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The extension problem for continuous linear operator

Thm (On extension of bdd operator). Let X be a norm-space, $Y \subseteq_{\text{lin}} X$ with $\overline{Y} = X$ and $A \in B(Y, Z)$ with Z - Banach space. Then $\exists! \ \tilde{A} \in B(X, Z)$ s.t. $\tilde{A}|_Y = A$. We call $\tilde{A}$ the continuous extension of A. Moreover $\|\tilde{A}\| = \|A\|$.

Proof. Uniqueness follows from the fact that Y is dense and $\tilde{A}$ is continuous.

Let $x \in X \Rightarrow \exists \ y_n \xrightarrow{\in Y} x$. $\tilde{A}$ is continuous operator $\Rightarrow \tilde{A} y_n = A y_n \longrightarrow \tilde{A} x$.
The idea is to define
$$\tilde{A} x := \lim_n A y_n$$

This is a good definition if this limit $\exists_{(1)}$ and it doesn't depend from the choice of the sequence $\{y_n\}_n$ that converges to $x_{(2)}$.

1. Let $\{y_n\}_n$ s.t. $y_n \xrightarrow[n \to +\infty]{} x \Rightarrow y_n$ is a Cauchy sequence. A is continuous, in particular it is Lip. (for the equivalent properties of bdd operator). So
$$\|A y_n - A y_m\| \leq \|A\| \ \|y_n - y_m\|$$
$\Rightarrow \{A y_n\}_n$ is C in Z. But Z is a Banach space $\Rightarrow \{A y_n\}_n$ is convergent.

2. Let $\{y_n\}_n$ and $\{y'_n\}_n$ be two sequences in Y s.t. $y_n, y_n' \xrightarrow[n \to +\infty]{} x$.
$$\Rightarrow \underbrace{y_n - y'_n}_{\in Y} \xrightarrow[n \to +\infty]{} 0 \Rightarrow A(y_n - y_n') \xrightarrow[n \to +\infty]{} 0$$
$$A(y_n - y_n') = A y_n - A y_n' \longrightarrow z - z'$$
$A y_n \xrightarrow{n \to +\infty} z$, $A y_n' \xrightarrow{n \to +\infty} z'$ (because we know from 1 that these limits exist)

For the unicity of the limit $z - z' = 0 \Rightarrow z = z'$.

Δ (a) $\tilde{A}$ is linear, (b) $\tilde{A}$ continuous, (c) $\tilde{A}|_Y = A$

Let's check that $\|\tilde{A}\| = \|A\|$.
Let $x \in X$, $\{y_n\}_n \in Y$ s.t. $y_n \to x \Rightarrow \|y_n\| \to \|x\|$.
We know that $A y_n \xrightarrow[n]{} \tilde{A} x$. It follows that
$$\|\tilde{A} x\| \xleftarrow[n]{} \|A y_n\| \leq \|A\| \ \|y_n\| \xrightarrow[n \to +\infty]{} \|A\| \ \|x\|$$
$$\Rightarrow \|\hat{A} x\| \leq \|A\| \ \|x\| \Rightarrow \|\tilde{A}\| \leq \|A\|.$$
The inverse inequality is trivial. In fact
$$\|\tilde{A}\| = \sup \{ \|\tilde{A} x\| : x \in X, \|x\| \leq 1 \} \geq \sup \{ \|\hat{A} x\| : x \in Y, \|x\| \leq 1 \} = \|A\|.$$
■

Def. A Banach space $\tilde{X}$ is the completion of a norm space X iff $\exists \ Y \subseteq_{\text{lin}} \tilde{X}$ s.t. $\overline{Y} = \tilde{X}$ and Y and X are isometric.

Remark It is easy to prove that if $\tilde{X}$ is a completion of X then there $\exists \ \tilde{\tilde{X}}$ - a Banach space such that X is a norm subspace of $\tilde{\tilde{X}}$ and $\overline{X} = \tilde{\tilde{X}}$.

Thm (On completion) If X is a norm space, then there exists a completion of X. Moreover all completions of X are isomorfic. More-moraver if for $j = 1, 2$ $\hat{X}_j$ is a completion of X, $\overline{Y_j} = \tilde{X}_j$ and $I_j : X \longrightarrow Y_j$ is isometry, then there exists a unique $J \in B(\hat{X_1}, \tilde{X_2})$ such that $J|_{Y_1} = I_2 \cdot I_1^{-1}$; The above J is an isometry.

Proof. Consider the set of all the C. sequences in X, and we define on it an equivalent relation s.t. $\{x_n\} \equiv \{x_n'\}$ iff $x_n - x_n' \longrightarrow 0$.

