Tight bounds for parameterized complexity of Cluster Editing

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Problem

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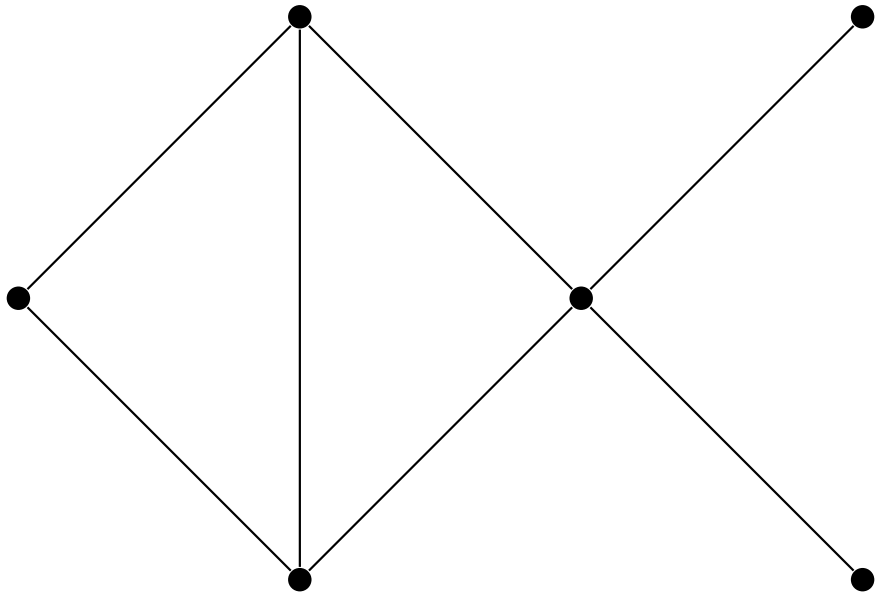
- **Parameter:** k
- **Question:** Can one make at most k edge additions/deletions to turn G into a cluster graph, i.e., a disjoint union of cliques?

Example



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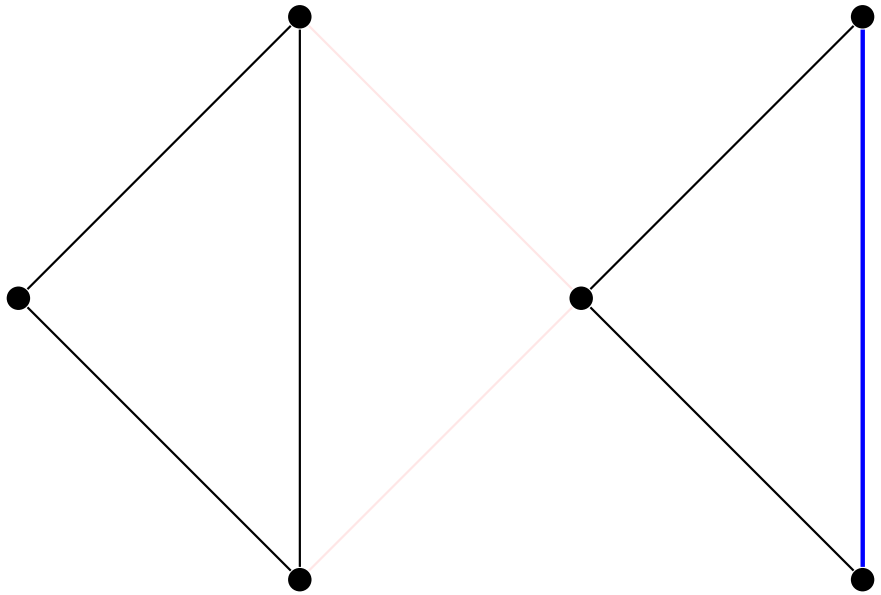

