21-10-2016

Linear density, separability

Let (X, τ) be a topological space

Def. A subset $C \subset X$ is dense iff $\overline{C} = X$.

Def. X is separable iff there exists $C \subset X$ that is dense and at most countable.

Let $(X, \|\cdot\|)$ be a normed space, $C \subseteq X$.

Def C is linearly-dense iff $\overline{\text{lin}\, C} = X$, id est $\text{lin}\, C$ is dense in X.

Remark. C - dense $\Rightarrow$ C - lin - dense. In fact, $C \subset \text{lin}\, C \Rightarrow X = \overline{C} \subset \overline{\text{lin}\, C} \Rightarrow \text{lin}\, C$ is dense.
The reverse is not true (in some sens C could be too small).

We have an intermediate result. Let $C \subseteq X$, we define

$$\text{lin}_{\mathbb{Q}}\, C := \begin{cases} \text{all lin. comb. of vectors from } C \text{ with coefficient in } \mathbb{Q}, & \text{when } \mathbb{K} = \mathbb{R} \\ \quad '' \quad\quad '' \quad\quad '' \quad \text{both real and complex in } \mathbb{Q}: \ \mathbb{Q} + i\mathbb{Q}, & \text{where } \mathbb{K} = \mathbb{Q} \end{cases}$$

Fact. C is lin - dense iff $\text{lin}_{\mathbb{Q}}\, C$ is dense, id est $\overline{\text{lin}\, C} = X \iff \overline{\text{lin}_{\mathbb{Q}}\, C} = X$

Proof. $(\Leftarrow)$ Obvious since $\text{lin}_{\mathbb{Q}}\, C \subseteq \text{lin}\, C$.

$(\Rightarrow)$ Assume that $\overline{\text{lin}\, C} = X$.

1) $\forall\, \varepsilon > 0,\ x \in X,\ \exists\, c_1, \dots, c_n \in C,\ \lambda_1, \dots, \lambda_n \in \mathbb{K}$ s.t. $\|x - \sum_{j=1}^{n} \lambda_j c_j\| \leq \frac{\varepsilon}{2}$.

2) We should find $\lambda_1', \dots, \lambda_n' \in \begin{cases} \mathbb{Q} & \text{if } \mathbb{K} = \mathbb{R} \\ \mathbb{Q} + i\mathbb{Q} & \text{if } \mathbb{K} = \mathbb{C} \end{cases}$ such that $\|\sum_{j=1}^{n} \lambda_j c_j - \sum_{j=1}^{n} \lambda_j' c_j\| \leq \frac{\varepsilon}{2}$.

Use triang. ine. to conclude $\triangle$

Fact (separability from linear density) A normed space is separable iff there exists an at most countable lin-dense subset in it.

Proof. $(\Rightarrow)$ If a normed space is separable, then exists $C \subseteq X$ at most countable that is dense $\Rightarrow$ C is lin. dense.

$(\Leftarrow)$ Assume that exists C at most countable and linearly dense. For the previous fact $\text{lin}_{\mathbb{Q}}\, C$ is lin. dense. But $\text{lin}_{\mathbb{Q}}\, C$ is at most countable (actually it is countable, only in the case in which $X = \langle 0 \rangle$ is less then countable). ∎

Example **0.** $\mathbb{K}^d$ with any norm. Any finite dimensional space is separable (in fact in a finite of dimd space we have a basis $\{v_1, \dots, v_d\}$ s.t. $\text{lin}(\{v_1, \dots, v_d\})$ = the space.

1. $\ell^p(\mathbb{N})$ is separable for any $p \in [1, +\infty)$.
We can infact consider, $\forall\, n \in \mathbb{N}$, $e_n \in \ell(\mathbb{N})$ where $e_n(k) = \begin{cases} 1 & \text{for } k = n \\ 0 & \text{otherwise} \end{cases}$

$C = \{e_n : n \in \mathbb{N}\}$ is lin.dense.

$\triangle$

2. $\ell^p_\omega(\mathbb{I})$ where $\mathbb{I}$ is a set at most countable, $\omega>0$.

$\triangle$ proof that it is separable

3. $\ell^\infty(\Omega)$ is not separable if Ω is an infinite set.

$\triangle$

4. $\mathcal{C}([a,b])$ is separable.
We can in fact consider $\mathcal{J} := \{x^n : n \in \mathbb{N}_0\}$, where $x^n(t) := t$, $x(t) := t$ and $0^0 = 1$.
$\text{lin}\,\mathcal{J} = \text{Pol}([a,b])$. and for Stone-Weierstrass thm each continuous function can be approximate uniformey by a sequence of polynoms.
$\Rightarrow$ $\text{lin}\,\mathcal{J}$ is dense in $\mathcal{C}([a,b])$ $\Rightarrow$ obtain the separability from the previous fact.

5. For typical measure (we will see this during exercise class), we have in general that $L^p(\Omega)$ is separable for $p<+\infty$, is not separable for $p=+\infty$.

Fact. ??? Vedi cogli appunti

Series in norm space. Schauder basis

Let $(X, \|\cdot\|)$ be a normed space, $\{x_n\}_{n\geq n_0}$ a sequence in X

For $\sum_{n=n_0}^{+\infty} x_n$ we indicate the sequence $(S_n)_{n\geq n_0}$ s.t. $S_n := \sum_{k=n_0}^{n} x_k$ for $n \geq n_0$.
The convergence of the series is equivalent at the convergence of $\{S_n\}_n$ as a seq.
If the series converges we denote its limit as $\sum_{n=n_0}^{+\infty} x_n$.

