2. Compact operators

We introduce and study here compact operators – a special class of bounded operators. They are considered to be "small" in some sense, if the dimension of the domain is infinite. But this "smallness" concerns not the norm, but the "type of the range". The typical examples are all bounded operators with finite dimensional range (so-called finite dimensional ops.). Some properties of compact operators resemble those known from matrix theory in finite dimensions.

2.1 The definition and some equivalent conditions

◆ A reminder of some metric space "round-compactness" notions and facts

Let (X, ρ) be a metric space (some of the notions below are used also for general topological spaces, but now, we need them only in norm spaces context):

- $P \subset X$, P is precompact*) iff $\overline{P}$ is compact
- $S, B \subset X$, $\varepsilon > 0$, S is an ε-net for B iff $S \subset B$ and $B \subset \bigcup_{s \in S} \overline{K}(s, \varepsilon)$**). We shall also use here the half ε-net notion – we require then only this second condition $B \subset \bigcup_{s \in S} \overline{K}(s, \varepsilon)$, and S can be not a subset of B.

**) sometimes $K(s, \varepsilon)$ are used, instead of $\overline{K}(s, \varepsilon)$...

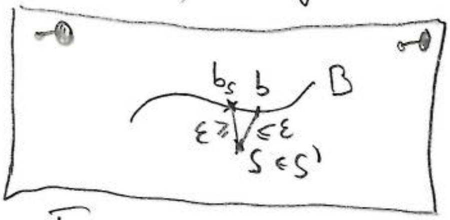
*) or conditionally compact; pol.: prezwarty, warunkowo zwarty. Note that this is a property of a subset of a fixed X, i.e., we can say also "precompact with respect to X" – see Remark and Fact pp. OTST 18–19...

• $B \subset X$, B is totally bounded *) iff for any $\varepsilon > 0$ there exists a finite ε-net for B.

The word "haf" in the "haf ε-net" can be associated with the convenient result below:

Lemma ("On 2ε-net")

If $S = \{s_1, \ldots, s_n\}$ is a half ε-net for B, then there exists some $\tilde{S} = \{\tilde{s}_1, \ldots, \tilde{s}_m\}$, with $m \le n$, being a 2ε-net for B.



Proof

Let $S' := \{s \in S : \exists_{b \in B}\ \rho(s, b) \le \varepsilon\}$. For any $s \in S'$ choose b_s, such that $b_s \in B$ and $\rho(s, b_s) \le \varepsilon$ and let $\tilde{S} := \{b_s : s \in S'\}$. So $\tilde{S} \subset B$. Let $b \in B$. S is a half ε-net for B, so we can choose $s \in S$ such, that $\rho(b, s) \le \varepsilon$ – thus $s \in S'$. Now $\rho(s, b_s) \le \varepsilon$, which gives $\rho(b, b_s) \le 2\varepsilon$ and $b_s \in \tilde{S}$. So $\tilde{S}$ is a 2ε-net and $\# \tilde{S} \le \# S' \le \# S = n$. $\blacksquare$

Fact ("On precompact sets", "the Hausdorff thm")

If P is precompact, then P is bounded.
P is precompact iff any sequence in P has a subsequence, which is convergent in X. Moreover, if X is a complete space, then P is precompact iff P is totally bounded.

This result should be well-known from topology, if not: $\longrightarrow$ $\triangle$.

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*) pol.: "całkowicie ograniczony".

Hint (for the last part) "$\Longrightarrow$": consider the family of balls $\{K(x, \frac{\varepsilon}{2})\}_{x\in P}$. "$\Longleftarrow$" For any sequence $\{x_n\}_{n\geq 1}$ in P construct (using appropriate ε-nets...) recursively a Cauchy subsequence.

It will be sometimes convenient to use the above result together with the following corollary related to the lemma "on 2ε-net")

Corollary

By lemma "on 2ε-net" $B\subset X$ is totally bounded iff for any $\varepsilon>0$ there exists a finite half ε-net for B in X.

