IV Linear Functionals

Subsections

1. The Hahn-Banach Theorem(s)

We shall ask here several questions on existence of some linear functionals, that satisfy some special conditions. One example of such a question for a normed space $X \neq \{0\}$ is:

Does there exist a non-zero continuous linear functional on X?

The above is "equivalent" to: Is $X^* \neq \{0\}$ for $X \neq \{0\}$?

– The answer is YES! – But the proof is far from being trivial... To get a convenient and quite general tool to prove many results like this – on existance of some special functionals – we shall prove first an abstract result, coming from Hahn and Banach.

1.1. Banach functionals and The Abstract Hahn-Banach Theorem in $\mathbb{R}$-spaces

The results which we obtain here will be applicable for both $\mathbb{R}$- and $\mathbb{C}$-linear (normed) spaces. But here we shall need only the "$\mathbb{R}$-part" of the linear structure. So, we assume here that X is $\mathbb{R}$-linear.

Note, that it is not a restriction in fact, because each $\mathbb{C}$-linear space is also a $\mathbb{R}$-space, in particular!

We shall need not only a space X, but also an "abstract extra object" for X, called Banach functional *) abbreviated here to B. f.

♦ Banach functionals

Definition

A function $p: X \longrightarrow \mathbb{R}$ is a Banach functional iff

(i) (subadditivity) $\forall_{x,y \in X} \quad p(x+y) \leq p(x) + p(y)$,

(ii) (semi-homogenity**) $\forall_{\substack{x \in X \\ t \geq 0}} \quad p(tx) = t\,p(x)$.

Observe, that $p(0) = 0$ for any B.f. p (since $p(0) = p(0 \cdot 0) = 0 \cdot p(0) = 0$.), but it is generally not true, that $p \geq 0$ (see Examples 1., 2. below)

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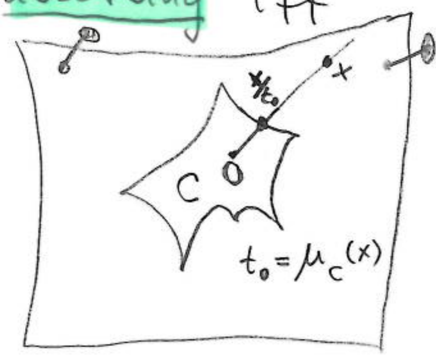
*) Note, that the word "functional" means here only that it is a scalar valued function, but the linearity is not assumed. In particular it is NOT this functional, we are looking for in our questions... . We can treat a Banach functional as something like "a norm", giving a kind of an extra "structure" for X.

**) Pol.: półjednorodność, semijednorodność

Examples

1. Let $\varphi \in X^{\#_{\mathbb{R}}}$, where $X^{\#_{\mathbb{R}}}$ denotes the set (and $\mathbb{R}$-space) of all $\mathbb{R}$-linear functionals $\varphi : X \longrightarrow \mathbb{R}$ (i.e. φ - additive, and $\mathbb{R}$-homogeneous: $\forall_{\substack{t\in\mathbb{R}\\ x\in X}} \ \varphi(tx) = t\varphi(x)$). Then (obviously...) φ is a B.f.
2. In $\ell^\infty(\mathbb{N})$ (the real seq. case) "sup" and "limsup" are B.f-s!
3. Let $\varphi \in X^{\#_{\mathbb{R}}}$ or $\varphi \in X^{\#}$, then $|\varphi|$ is (also obviously...) a B.f.
4. Let $|||\cdot|||$ be a seminorm in X, then it is a B.f.
5. Let $C \subset X$. We call C absorbing iff

$$\forall_{x\in X} \ \exists_{t>0} \ tx \in A.$$



(in particular $0 \in X$ then). If C is absorbing, then denote by μ_C the function $\mu_C : X \longrightarrow \mathbb{R}$ given by

$$\mu_C(x) := \inf\left\{t>0 : \frac{x}{t} \in C\right\}, \quad x \in X.$$

Note, that $C_x := \left\{t>0 : \frac{x}{t} \in C\right\}$ is nonempty (by "absorbity") and bounded from below by 0, so "inf" has sense.

It can be easily proved that for any $x \in X$ also

$$\mu_C(x) := \left(\sup \tilde{C}_x\right)^{-1}, \text{ where } \tilde{C}_x := \{t > 0 : tx \in C\}$$

(and we set $(+\infty)^{-1} := 0$) $\longrightarrow \triangle$.