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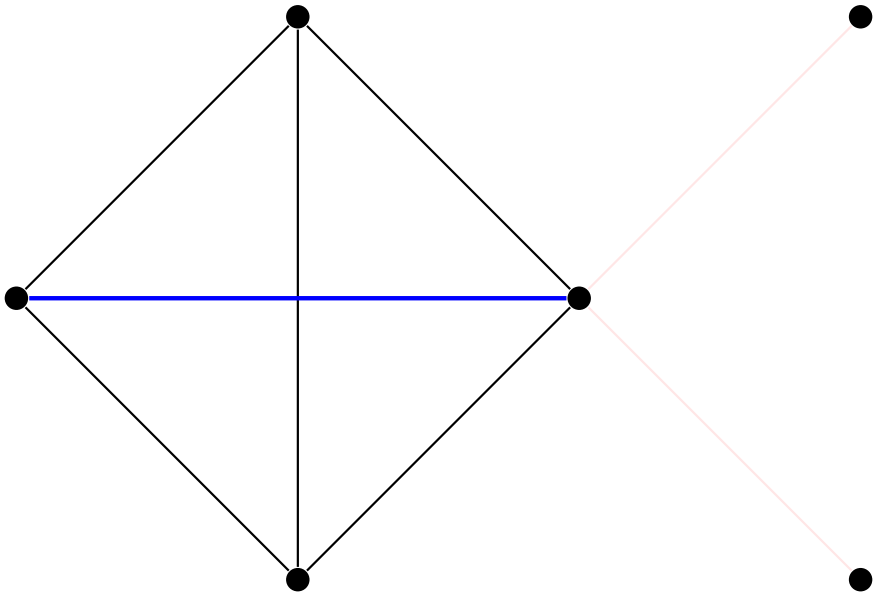

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  - machine learning.
- **Reason**: Models the natural problem of clustering objects, where a simple similarity relation is given.

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  - kernel with at most $2k$ vertices [Cao, Chen, Meng],
  - no $\mathcal{O}^\star(2^{o(k)})$ algorithm under ETH [this work; Komusiewicz].

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- There is a PTAS, if we ask for a cluster graph with at most p clusters, where p is a constant! [Giotis, Guruswami]
- **Running time**: $n^{\mathcal{O}(9^p/\epsilon^2)}$
- **Our goal**: Study parameterized complexity of Cluster Editing, when the target number of clusters is bounded by a sublinear function of k.

Problem definition(s)

p-Cluster Editing

- **Input:** Graph G, integers p, k
- **Parameters:** p, k
- **Question:** Can one make at most k edge additions/deletions to turn G into a cluster graph with **exactly** p clusters?

Problem definition(s)

$p_{\leqslant}$-Cluster Editing

- **Input:** Graph G, integers p, k
- **Parameters:** p, k
- **Question:** Can one make at most k edge additions/deletions to turn G into a cluster graph with **at most** p clusters?

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- **Question:** Can one make at most k edge additions/deletions to turn G into a cluster graph with **at most** p clusters?

- p-Cluster Editing is at least as hard as $p_{\leqslant}$-Cluster Editing, but the converse is unclear.