The main topological notion used to define compact operators is not the notion of compact set (space), but of the precompact set. It is somewhat delicate, since it is not as independent of the choice of the "superspace" $X'\subset X$ as it is for compact sets. But it is slightly independent...

Remark ("On independence of precompactness")

Let $X\supset X'\supset P$. If X' is a closed subset of X, then P is precompact as a subset of the space X iff
P — " — — " — — " — X'.

Proof

$\overline{P}^X\subset X'$ so $\overline{P}^X = X'\cap\overline{P}^X$ - hence $\overline{P}^X$ is X'-closed and $P\subset\overline{P}^X$. Thus $\overline{P}^{X'}\subset\overline{P}^X$. But also $\overline{P}^{X'} = X'\cap F$ for some X-closed F and X' is X-closed, so $\overline{P}^{X'}$ is X-closed and $P\subset\overline{P}^{X'}$. Hence $\overline{P}^X\subset\overline{P}^{X'}$, therefore $\overline{P}^X = \overline{P}^{X'}$. ▥

The above remarks means, that if we study the precompactness <u>only</u> with respect to the <u>closed</u> subspaces of X, the choice of the subspace is not important. But we have also:

Fact ("On extention of precompactness")

Let $X \supset X' \supset P$. If P is precompact with respect to X', then P is also precompact with respect to X.

Proof

If $\overline{P}^{X'}$ is compact, then this is compact in a "universal" sense (w.r. to both X and X') — hence it is a closed set in the X-sense, too. Thus $\overline{P}^{X'} = \overline{P}^{X}$. ☐

♦ Compact operator

Possibly "precompact" would be a better name than "compact" in the context of the definition below...

Definition

Let X, Y - be normed spaces and $C \in \mathcal{L}(X,Y)$. Then C is compact iff $C(\overline{K}(0,1))$ is precompact in Y.

The class of all compact operators from X into Y is denoted by

$$\mathcal{C}(X,Y); \qquad \mathcal{C}(X) := \mathcal{C}(X,X).$$

Of course, since any precompact set is bounded, $\mathcal{C}(X,Y) \subset B(X,Y)$. Moreover (see below) the choice of <u>closed</u> ball and of the radius 1 is not very important.

Fact

If X, Y are normed spaces $(r>0)$ and $C \in \mathcal{L}(X,Y)$, then TFCAE:

(i) C is compact;

(ii) $C(\overline{K}(0,r))$ is precompact in Y;

(iii) $C(K(0,r))$ is precompact in Y;

(iv) $C(B)$ is precompact for any bounded $B \subset X$,

(v) for any $\{x_n\}_{n\geq 1}$ in X such that $\forall_{n\geq 1}\ \|x_n\| \leq r$ there exits a subsequence of $\{Ax_n\}_{n\geq 1}$ which is convergent.

Proof

(i) $\Rightarrow$ (ii) is obvious (because $X \ni x \longmapsto rx \in X$ is a homeomorphism of X).

(ii) $\Rightarrow$ (iv) if B - bounded, then $B \subset \overline{K}(0,r)$ for some $r>0$, hence $\overline{C(B)} \subset \overline{C(\overline{K}(0,r))}$ in Y, so $\overline{C(B)}$ is compact, being a closed subset of a compact set.

(iv) $\Rightarrow$ (iii), (ii), (i) is obvious.

(iii) $\Rightarrow$ (iv) — the proof is like "(ii) $\Rightarrow$ (iv)".

So any two from (i) — (iv) are equivalent. And (iv) $\Leftrightarrow$ (v) is just the "second sentence" of fact "on precompact sets" (p. OTST-17).

$\boxed{\text{Q}}$

2.2. The class $\mathcal{C}(X,Y)$

We prove here several "algebraic" and "topological" properties of the class $\mathcal{C}(X,Y)$ (see (i), (iii) below) as a subset of $B(X,Y)$.

We also prove an important property of each "individual" compact operator (– see (iv) below).

Theorem ("On properties of $\mathcal{C}(X,Y)$")

Let X, Y, Z be Banach spaces. Then:

(i) $\mathcal{C}(X,Y)$ is a closed linear subspace of $B(X,Y)$ (i.e., $\mathcal{C}(X,Y)$ is a Banach space with the operator norm).