1. $\equiv$ is an equivalent relation. $\triangle$

We define $\tilde{X} = \{ [\{x_n\}]_{\equiv} \mid \{x_n\} - \text{is C. in } X \}$

Let X, Y be n.s. It is easy to check that $B(X,Y)$ is a linear space. We have defined a function called operator norm on $B(X,Y)$. Is this a norm?

Fact. $B(X,Y) \underset{\text{lin}}{\subseteq} \mathcal{L}(X,Y)$ and $\|\cdot\|$ is a norm in $B(X,Y)$. Moreover if Z is a n.s. and $A \in B(X,Y)$, $B \in B(Y,Z)$, then $B \cdot A \in B(X,Z)$ and

$$\|BA\| \leq \|B\| \cdot \|A\|.$$

In particular $B(X) \equiv B(X,X)$ is a norm $\mathbb{K}$-algebra with unity if $X \neq \{0\}$, where multiplication is the composition of operator and the unity is I.

Remark. A norm algebra with unity $\mathbb{1}$ is a norm space s.t.

$\|x \cdot y\| \leq \|x\| \cdot \|y\|$

$\|\mathbb{1}\| = 1.$

Proof. We prove only $\|BA\| \leq \|B\| \, \|A\|$. The other things are for ex.

$$\|(BA)x\| = \|B(Ax)\| \leq \|B\| \, \|Ax\| \leq (\|B\| \cdot \|A\|)\|x\|$$

$\Rightarrow$ by equiv. cond. of def. of $\|\cdot\|_{op}$ $\quad \|BA\| \leq \|B\| \cdot \|A\|$

Now we know that $B(X,Y)$ is a normspace. Is it a Banach space?

Thm (on completness of $B(X,Y)$). If Y is Banach, then $B(X,Y)$ is Banach.

Proof. Let $\{A_n\}_{n\geq 1}$ be a C. sequences in $B(X,Y)$. Let $x \in X$. Then

$\|A_n x - A_m x\| = \|(A_n - A_m)x\| \leq \|A_n - A_m\| \|x\|$

$\Rightarrow \{A_n(x)\}_n$ is a C. sequences in Y. But Y is complete $\Rightarrow A_n x \xrightarrow[n\to+\infty]{} A(x)$.

We can define:

$$\begin{aligned} A: X &\longrightarrow Y \\ x &\longmapsto \lim_{n\to+\infty} A_n(x) =: A(x) \end{aligned}$$

By definition A is the pointwise limit of $\{A_n\}_n$ $\forall x \in X$.
We have to check that $A \in B(X,Y)$ and that $A_n \xrightarrow[\|\cdot\|_{op}]{} A$.

- $A \in \mathcal{L}(X,Y) \longrightarrow \Delta$

- Let $\varepsilon > 0$, $N \in \mathbb{N}$ s.t. $\forall n,m \geq N$ $\|A_n - A_m\| < \frac{\varepsilon}{2}$ (we can do this from def. of C. sequences).
  Then if $m \geq N$, then for large enough

  $\|A_n x - A_m x\| \leq \|A_n - A_m\| \|x\| \leq \frac{\varepsilon}{2}\|x\|$ $\forall x \in X$.

  Observe that, by continuity of norm, $\forall x \in X, \forall m \geq 1$ $A_n x - A_m x \xrightarrow[n]{} Ax - A_m x$
  $\Rightarrow \|A_n(x) - A_m(x)\| \xrightarrow[n]{} \|Ax - A_m x\|$.

  It follows that $\forall x \in X$

  $\lim_n \|A_n x - A_m x\| = \|Ax - A_m x\| \leq \frac{\varepsilon}{2}\|x\|$

  $\rightarrow$ Take $m = N$ $\|A - A_N\| \leq \frac{\varepsilon}{2}$ $\Rightarrow A - A_N \in B(X,Y)$, $A_N \in B(X,Y)$
  $\rightarrow$ $A = (A - A_N) + A_N \in B(X,Y)$.

- In the previous point we have just proved that $\forall m \geq N$ $\|A - A_m\| < \varepsilon$
  $\Rightarrow A_m \xrightarrow[m\to+\infty]{} A$ in $\|\cdot\|_{op}$.

Remark. If $X = \{0\}$, $B(X,Y) = \{0\}$ $\forall Y$, so we don't need completness on Y.

Ex. If $X \neq \{0\}$, the completness of Y is also a necessary condition in order to have completness of $B(X,Y)$? Not so easy

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