The function μ_C is called the Minkowski functional of C (for C - absorbing).

$\boxed{\text{Fact}}$ ("On the Minkowski functional")

If C is absorbing and convex subset of X, then:

(1) μ_C is a B.f.

(2) if $C' := \{x \in X : \mu_C(x) < 1\}$ and $\tilde{C} := \{x \in X : \mu_C(x) \leq 1\}$ then $C', \tilde{C}$ are absorbing, $C' \subset C \subset \tilde{C}$ and $\mu_{C'} = \mu_C = \mu_{\tilde{C}}$.

(3) if C is also balanced, i.e $\forall_{\lambda \in \mathbb{K}} (|\lambda| \leq 1 \Rightarrow \lambda C \subset C)$ *), then μ_C is a seminorm.

$\boxed{\text{Proof}}$ $\longrightarrow \triangle$.

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*) This name works both in $\mathbb{K} = \mathbb{R}$ and $\mathbb{C}$ cases (the $\mathbb{C}$ case is "more restrictive"...). If X is $\mathbb{C}$-linear then we can consider it in two different meanings (for $\mathbb{C}$ and for $\mathbb{R}$) in fact...

Dominating by B.f. and 1-dimensional extensions

As we shall see soon, several natural properties of linear functionals can be expressed in terms of domination by an appropriately chosen (for the property) B. f.

Definition

Let $\varphi \in X^{\#_{\mathbb{R}}}$ and let p be a B. f. on X. Then φ is dominated by p iff $\varphi \leq p$. *) We use also the same name (φ is d. by p) for $\varphi \in Y^{\#_{\mathbb{R}}}$, where $Y \underset{\mathrm{lin}_{\mathbb{R}}}{\subset} X$, **) and it means $\varphi \leq p_{|Y}$.

Observation

If $\varphi \leq p$, then for any $x \in X$ $\quad -p(-x) \leq \varphi(x) \leq p(x)$.

Proof $\quad \varphi(x) = -\varphi(-x) \geq -p(-x)$ ∎

Note, that for some B.f. p the identity $p(-x) = p(x)$ holds for any $x \in X$ (but not for each p – see e.g. Example 1 ...). In such a case the above result means exactly that $|\varphi(x)| \leq p(x)$ for any $x \in X$.

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*) But, it is shorter to write "$\leq$" than "is dominated by", so we shall use "$\leq$" in place of it often...

**) $Y \underset{\mathrm{lin}_{\mathbb{R}}}{\subset} X$: Y is a $\mathbb{R}$-linear subspace of X. Also $\mathrm{lin}_{\mathbb{R}}(M)$ denotes the $\mathbb{R}$-linear subspace spanned by M.

We shall prove first, that the dominating which is not "on the whole space X" can be always "slightly" extended...

Lemma ("On 1-dim. dominated extension")

Suppose that $Y \underset{\mathrm{lin}_{\mathbb{R}}}{\subset} X$, p is a B.f. on X, $\varphi \in Y^{\#_{\mathbb{R}}}$ and $\varphi \leq p_{|Y}$. If $x_0 \in X \setminus Y$, then there exits $\tilde{\varphi} \in \tilde{Y}^{\#_{\mathbb{R}}}$, where $\tilde{Y} := \mathrm{lin}_{\mathbb{R}}(Y \cup \{x_0\})$, s.t. $\tilde{\varphi}_{|Y} = \varphi$ and $\tilde{\varphi} \leq p_{|\tilde{Y}}$.

Proof

Observe that $\tilde{Y} = \{y + t x_0 : t \in \mathbb{R}, y \in Y\}$. Hence each $\tilde{\varphi} \in \tilde{Y}^{\#_{\mathbb{R}}}$ extending φ has the (unique) form

$$\underset{t \in \mathbb{R}, y \in Y}{\forall} \tilde{\varphi}(y + t x_0) = \varphi(y) + t z, \qquad (1)$$

for some $z \in \mathbb{R}$ (with $z = \tilde{\varphi}(x_0)$), and conversly, for any $z \in \mathbb{R}$ (1) defines a linear extension of φ (since $x_0 \notin Y$). So, we are looking for such $z \in \mathbb{R}$ that $\tilde{\varphi}$ given by (1) is dominated by p. This condiolion is equivalent to $\forall_{t \in \mathbb{R}}$ (2_t), with:

$$\underset{y \in \mathbb{R}}{\forall} \varphi(y) + t z \leq p(y + t x_0) \qquad (2_t)$$

Observe, that (2_0) holds by $\varphi \le P|Y$.