(ii) If $A \in B(X,Y)$ and $\dim A(X) < +\infty$, then $A \in \mathcal{C}(X,Y)$.

(iii) If $A_1 \in B(Y,Z)$, $A_2 \in B(X,Y)$ and A_1 or A_2 is compact, then $A_1 A_2$ is compact.

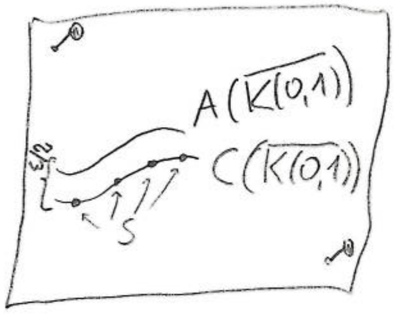
(iv) If $A \in \mathcal{C}(X,Y)$ and $\dim A(X) = +\infty$, then $A(X)$ is not closed.

Proof

(i) Suppose that $A, B \in \mathcal{C}(X,Y)$, $\lambda_1, \lambda_2 \in \mathbb{K}$ and let $\{x_n\}_{n\geq 1}$ be a bounded sequence in X, hence $\{Ax_n\}_{n\geq 1}$ has such a subsequence $\{Ax_{k_n}\}_{n\geq 1}$, that $Ax_{k_n} \to y \in Y$, $k_n \nearrow +\infty$. Now, similarly $\{Bx_{k_n}\}_{n\geq 1}$ possesses a subsequence $\{Bx_{k_{\ell_n}}\}_{n\geq 1}$ such, that $Bx_{k_{\ell_n}} \to z \in Y$, $\ell_n \nearrow +\infty$. Denoting $m_n := k_{\ell_n}$, we see that $Ax_{m_n} \to y$ and $Bx_{m_n} \to z$, so $(\lambda_1 A + \lambda_2 B)x_{m_n} \to \lambda_1 y + \lambda_2 z$, $m_n \nearrow +\infty$. So, $\lambda_1 A + \lambda_2 B \in \mathcal{C}(X,Y)$, i.e., $\mathcal{C}(X,Y) \underset{lin}{\subset} B(X,Y)$. We shall prove that $\mathcal{C}(X,Y)$ is closed in $B(X,Y)$.

Let $A \in \overline{\mathcal{C}(X,Y)}$ - it suffices to prove, that $A \in \mathcal{C}(X,Y)$.

For $\varepsilon > 0$ choose $C \in \mathcal{C}(X,Y)$ with $\|A - C\| < \varepsilon/2$ and let S be a finite $\varepsilon/2$-net for $C(\overline{K(0,1)})$.*) If $y \in A(\overline{K}(0,1))$ then $y = Ax$ for some $x \in \overline{K}(0,1)$,



so $\|y - Cx\| = \|Ax - Cx\| \leq \varepsilon/2 \|x\| \leqslant \varepsilon/2$.

Let $s \in S$ be such that $\|Cx - s\| \leq \varepsilon/2$;

then $\|y - s\| \leq \|y - Cx\| + \|Cx - s\| \leq \varepsilon$.

We have proved that for any $y \in A(\overline{K}(0,1))$ there exists $s \in S$ such, that $\|y - s\| \leq \varepsilon$. So, S is a half ε-net for $A(\overline{K}(0,1))$ and $A(\overline{K}(0,1))$ is precompact, by Coroll. p. OTST-18 and Fact p. OTST-17, i.e. $A \in \mathcal{C}(X,Y)$.

(ii) - is almost obvious: If $\dim A(X) < +\infty$ and $A \in B(X,Y)$, then $D := A(\overline{K}(0,1))$ is a bounded subset of a closed linear subspace $Y' := A(X) \subset Y$ (see facts p. PB-19 and PB-30). So, by remark "on independance..." p. OTST-18 it suffices to check, that D is precompact w.r. to Y'. But D is bounded and $\dim Y'$ is finite - thus this is true, by Theorem "on compactness of the ball" p. PB-11.