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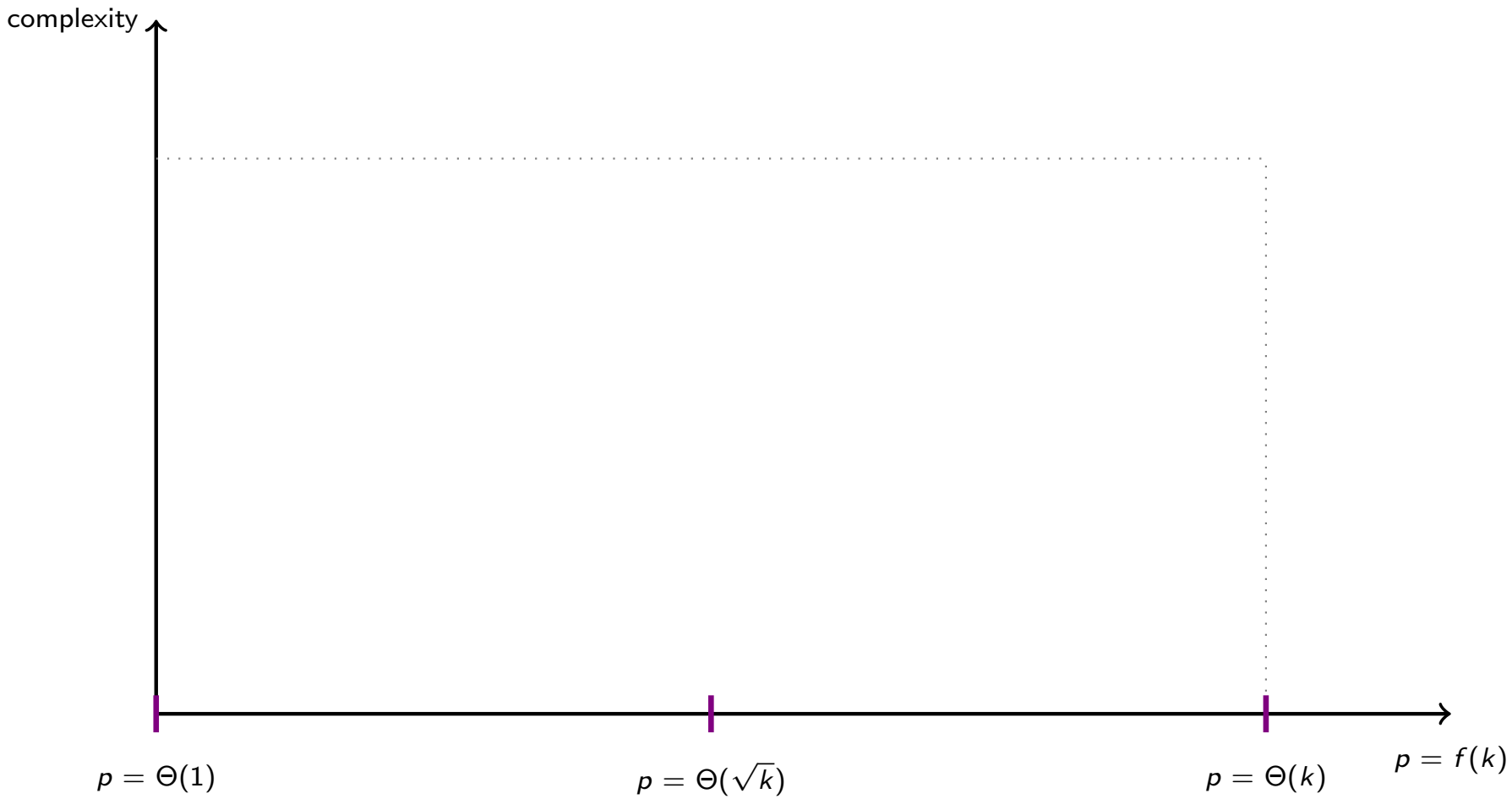
- p-Cluster Editing is at least as hard as $p_{\leqslant}$-Cluster Editing, but the converse is unclear.
- There is a polynomial-time reduction that ensures $p \leqslant 6k$ in both variants (technical).

Upper bound

Theorem 1

p-Cluster Editing can be solved in $\mathcal{O}^\star(2^{\mathcal{O}(\sqrt{pk})})$ time.