If $t > 0$, then (2_t) is equivalent to

$$\forall_{y \in Y} \varphi(y/t) + z \le p(y/t + x_0)$$

and the above is equiv. to

$$z \le \inf_{y \in Y} \Big(p(y + x_0) - \varphi(y) \Big). \qquad (2_+)$$

If $t < 0$, then $t = -s$, $s > 0$, so (2_t) is equiv. to

$$\forall_{y \in Y} \varphi(y/s) - z \le p(y/s - x_0)$$

which is equiv. to

$$\sup_{y \in Y} \Big(\varphi(y) - p(y - x_0) \Big) \le z. \qquad (2_-)$$

Thus, to finish the proof (i.e. to find z satisfying (2_+) and (2_-)) it suffices to check that

$$\sup A \le \inf B, \qquad (3)$$

where $A := \{ \varphi(y) - p(y - x_0) : y \in Y \}$, $B := \{ p(y + x_0) - \varphi(y) : y \in Y \}$.

But (by Analysis I ...) (3) is equivalent to

$$A \le B^{*)} \qquad (3')$$

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*) i.e. $\forall_{\substack{a \in A \\ b \in B}} a \le b$. Note also, that $A, B \neq \emptyset$ here, because $0 \in Y$.

But for any $y_1, y_2 \in Y$

$$\varphi(y_1)+\varphi(y_2)=\varphi(y_1+y_2)\leq p(y_1+y_2)=$$
$$=p((y_1-x_0)+(x_0+y_2))\leq p(y_1-x_0)+p(y_2+x_0),$$

i.e.

$$\varphi(y_1)-p(y_1-x_0)\leq p(y_2+x_0)-\varphi(y_2)$$

which proves (3'). $\boxed{G}$

The extension onto the whole X – the Abstract Hahn-Banach Theorem.

Our goal now is to prove, that the extension of a $\varphi_0 \in Y^{\#_{\mathbb{R}}}$ which is dominated by p can be make even onto the whole space X (with the domination property). So assume here, that p is a B.f., $Y_0 \underset{\text{lin}_{\mathbb{R}}}{\subset} X$ and $\varphi_0 \in Y_0^{\#_{\mathbb{R}}}$ Denote by $Ex_p(\varphi_0)$ the set of all the $\mathbb{R}$-linear extensions of φ_0 to $\mathbb{R}$-linear subspaces of X which are dominated by p, i.e. $\varphi \in Ex_p(\varphi_0)$ iff $\varphi \in Y^{\#\mathbb{R}}$ for some $Y_0 \subset Y \subseteq_{lin_{\mathbb{R}}} X$, $\varphi_{|Y_0} = \varphi_0$ and $\varphi \leq p_{|Y}$.

Observe, that each $\varphi \in Ex_p(\varphi_0)$ is (as a function, which is equal to its graph, by the definition of function...) a subset of $X \times \mathbb{R}$. So $Ex_p(\varphi_0)$ is ordered by the usual inclusion relation $\subset$!

The relation $\varphi \subset \varphi'$ can be translated into the common function notation as follows:

$$\mathrm{Dom}\,\varphi \subset \mathrm{Dom}\,\varphi' \text{ and } \varphi'_{|\mathrm{Dom}\,\varphi} = \varphi,$$

where $\mathrm{Dom}\,\varphi$ denotes the domain of the function(al) φ.

Lemma ("On maximal dominated extension")

There exists a maximal element in $Ex_p(\varphi_0)$ (with respect to $\subset$).

Proof

Observe first that $Ex_p(\varphi_0) \neq \emptyset$, since $\varphi_0 \in Ex_p(\varphi_0)$. Thus, by the Kuratowski - Zorn Lemma, it suffices to prove, that for any chain-system $\{\varphi_\alpha\}_{\alpha \in A}$ in $Ex_p(\varphi_0)$ (i.e. $\{\varphi_\alpha : \alpha \in A\}$ is a chain $\Leftrightarrow$ it is linearly ordered subset of $Ex_p(\varphi_0)$) there exists its upper bound in $Ex_p(\varphi_0)$.