Of course, we could argue also using a bounded sequence in X, as in (i) (and the proof would be slightly shorter...).

(iii) Let $\{x_n\}_{n\geqslant 1}$ be bounded in X. If $A_2 \in B(X,Y)$ and A_1 is compact, then $\{A_2 x_n\}_{n\geqslant 1}$ is bounded. Thus convergent subsequence can be choosen for $\{A_1(A_2 x_n)\}_{n\geqslant 1}$, so $A_1 A_2$ is compact. If $A_1 \in B(X,Y)$ and A_2 is compact then choose a subsequence

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*) We use that it is totally bounded, by Fact p. OTST-17.

$\{A_2 x_{k_n}\}_{n\geq 1}$ of $\{A_2 x_n\}_{n\geq 1}$, $k_n \nearrow +\infty$, such that $A_2 x_{k_n} \longrightarrow y \in Y$. Then $A_1(A_2 x_{k_n}) \longrightarrow A_1 y$, so $\{(A_1 A_2) x_{k_n}\}_{n\geq 1}$ is a convergent subsequence of $\{(A_1 A_2) x_n\}_{n\geq 1}$, and $A_1 A_2$ is compact.

(iv) Suppose that A is compact and $\tilde{Y} := A(X)$ is infinite dim. closed subspace of Y. Denote $\tilde{X} := X/\mathrm{Ker}A$. There exists a "factorisation" $\tilde{A} \in B(\tilde{X}, \tilde{Y})$, given by $\tilde{A}([x]) := Ax$ for any $x \in X$ (see fact "on factorization" p. OF-19). Moreover, now $\tilde{A}$ is a bijection!

$$\begin{tikzcd} X \arrow[r, "A"] \arrow[d, "\pi"] & \tilde{Y} \subset Y \\ \tilde{X} = X/_{\mathrm{Ker}A} \arrow[ur, "\tilde{A}"'] & \end{tikzcd}$$

We know, that $\tilde{X}$ is a Banach space, and $\tilde{Y}$ - too (as a closed subspace of Y). Thus, by the Inverse Mapping Thm., $\tilde{A}$ is an isomorphism of $\tilde{X}$ onto $\tilde{Y}$. But $A = \tilde{A} \circ \pi$, so

$$A(K_X(0,1)) = \tilde{A}(\pi(K_X(0,1)))$$ — recall, that $\pi(K_X(0,1)) = K_{\tilde{X}}(0,1)$ (see thm. "On quotient space" p. PB-52). Hence

$$P := A(K_X(0,1)) = \tilde{A}(K_{\tilde{X}}(0,1)).$$ Thus $K_{\tilde{X}}(0,1) = \tilde{A}^{-1}(P)$, and P is precompact.*) So $K_{\tilde{X}}(0,1)$ also is, since $\tilde{A}^{-1}$ is, in particular, a homeomorphism of the "whole" $\tilde{Y}$ onto $\tilde{X}$. Therefore $\overline{K_{\tilde{X}}(0,1)} = \overline{K}_{\tilde{X}}(0,1)$ **) is compact.

But the space $\tilde{X}$ is infinite-dimensional, because $\tilde{A}$ is also a linear isomorphism (and $\dim \tilde{X} = +\infty$) — this gives a contradiction, since $\overline{K}(x,r)$ $(r>0)$ is never compact in infinite-dimensional spaces (see thm. "On non-compactness of the ball", p. PB-16).

*) by Fact from p. OTST-20.

**) Recall that $\overline{K(x,r)} = \overline{K}(x,r)$ for any $r>0$ in normed spaces (however not in any metric space...).

[4]

Remarks

1. In some parts of the theorem the assumptions on the completness ("Banach spaces...") of some of X, Y, Z can be omited, e.g., using the fact "on extension of precompactness" (p. OTST-19) to the appropriate completions of the spaces. (Try to find such parts ... → △.)

2. The parts (i) and (iii) together, means that for $X=Y$ we obtain $\mathcal{C}(X)$ being not only a closed subspace of $B(X)$, but also a closed subalgebra – and even more: a closed two-sided ideal!