Def. $\sum_{n=n_0}^{+\infty} x_n$ is absolutely convergent iff $\sum_{n=n_0}^{+\infty} \|x_n\|$ is convergent.

Notice that in the real case, a series admits summation that could be also $\pm\infty$, and in this case we say that it is not convergent but it doesn't admit finite summation. In a general norm space we don't have this possibility.

Thm. (On absolute convergence in normed space) X is a Banach space iff each absolutely convergent series in X is convergent. Listen at minute 55
For the proof see exercise class (pag 3)

Remark. Let X be a normed space If $\sum_{n=n_0}^{+\infty} x_n$ converges and absolutely converges, then
$\|\sum_{n=n_0}^{+\infty} x_n\| \leq \sum_{n=n_0}^{+\infty} \|x_n\|$

$\triangle$

Example. Consider the space $\ell(\mathbb{N}_1)$ and the sequence $\{x_n\}_{n\geq 1}$ s.t. $x_n := \frac{1}{n} e_n$

It is convergent in each $\ell^p(\mathbb{N})$ $\forall p>1$
It is divergent in $\ell^1(\mathbb{N})$.
In both the spaces it is not absolutely convergent.

$\triangle$

Let $(X, \|\cdot\|)$ be a normed space and let $\{x_n\}_{n \geq n_0}$ a sequence in X.

Def. $\{x_n\}_{n \geq n_0}$ is a Shauder base iff $\forall x \in X \quad \exists! \; a \in \ell(\mathbb{N}_{n_0})$ s.t. $\sum_{n=n_0}^{+\infty} a_n x_n = x$.

Remark. 1. Shauder basis is a linear independent sequence

$\triangle$

2. $\{x_n\}_{n \geq n}$ a s.b. then the set of all terms of it is lin.-dense.
Be carefull! The reverse is not true.

Example. 1. $\{x^n\}_{n \geq 0}$ for $\mathcal{C}([0,1])$ it is not a s.b.

$\triangle$

Fact. If there exists a S.b. then the space is separable.

2. Bounded operators, functionals

Def. Let X, Y be two linear space. We indicate with $\mathcal{L}(X,Y)$ the set of linear operator from X to Y. If $A \in \mathcal{L}(X,Y)$ we write $Ax := A(x)$. $\forall x \in X$.
If $Y = \mathbb{K}$ we call the linear operator functional. We denote the space of functionals as $X^{\#}$.
When $Y = X$ we indicat the space of linear operators of X into itself as $\mathcal{L}(X)$. Notice that $0, I \in \mathcal{L}(X)$ (and $0 \neq I$ if $X \neq \langle 0 \rangle$).

Let X, Y be normed space, $A \in \mathcal{L}(X,Y)$.

Def. A is a bounded operator iff for any bounded set $C \subset X$, $A(C)$ is bounded (in Y).

Remark. Notice that A is also a function and so we have for it a concept of boundness. But this notion is different from the notion of boundness as operator. In fact, in the first case we need that $A(X)$ (where X could be unbounded) is bounded, while in the second one we need that $A(C)$ is bounded for each bounded C.

Fact. If $A \in \mathcal{L}(X,Y)$ and $X \neq \{0\}$, then $\sup\limits_{\|x\| \leq 1} \|Ax\| = \sup\limits_{\|x\| \leq 1} \|Ax\| = \sup\limits_{\|x\| = 1} \|Ax\|$

Proof. $\triangle$ (or look the notes)

Fact. A is bounded iff $A(\overline{K}(0,1))$ ($\Leftrightarrow A(K(0,1))$ or $A(S(0,1))$ is bounded in Y.

Proof. $\triangle$

Corollary. A is bounded operator $\Leftrightarrow A|_{\overline{K}(0,1)}$ is a bounded function.

Fact (On conditions equivalent to continuity) If $A \in \mathcal{L}(X,Y)$, then the following conditions are equivalent:

(i) A is bounded;
(ii) A is continuous at 0;
(iii) A is continuous;
(iv) A is uniformly continuous;
(v) A is Lipschitz
(vi) $\exists\, C \in \mathbb{R} : \forall x \in X \quad \|Ax\| \leq C\|x\|$
(vii) $\sup\limits_{\|x\| \leq} \|Ax\| < +\infty$

Try to make the proof or look at the notes

Definition: • $B(X,Y) := \{ A \in \mathcal{L}(X,Y) : A \text{ is bounded operator} \}$.

• Let $A \in \mathcal{L}(X,Y)$, we can define the operator norm of A as

$$\|A\|_{op} := \sup_{\|x\| \leq 1} \|Ax\|$$

Fact (On Lipsch. constant) If $A \in B(X,Y)$ is a bounded operator, then

1) $\forall x \in X \quad \|Ax\| \leq \|A\| \cdot \|x\| \quad$ (id est $\|Ax\|_Y \leq \|A\|_{op} \cdot \|x\|_X$)

2) $\|A\| = \sup_{\|x\| \leq 1} \|Ax\| = \sup_{\|x\| < 1} \|Ax\| = \sup_{\|x\| = 1} \|Ax\| = \inf G$,

where $G := \{ c \in [0, +\infty) : \forall x \in X \ \|Ax\| \leq c \|x\| \}$. Moreover G posses the smallest element.

Remark. But $\{ \|Ax\| : \|x\| \leq 1 \}$ may not posses a biggest element.

Find an example as exercise + Section 2 [illegible] p. 12