If $\mathcal{A} = \emptyset$, then φ_0 is such an upper bound. And if $\mathcal{A} \neq \emptyset$ then define $\tilde{\varphi} := \bigcup_{\alpha \in \mathcal{A}} \varphi_\alpha$. *) Note that a priori $\tilde{\varphi}$ is just a subset of $X \times \mathbb{R}$, and we shall check that it is an $\mathbb{R}$-linear functional and $\tilde{\varphi} \in Ex_p(\varphi_0)$. Let $(x, z) \in \tilde{\varphi}$ i.e. $x \in Dom(\varphi_\alpha)$ and $z = \varphi_\alpha(x)$ for some $\alpha \in \mathcal{A}$, moreover

$$\tilde{\varphi} = \{(x', z') \in X \times \mathbb{R} : \exists_{\alpha \in \mathcal{A}}\; x' \in Dom(\varphi_\alpha) \wedge \varphi_\alpha(x') = z'\};$$

in particular $x \in D := \bigcup_{\alpha \in \mathcal{A}} Dom(\varphi_\alpha)$. So, we easily prove now that $\tilde{\varphi}$ is a function from D into $\mathbb{R}$: if $x \in D$ then $x \in Dom\, \varphi_\alpha$ for some $\alpha \in \mathcal{A}$ and so $(x, \varphi_\alpha(x)) \in \tilde{\varphi}$, moreover if $(x, z_1), (x, z_2) \in \tilde{\varphi}$ then we can find a common $\alpha \in \mathcal{A}$ (by the linearity of the order) such that $(x, z_1), (x, z_2) \in \varphi_\alpha$ so $z_1 = \varphi_\alpha(x) = z_2$. We have proved that $D = Dom\, \tilde{\varphi}$ and $\tilde{\varphi} : D \longrightarrow \mathbb{R}$. Again, by linearity of the order, the $\mathbb{R}$-linearity of subspaces $Dom(\varphi_\alpha)$ and $\mathbb{R}$-linearity of φ_α we easily get that $\tilde{\varphi} \in D^{\#_{\mathbb{R}}}$ (in particular $D \subset_{lin_{\mathbb{R}}} X$) $\longrightarrow \triangle$ Obviously $\varphi_0 \subset \tilde{\varphi}$ since $\mathcal{A} \neq \emptyset$ and each $\varphi_\alpha \in Ex_p(\varphi_0)$, so in particular $\varphi_0 \subset \varphi_\alpha$. Hence $\tilde{\varphi}$ is a linear extension of φ. It remains to prove that $\tilde{\varphi} \leq p_{|D}$, but it is also obvious: for $x \in D$ $(x, \tilde{\varphi}(x)) \in \tilde{\varphi}$, so $(x, \tilde{\varphi}(x)) = (x, \varphi_\alpha(x))$ for some $\alpha \in \mathcal{A}$, so $\tilde{\varphi}(x) = \varphi_\alpha(x) \leq p(x)$. $\square$

*) The rest of the proof is really <u>standard</u>! I give almost all (the) details, but check it by yourself, please...

Now, having both last lemmas, we easily get our main result.

Theorem ("The Abstract Hahn-Banach Theorem")

If p is a B. f. on a $\mathbb{R}$-linear space X and φ is an $\mathbb{R}$-linear functional on a $\mathbb{R}$-linear subspace Y of X which is dominated by p, then there exists $\tilde{\varphi} \in X^{\#_{\mathbb{R}}}$ such that $\tilde{\varphi}_{|Y} = \varphi$ and $\tilde{\varphi} \leq p$.

Proof

Let $\tilde{\varphi}$ be a maximal element in $Ex_p(\varphi)$, which exits by Lemma "On max. dom. ext." By Lemma "On 1-dim. dom. ext." $\operatorname{Dom} \tilde{\varphi} = X$ (if not, we could extend it to a strictly larger domain ...). So $\tilde{\varphi}$ satisfies the condition of the theorem. $\square$

1.2. The Continuous Hahn-Banach Theorem and some special continuous functionals

We shall use now the abstract tool from the subsection 1.1 to prove some "more concrete" results on continuous linear functionals in normed spaces.

Real and complex linear functionals in complex linear spaces

To get easily the proper results also for $\mathbb{K} = \mathbb{C}$ we need a result which "join" real and complex linear functionals in the case of X being complex space. Recall, that in such a case we distinguish:

$X^{\#}$ – the space of all linear ($= \mathbb{C}$ – linear) functionals on X and

$X^{\#_{\mathbb{R}}}$ – the space of all $\mathbb{R}$-linear functionals on X.

We shall use also the analog notation for the subspaces of continuous functionals:

X^{*} (– for cont. $\mathbb{C}$-linear) and $X^{*_{\mathbb{R}}}$ (– for cont. $\mathbb{R}$-lin.).