3. The closedness from (i) with (ii) give a convenient method of proving, that a particular bounded operator A is compact:
it suffices to find a sequence of finite-dimensional (bounded) operators A_n, approximating A in norm (i.e., $A_n \to A$ in $B(X,Y)$). Sometimes, finding such A_n-s consists on "modifying" (e.g. "cutting" etc...) A in a special way...

4. The part (iv) says, that the range of any compact operator <u>cannot</u> be at the same time closed and infinite-dimensional! So, (all) the closed range cases are described <u>exactly</u> by the part (ii)...

2.3. Some simple examples of compact operators

We show here only some simple examples, however you can encounter compact operators in many applications of operator theory to some other parts of analysis (– e.g. differential equations)

$\boxed{\text{Example 1}}$ ("Multiplication by some sequences")

Consider the operator M_a of multiplication by a sequence $a \in \ell^\infty(\mathbb{N})$ in $X = \ell^p(\mathbb{N})$ for $p \in [1; +\infty]$ or in $X = c$ or c_0 (see pp. OF-22-23), i.e., $M_a f = a \cdot f$, for $f \in X$.
Using the method described in Remark 3, (p. OTST-24) one can easily prove that

$$M_a \in \mathcal{C}(X), \text{ if } a \in c_0.$$

($\longrightarrow$ △).

$\boxed{\text{Example 2}}$ ("The embedding of C^1 in C")

Consider the space $X := C^1([a;b])$ for $a < b$, $a, b \in \mathbb{R}$ and two operators from X to $C([a;b])$ (see p. OF-29) given by

$$I_1 f := f, \quad Df := f', \qquad f \in X$$

By the use of Arzela-Ascoli thm. (being in fact an "iff" criterion for precompactness i $C(K)$ for K-compact metric space) one can prove ($\longrightarrow$ △) that I_1 is compact, but D is not.

$\boxed{\text{Example 3}}$ ("The Volterra operator in $C([a;b])$")

Similarly, using the Arzela-Ascoli thm., one can prove ($\longrightarrow$) that the Volterra operator $V \in B(C([a;b]))$, which is given by $(Vf)(t) := \int_a^t f(s)ds$ ($f \in C([a;b])$, $t \in [a;b]$), is compact (see p. OF-30).

$\boxed{\text{Example 4}}$ ("Hilbert-Schmidt operators in L^2")

Let $(\Omega, \mathfrak{M}, \mu)$ be such a measure space, that $X := L^2(\Omega, \mu)$ is a (nontrivial) separable space (e.g., μ - a Borel σ-finite measure on $\mathbb{R}$), and let $k \in \tilde{L}^2(\Omega \times \Omega, \mu \times \mu)$ *). Consider the integral operator in X with the kernel k given by the formula

$$K[f] := [\tilde{K} f], \quad \text{where } \tilde{K} : \tilde{L}^2 \to \tilde{L}^2 \text{ **) is given by}$$

$$(\tilde{K} f)(t) := \int_\Omega k(t,s) f(s) d\mu(s)$$

for μ-a.e. $t \in \Omega$ (for such t, that $s \mapsto k(t,s)$ is an $\tilde{L}^2(\Omega, \mu)$ function; by Fubini's thm the set of such t has "full-μ" measure)

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*) $\mu \times \mu$ is the product measure of μ and μ.

**) $\tilde{L}^2 := \tilde{L}^2(\Omega, \mu)$, here, and not $\tilde{L}^2(\Omega \times \Omega \ldots)$.

Using the Hölder ineq. and Fubini's thm. one easily check ($\longrightarrow \triangle$) that $\tilde{K}$ is half-norm bounded (see p. OF-17) – hence K is bounded (see Fact "on bdd operator from a semi-bdd one", p. OF-17), moreover $\|K\| \leq |||k|||_2$ (i.e. $|||\ |||$ in $\tilde{L}^2(\Omega\times\Omega, \mu\times\mu)$). It can be proved, that K is compact. The method is the one described in Rem. 3. p. OTST-24, but more precisely:

(i) We choose an orthonormal base $\{[\varphi_n]\}_{n\in\mathcal{A}}$ for X, where $\mathcal{A} = \mathbb{N}$ if X is not a finite-dimensional space and $\mathcal{A} = \{1, \dots N\}$ if $\dim X = N < +\infty$ (recall that X is separable by our assumption ...), but $N < +\infty$ is trivial ... – see next example!