Complexity



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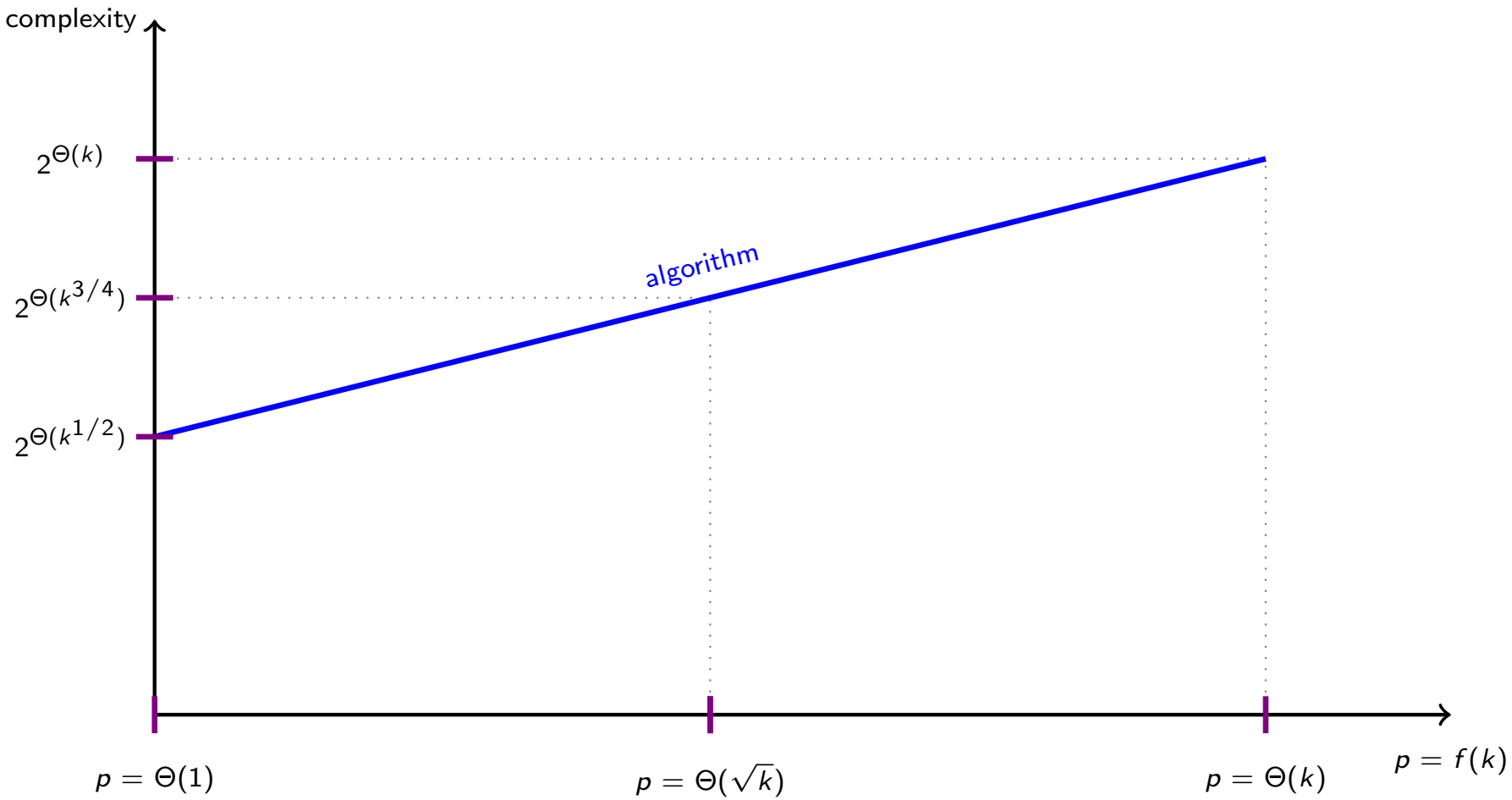
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$p = \Theta(1)$
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Lower bound

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Corollary

Unless ETH fails, for every $\sigma \in [0,1]$ there exists $p(k) = \Theta(k^\sigma)$ such that there is no algorithm with running time $\mathcal{O}^\star(2^{o(\sqrt{pk})})$ solving $p_{\leqslant}$-Cluster Editing.

