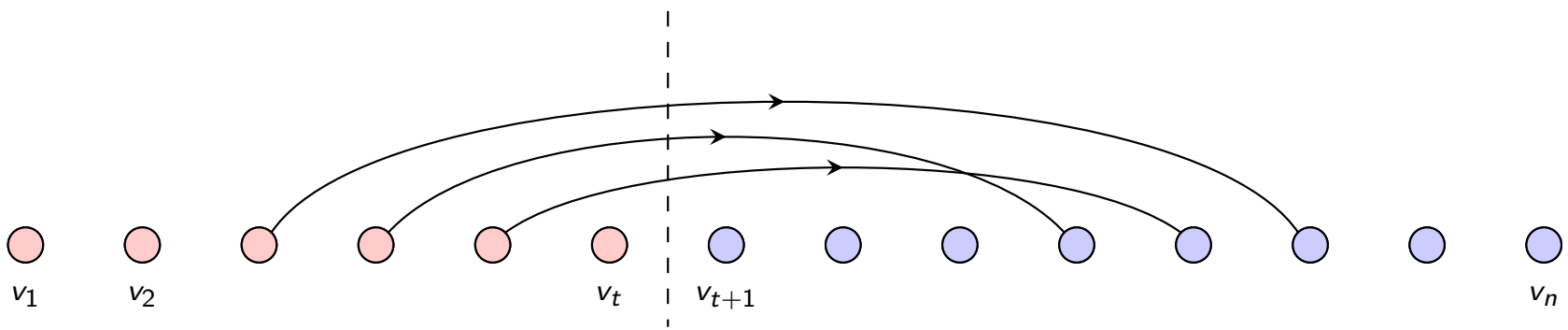

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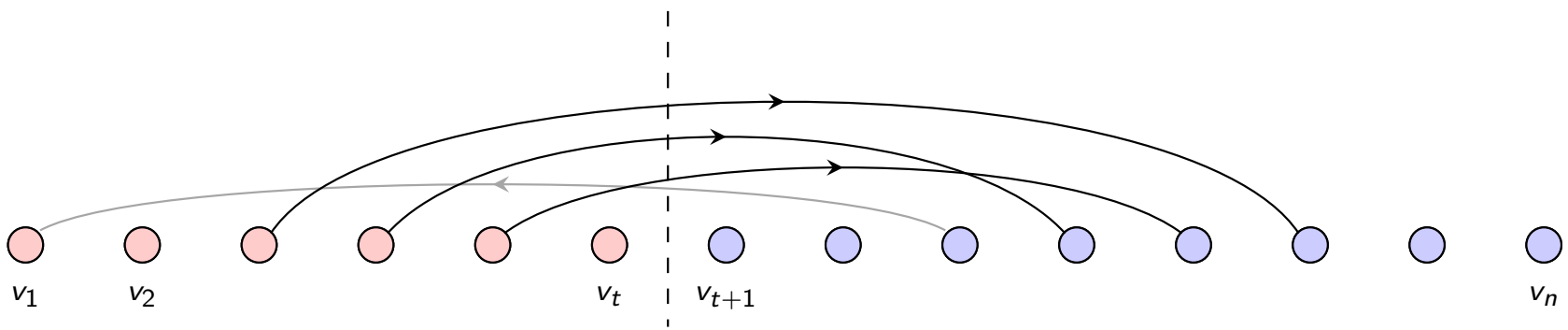

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- **This talk**: only tournaments.

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- (Chudnovsky, Seymour): The immersion order is a well-quasi-ordering of tournaments.
- (P.): An exact FPT algorithm working in $2^{\mathcal{O}(k)} \cdot n^2$ time, where k is the value of cutwidth.

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- Simpler than previous algorithms (Alon, Lokshtanov, Saurabh; Karpinski and Schudy; Feige)

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- (X_2, Y_2) *extends* (X_1, Y_1) if $X_2 = X_1 \cup \{v\}$ and, equivalently, $Y_2 = Y_1 \setminus \{v\}$.

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- Let D be a digraph with $V(D) = \mathcal{N}$, where we put an arc from (X_1, Y_1) to (X_2, Y_2) if (X_2, Y_2) extends (X_1, Y_1).

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- **Observation 1**. Paths from $(\emptyset, V(T))$ to $(V(T), \emptyset)$ correspond to orderings of width at most k.
- **Observation 2**. If we put weights $|E(X_1, \{v\})|$ on the arc from (X_1, Y_1) to (X_2, Y_2), then paths from $(\emptyset, V(T))$ to $(V(T), \emptyset)$ of total weight at most k correspond to orderings with at most k forward arcs.

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- **Corollary**. If we are able to construct D fast, then we are able to solve both problems fast.

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- **Algorithms**:
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  - Otherwise, construct D and run DFS/Dijkstra.

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- How many k-cuts can a transitive tournament have?