(ii) We prove that $\{[\varphi_n \otimes \overline{\varphi_m}]\}_{(n,m)\in\mathcal{A}\times\mathcal{A}}$ is an orthonormal base for $S := L^2(\Omega\times\Omega, \mu\times\mu)$, where the "multiplication" $\otimes$ is given by $(f\otimes g)(t_1, t_2) := f(t_1)\cdot g(t_2)$, $t_1, t_2 \in \Omega$. ($\longrightarrow \triangle$).

(iii) We define
$$k_n := \sum_{k_1,k_2=1,\dots n} \alpha_{k_1,k_2}(\varphi_{k_1}\otimes\overline{\varphi_{k_2}}),$$
where $\alpha_{k_1,k_2} := \left([k], [\varphi_{k_1}\otimes\varphi_{k_2}]\right)_S$ for $n \geq 1$ (here we study the case $\mathcal{A} = \mathbb{N}$ only) and K_n is the integral oper. in X with the kernel k_n (see the definition of K ...).

(iv) It suffices now to prove, that $K_n \to K$ in $B(X)$ ($\longrightarrow \triangle$) and that K_n is a finite-dimensional operator for each n ($\longrightarrow \triangle$).

$\boxed{\text{Example 5}}$ ("Linear algebra finite matrices")

For any finite-dim. normed spaces X, Y and $A \in \mathcal{L}(X,Y)$ we have obviously $A \in \mathcal{C}(X,Y)$ (A is bounded and $\dim(A(X)) \leqslant \dim Y < +\infty$). In particular any operator given by scalar rectangular matrix is a compact operator between the Euclidean spaces $\mathbb{K}^n$ and $\mathbb{K}^m$ for the appropriate n, m.

2.4. On $\mathcal{C}(X)$ operators

This part of the "Compact operators" section is devoted to compact operators acting on a space X into the same X. Our goal is to prove one "main" theorem describing some specific "perturbation" properties of such operators. However, the method of the proof forces us to start here from a digression...

♦ On complemented*) subspaces

If X is a normed space and $Y \underset{\text{lin}}{\subset} X$ is closed, then

*) pol.: complemented = dopełnialne

Y is complemented (in X) iff there exists such a closed $Z \underset{lin}{\subset} X$, that $X = Y \oplus Z$.

In general this is not true, that each closed Y is complemented*), however it can be always "algebraically complemented" (i.e, there exists just a linear subspace Z, such that $X = Y \oplus Z$, and if Y was not closed, it would be also true – we can just use the appropriate results on linear bases). Surely, each closed subspace in Hilbert space is complemented ($Y \oplus Y^{\perp} = X$). Also:

Fact ("On complementary**) to $\dim < \infty$ - subspace")

Each finite-dimensional subspace in a normed space is complemented.

Proof

Let Y be an d-dimensional linear subspace of X. Choose a base $\{y_1, \dots y_d\}$ for Y. Let $\tilde{\varphi}_n : Y \to \mathbb{K}$ be defined for $x \in Y$ by $\sum_{j=1}^{d} \tilde{\varphi}_j(x) y_j = x$, for $n = 1, \dots d$. This is a proper definition, by the definition of a base of a linear space, and each φ_j is linear, by the uniqueness of the choice of the scalar coefficients for the base-linear combination for x. Moreover, by linearity, each φ_j is continuous, since $\dim Y$ is finite.

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*) It was proved in 1937 that for any $p > 1, p \neq 2$ the space $\ell^p(\mathbb{N})$ possesses an uncomplemented subspace (closed) – [Murray]. In 1940 Phillips showed, that c_0 is uncompemented in $\ell^{\infty}(\mathbb{N})$...