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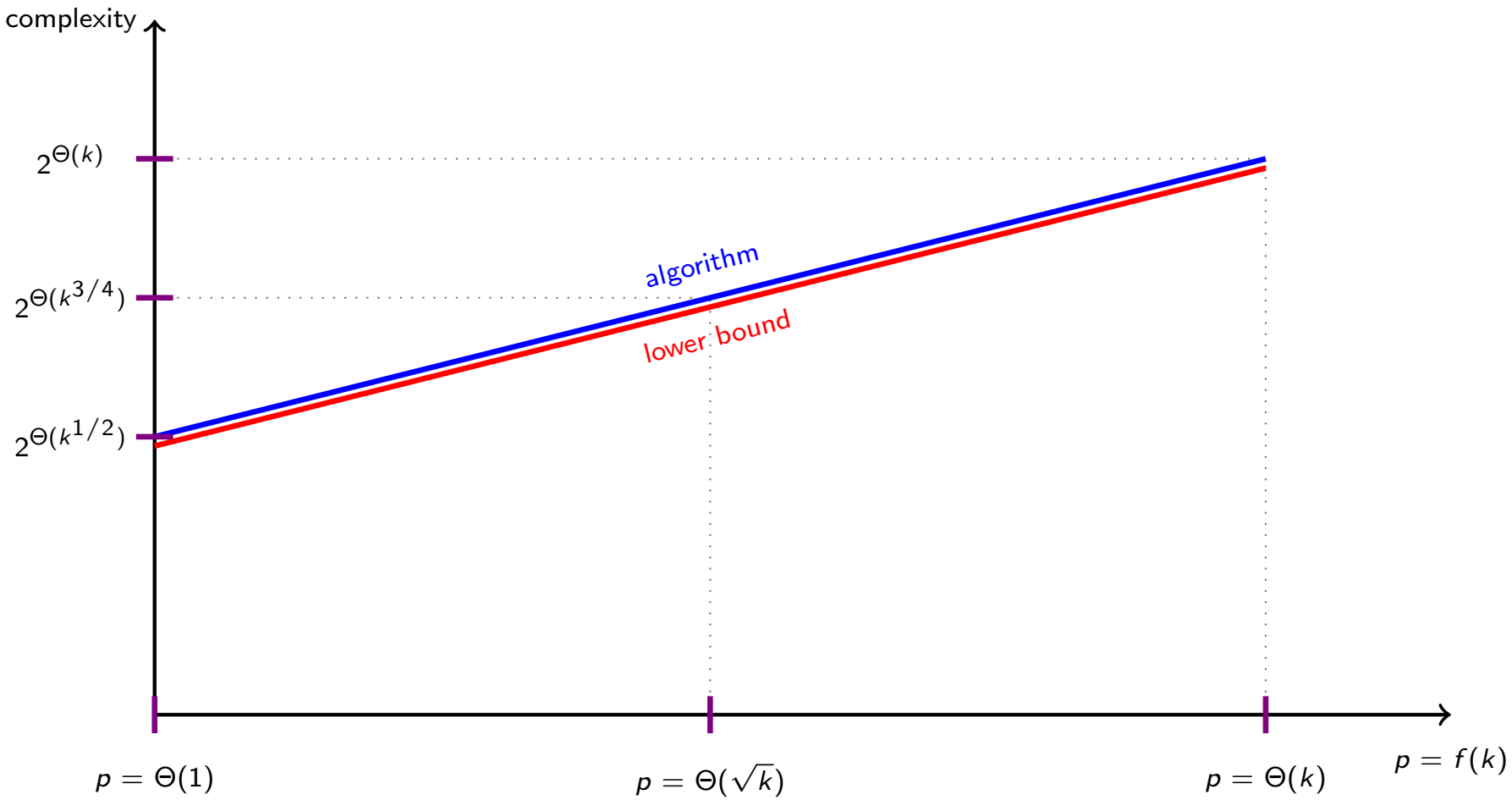
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- **Result**: $D[V(G)][p]$.
- **Time complexity**: for every subset we search through its subsets, so $\mathcal{O}^\star(3^n)$.

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Partition (X, Y) of $V(G)$ is a *k-cut* if $|E(X, Y)| \leqslant k$.

- **Observation**: if someone gave us the set $\mathcal{N}$ of k-cuts, then we can perform the DP in $\mathcal{O}^\star(|\mathcal{N}|^2)$ time.

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- **Algorithm**:
  - we enumerate the k-cuts with polynomial delay;
  - if we exceed the combinatorial bound, we say NO;
  - otherwise, we employ the DP on the set of k-cuts $\mathcal{N}$.

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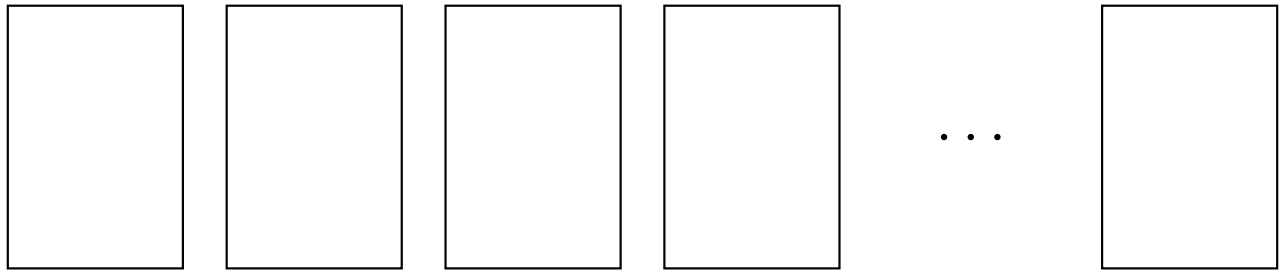
- Hence, we need to prove that the number of k-cuts of a cluster graph with p clusters is at most $2^{\mathcal{O}(\sqrt{pk})}$.
- **Note**: this is a purely combinatorial question!