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- Partition number $p(k)$ is the number of different multisets of positive integers summing up to k. $(10 = 1 + 1 + 1 + 3 + 4)$
- $p(k) \leqslant \frac{A}{k+1} \cdot e^{C \cdot \sqrt{k}}$ for some constant A, where $C = \pi \cdot \sqrt{\frac{2}{3}}$ (Hardy, Ramanujan, 1919).

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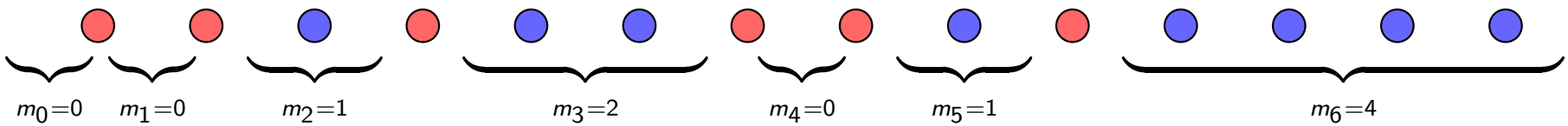

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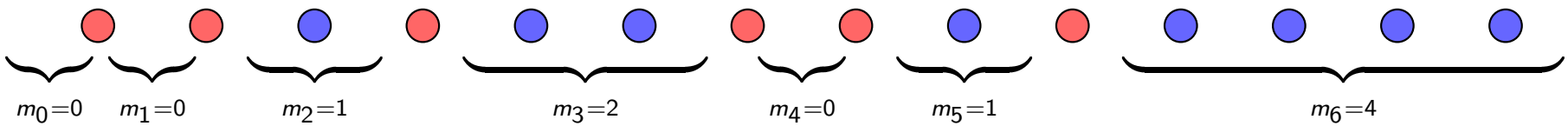

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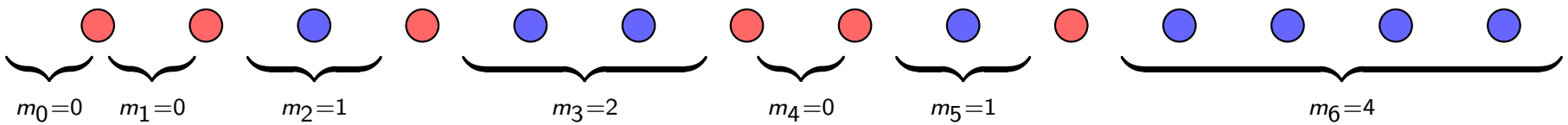

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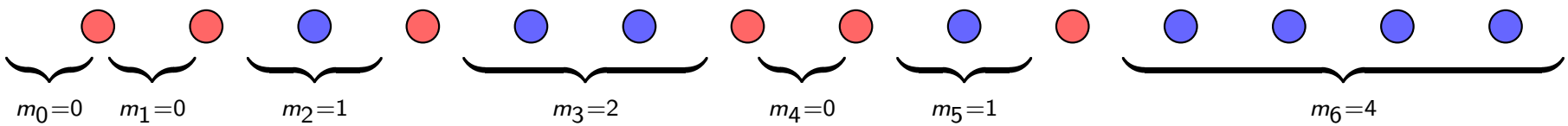
- Then $1 \cdot m_{a-1} + 2 \cdot m_{a-2} + 3 \cdot m_{a-3} + \ldots + a \cdot m_0 = k' \leqslant k$.
- $(X, Y) \to \{m_{a-1} \times 1, m_{a-2} \times 2, \ldots, m_0 \times a\}$, and (X, Y) can be easily reconstructed from the partition.



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- Summing through different a gives n overhead.
- For tournaments of small cutwidth we need to be a little bit smarter.

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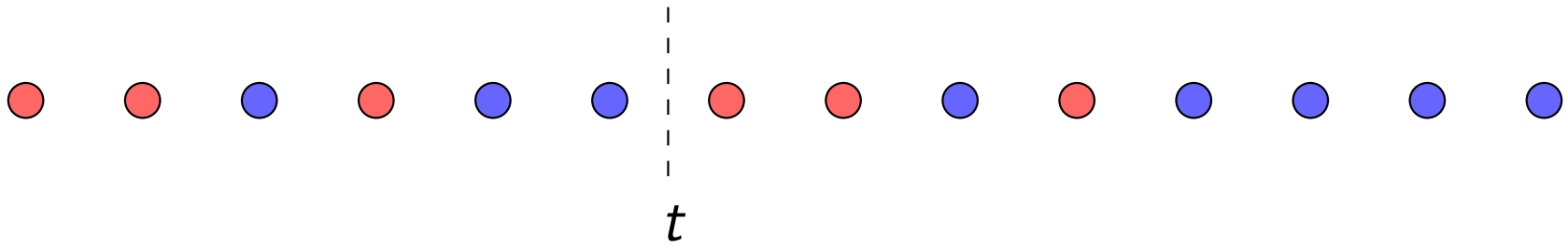
Lemma

Assume that for every t, the number of bad pairs (a, b) with $a \leqslant t < b$ is at most k. Then the total number of bad pairs is at most $k \cdot (1 + \ln k)$.