**) Each closed subspace Z such that $X = Y \oplus Z$ is called complementary to Y (in X, for Y - closed).

So, using the Hahn-Banach thm. let us extend each $\tilde{\varphi}_n$ to some $\varphi_n \in X^*$. Define:

$$Z := \bigcap_{n=1}^{d} \operatorname{Ker} \varphi_n .$$

Z is a closed subspace, as a finite intersection of such subspaces (φ_n are continuous...) and if $x \in Y \cap Z$, then we have $Y \ni x = \sum_{n=1}^{d} \tilde{\varphi}_n(x) y_n = \sum_{n=1}^{d} \varphi_n(x) y_n = \sum_{n=1}^{d} 0 y_n = 0$, because $x \in Z$. Hence $Y \cap Z = \{0\}$. Moreover $Y + Z = X$, since if $x \in X$, then for any $n_0 \in \{1, \dots, d\}$ we have

$$\varphi_{n_0}\Big(x - \sum_{j=1}^{d} \varphi_j(x) y_j\Big) = \varphi_{n_0}(x) - \sum_{j=1}^{d} \varphi_j(x) \cdot \tilde{\varphi}_{n_0}(y_j) =$$

$$= \varphi_{n_0}(x) - \sum_{j \neq n_0} 0 - \varphi_{n_0}(x) \cdot 1 = 0,$$ which means that

$z := \big(x - \sum_{j=1}^{d} \varphi_j(x) y_j\big) \in Z$, but $y := \sum_{j=1}^{d} \varphi_j(x) y_j \in Y$, hence $z = x - y$, i.e., $x = y + z \in Y + Z$. So, $X = Y \oplus Z$. $\boxed{\text{tu}}$

$\boxed{\text{Remark}}$ (for the proof)

Observe that defining above $Px := \sum_{j=1}^{d} \varphi_j(x) y_j$, we get $P \in B(X)$, for $y \in Y$ $Py = \sum_{j=1}^{d} \tilde{\varphi}_j(y) y_j = y$, and $\operatorname{Ran} P \subset Y$. Thus $P^2 = P$ and $\operatorname{Ran} P = Y$, moreover $Z = \operatorname{Ker} P$. I.e., P is the continous projection uniquely related to the decomposition $X = Y \oplus Z$ by 3. from fact "On linear projection" p. HS-28.

The main result

Theorem ("On compact perturbation of the identity")

Let X be a Banach space and $K \in \mathcal{C}(X)$. Then:

1) $\dim \operatorname{Ker}(I+K) < +\infty$.

2) $\operatorname{Ran}(I+K)$ is closed.

3) $\dim \operatorname{Ker}(I+K) = \operatorname{codim} \operatorname{Ran}(I+K)$. *)

Proof – of 1) and 2) only! – We omit the proof of 3)!

Denote $X_0 := \operatorname{Ker}(I+K)$ – it is a Banach subspace of X – and let $K_0 := K|_{X_0}$. So, $K_0 \in B(X_0, X)$. We have $\overline{K}_{X_0}(0,1) \subset \overline{K}_X(0,1)$, so K_0 is compact (any subset of a precompact set in X is a precompact set in X). But we have:
$K_0 x = (I+K_0)x - x = (I+K)x - x = -x$ for $x \in X_0$, so $K_0 = -I|_{X_0}$. Thus $\operatorname{Ran} K_0 = -X_0 = X_0$, which means, that K_0 is a compact operator with $\operatorname{Ran} K_0$ being closed, hence $\operatorname{Ran} K_0 = X_0$ is finite-dimensional, by thm. "On properties of $\mathcal{C}(X,Y)$" (p. OTST-21). – We have proved 1).

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*) Recall (p. OF-54) that $\operatorname{codim} Y := \dim X/Y$ for $Y \underset{lin}{\subset} X$ and X – a linear space.