k-cuts of a cluster graph



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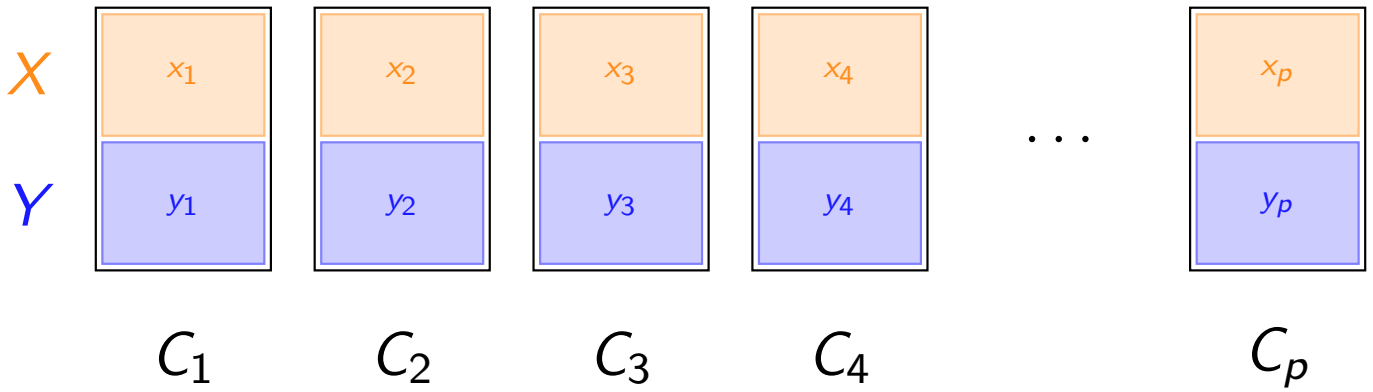

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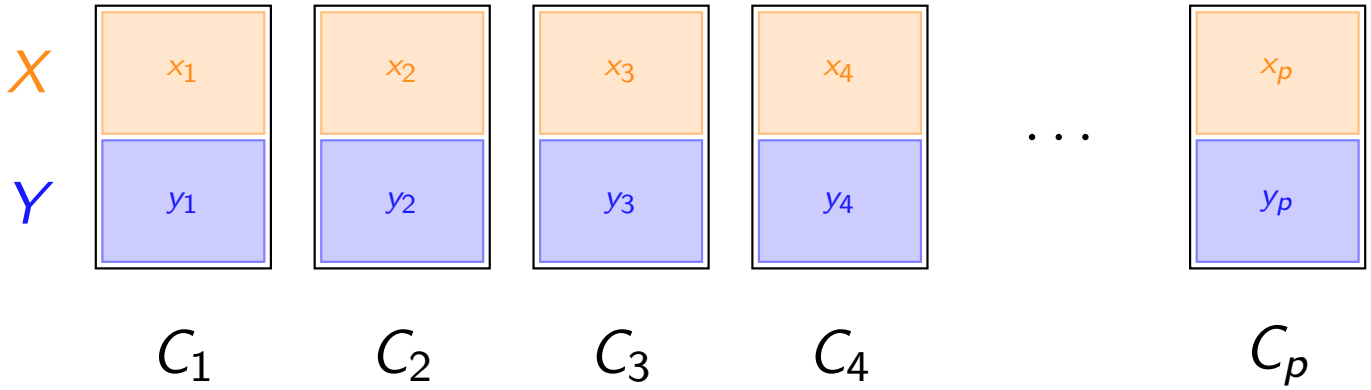