Bad pairs lemma: proof

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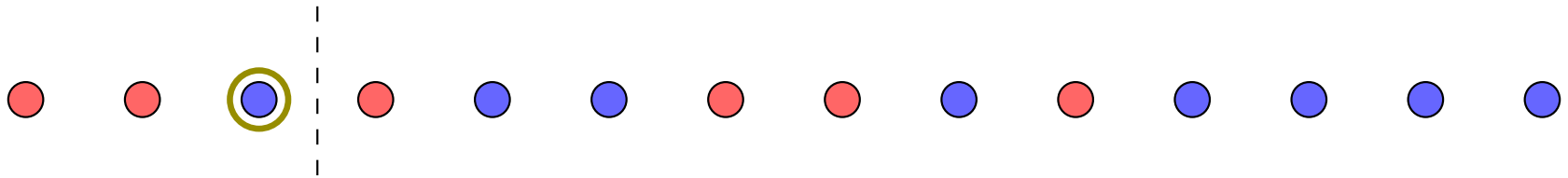

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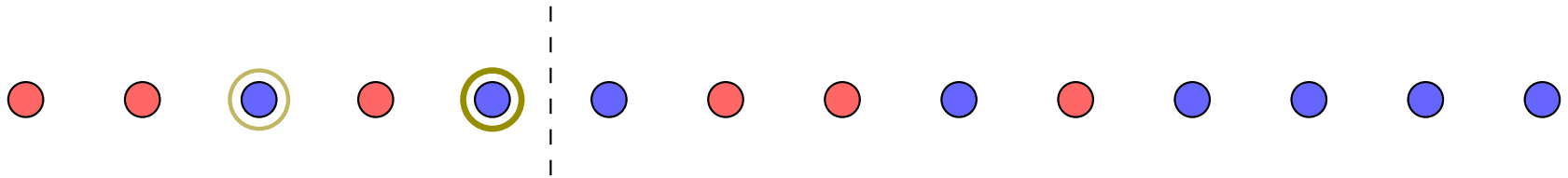

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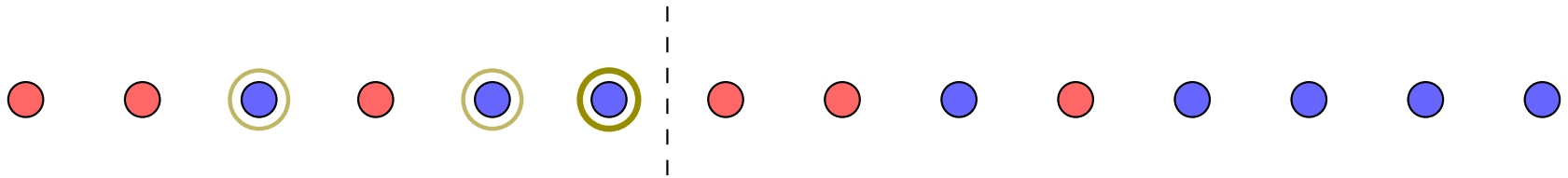

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- The number of bad pairs is hence bounded by

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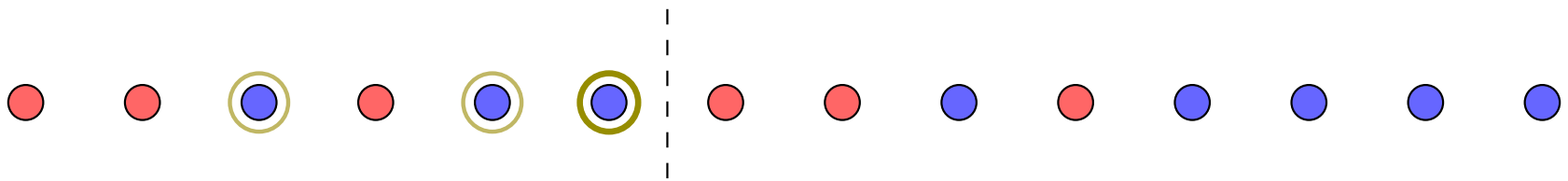

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  - at most k that were present in T;
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- Bad Pairs Lemma $\Rightarrow$ (X, Y) is a $2k \cdot (1 + \ln 2k)$-cut of T'.

Bounding $|\mathcal{N}|$ for small cutwidth

- Let σ be an ordering of width at most k, and let T' be a transitive tournament s.t. $(b, a) \in E(T')$ iff $a <_\sigma b$.
- Let (X, Y) be a k-cut of T. Note that arcs of T' from X to Y correspond to bad pairs.
- For every t, the number of arcs from X to Y in T' that cross the vertical line at t is at most $2k$:
  - at most k that were present in T;
  - at most k arcs crossing the line at t can have different directions in T and T'.
- Bad Pairs Lemma $\Rightarrow$ (X, Y) is a $2k \cdot (1 + \ln 2k)$-cut of T'.
- The number of such cuts is bounded by $\mathcal{O}(2^{\mathcal{O}(\sqrt{k \log k})} \cdot n)$.

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- **Thank you!**