Now, using the fact "on complementary to $\dim < +\infty$ - subspace" (p. OTST-) we choose a closed subspace X_1 such that $X = X_0 \oplus X_1$. Denote now $S := I + K$ and $S_i := S_{|X_i}$ for $i = 0, 1$. But we have $S_0 = 0$, because $S_0 = S_{|X_0} = I_{|X_0} + K_0 = I_{|X_0} - I_{|X_0} = 0$. Hence for any $x \in X$ we have $x = x_0 + x_1$ for some $x_0 \in X_0$, $x_1 \in X_1$ and $Sx = S_0 x_0 + S_1 x_1 =$

$= S_1 x_1$. Thus $\mathrm{Ran}\, S_1 = \mathrm{Ran}\, S$, and it suffices to show that $\mathrm{Ran}\, S_1$ is closed, to obtain 2). We shall prove that there exsists such a $\delta > 0$, that

$$\forall_{x \in X_1} \quad \delta \|x\| \leq \|S_1 x\|. \qquad (1)$$

Suppose that this is not true; then we can construct a sequene $\{\tilde{x}_n\}_{n \geq 1}$ in X_1 (by taking "$\delta_n := \frac{1}{n}$") satisfying:

$$\frac{1}{n}\|\tilde{x}_n\| > \|S_1 \tilde{x}_n\|, \quad n \geq 1 \qquad (2)$$

in particular $\|\tilde{x}_n\| \neq 0$ (because of "$>$" above), so let $x_n := \frac{\tilde{x}_n}{\|\tilde{x}_n\|}$, then $\|x_n\| = 1$ for any $n \geq 1$, and by (2)

$$0 \leq \|S_1 x_n\| < \frac{1}{n}, \qquad n \geq 1.$$

We have $S_1 x_n \longrightarrow 0$, and we choose now $k_n \nearrow +\infty$ with $K x_{k_n} \longrightarrow y \in X$, because K is compact. Denote $x_n' := x_{k_n}$. So we have:

$$\forall_{n \geq 1} \quad x_n' \in X_1, \; \|x_n'\| = 1 \text{ and } K x_n' \longrightarrow y, \; S_1 x_n' \longrightarrow 0.$$

Therefore $x_n' = (I+K)x_n' - Kx_n' = S_1 x_n' - K x_n' \longrightarrow 0 - y = -y \in X_1$ (because X_1 is closed), and $S_1 x_n' \longrightarrow S_1(-y)$. But also $S_1 x_n' \longrightarrow 0$, so $-S_1 y = 0$, i.e. $y \in \operatorname{Ker} S_1 \subset \operatorname{Ker} S = \operatorname{Ker}(I+K) = X_0$. This means that $y \in X_1 \cap X_0 = \{0\}$, i.e $y = 0$, which is a contradiction, because $1 = \|x_n'\| \to \|y\| = 0$. We have proved (1) for some $\delta > 0$. It follows that $\operatorname{Ran} S_1$ is a complete space – to prove this, take $\{y_n\}_{n \geq 1}$ – a Cauchy seq. in $\operatorname{Ran} S_1$. Then for some $\{u_n\}_{n \geq 1}$ in X_1 we have $\forall_{n \geq 1} S_1 u_n = y_n$, so $\{u_n\}_{n \geq 1}$ is also Cauchy seq., by (1). Thus $u_n \to u \in X_1$, because X_1 is a Banach space ($X_1 = \overline{X_1}$), and now $y_n = S_1 u_n \to S_1 u$ (S_1 is continuous, because $S_1 = S_{|X_1}$ and S – continuous), so $\{y_n\}_{n \geq 1}$ is convergent in $\operatorname{Ran} S_1$. But Banach subspace of a Banach space is closed. $\boxed{\text{u}}$

$\boxed{\text{Remark}}$

Observe that in the case of "the simplest" compact K, i.e, when $K = 0$, the assertion of the theorem is trivial: $\dim \operatorname{Ker} I = \dim \{0\} = 0 < +\infty$, $\operatorname{Ran}(I) = X$ – a closed subspace of X and $\dim \operatorname{Ker} I = 0 = \operatorname{codim} \operatorname{Ran} I = \operatorname{codim} X = \dim X/X = \dim \{0\}$. When we perturb I by any compact K, instead of 0, the situation is "partially" preserved.