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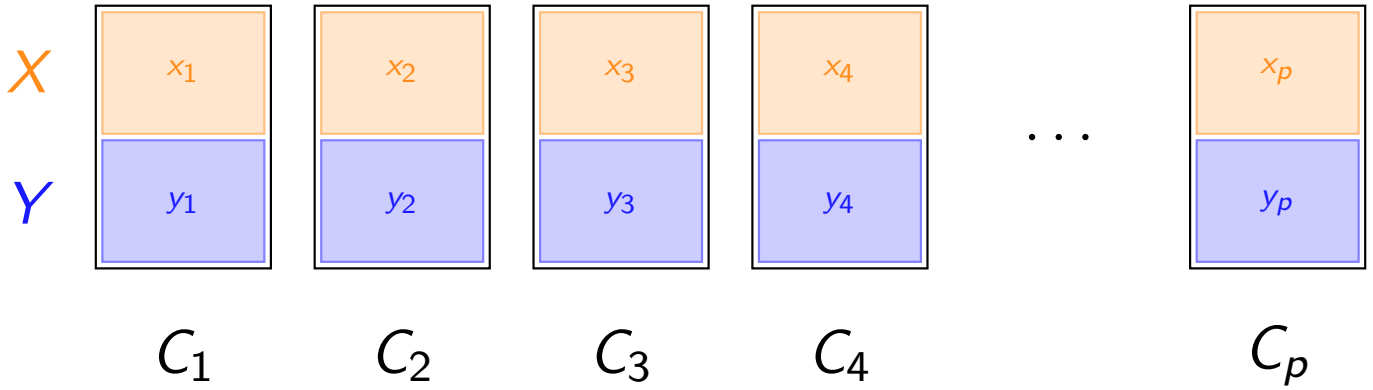

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- **Note**: $\sum_{i=1}^{p} x_i y_i \leqslant k$.

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- The number of choices (x_i, y_i) is also bounded by $2^{\mathcal{O}(\sqrt{pk})}$ (easy, but we need $p = \mathcal{O}(k)$).

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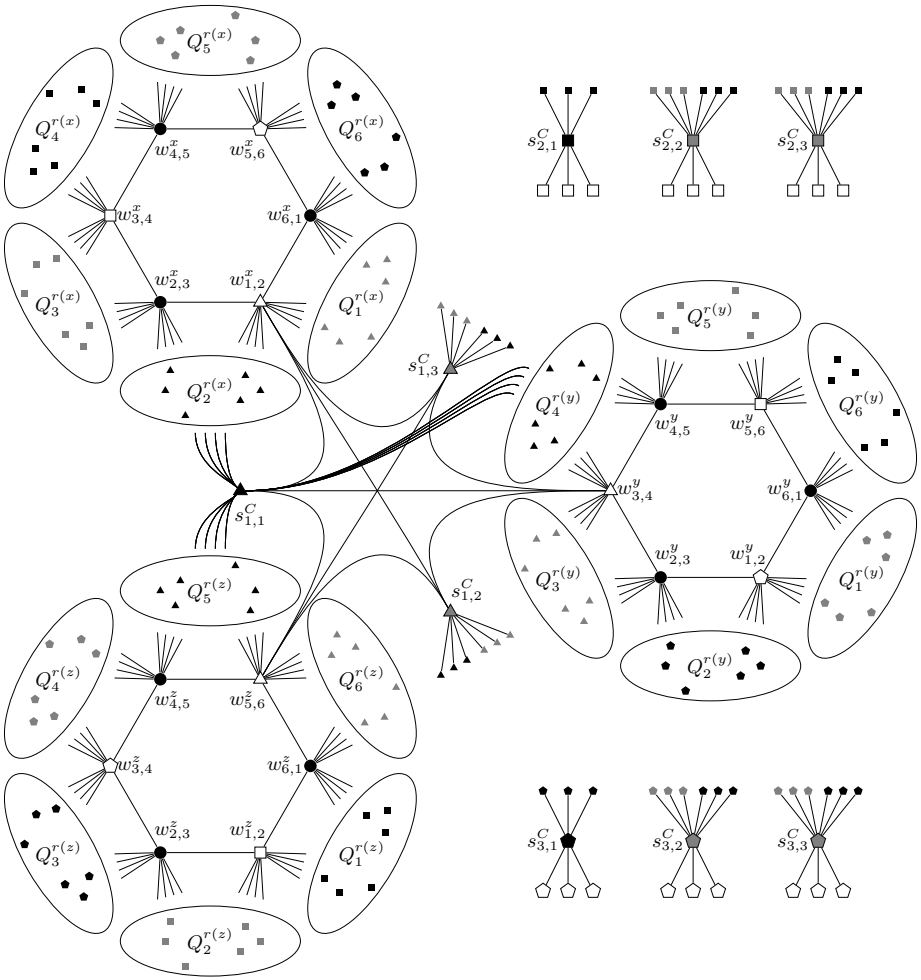
- Hence, we will present just some pictures of gadgets.

Gadgets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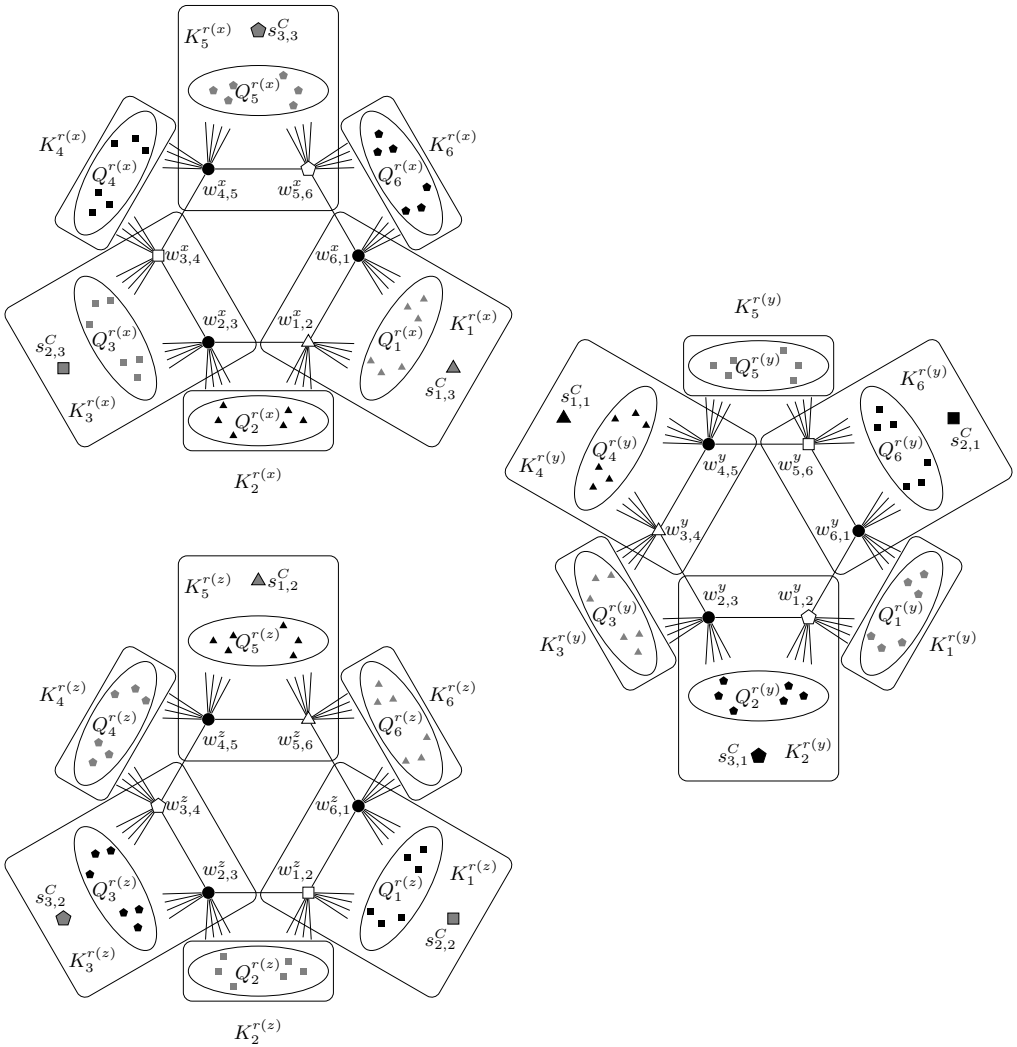
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  - **Open**: What about $p < 6$?

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