Warning! One coud say: "it is easier to be $\mathbb{R}$-linear than to be $\mathbb{C}$-lin., so $X^{\#} \subset X^{\#_{\mathbb{R}}}$ and $X^{*} \subset X^{*_{\mathbb{R}}}$ ..." – but this is NOT true ...!!! *)

*) Since $\varphi \in X^{\#} \Rightarrow \varphi : X \to \mathbb{C}$, and not $X \to \mathbb{R}$, if $\varphi \neq 0$ ...

Suppose now, that X is a $\mathbb{C}$-linear space and $\varphi \in X^{\#}$. Let $\varphi_1 := \mathrm{Re}\,\varphi$, $\varphi_2 := \mathrm{Im}\,\varphi$. Then for any $x \in X$ we have:

$$\begin{array}{l} \varphi(ix) = \varphi_1(ix) + i\,\varphi_2(ix) \\ \quad \| \\ i\varphi(x) = i\big(\varphi_1(x) + i\varphi_2(x)\big) = -\varphi_2(x) + i\,\varphi_1(x). \end{array}$$

So (using $\varphi_1(\ldots), \varphi_2(\ldots) \in \mathbb{R}$) we get

$$\forall_{x \in X} \quad \varphi_2(x) = -\varphi_1(ix) \qquad (1)$$

and

$$\forall_{x \in X} \quad \varphi_1(x) = \varphi_2(ix), \qquad (1')$$

but (1') follows from (1), since by (1) $\varphi_2(ix) = -\varphi_1(i\cdot ix) = -\varphi_1(-x) = \varphi_1(x)$. Note, that the last "=" follows from the fact that

$$\mathrm{Re}\,\varphi,\ \mathrm{Im}\,\varphi \in X^{\#_{\mathbb{R}}}$$

(since $\mathrm{Re}, \mathrm{Im} : \mathbb{C} \to \mathbb{R}$ are $\mathbb{R}$-linear ...).

It turns out that (1) can be "inverted".

Fact / **Definition** ("On complex complement of $X^{\#_{\mathbb{R}}}$")

Suppose that X is a $\mathbb{C}$-linear space. Then:

(i) If $\psi \in X^{\#}$, then $\left(\begin{array}{l}\operatorname{Re}\psi, \operatorname{Im}\psi \in X^{\#_{\mathbb{R}}} \text{ and} \\ \forall_{x\in X}\ (\operatorname{Im}\psi)(x) = -(\operatorname{Re}\psi)(ix);\end{array}\right.$

(ii) If $\varphi \in X^{\#_{\mathbb{R}}}$, then there exists a unique $\psi \in X^{\#}$ such that $\varphi = \operatorname{Re}\psi$ — we call it the complex complement of φ and denote by $\varphi_{\mathbb{C}}$. Moreover, if $\varphi \in X^{\#_{\mathbb{R}}}$, then

$$\forall_{x\in X} \quad \varphi_{\mathbb{C}}(x) = \varphi(x) - i\,\varphi(ix). \qquad (2)$$

(iii) Suppose that $\varphi \in X^{\#_{\mathbb{R}}}$, and X is a complex normed space.

Then $\varphi \in X^{*_{\mathbb{R}}} \iff \varphi_{\mathbb{C}} \in X^{*}$,

moreover if $\varphi \in X^{*_{\mathbb{R}}}$, then $\|\varphi\| = \|\varphi_{\mathbb{C}}\|$. *)

(iv) If $\psi \in X^{\#}$ (and X is $\mathbb{C}$-normed, as in (iii)), then $(\psi \in X^{*} \iff \operatorname{Re}\psi \in X^{*_{\mathbb{R}}})$, moreover if $\psi \in X^{*}$, then $\|\operatorname{Re}\psi\| = \|\psi\|$. *)

$\boxed{\text{Proof}}$

(i) - was just proved on p. LF-14 (see (1)).

(ii) - For $\varphi \in X^{\#_{\mathbb{R}}}$ define ψ by the RHS of (2) — then $\varphi = \operatorname{Re}\psi$. We have to prove, that $\psi \in X^{\#}$ — it is a simple calculation (similar to this from p. LF-14) $\longrightarrow$ $\triangle$. Observe, that (i) proves that such $\psi \in X^{\#}$ that $\varphi = \operatorname{Re}\psi$ can be at most one.

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*) Here the LHS is in the sense of the real normed space $B(X, \mathbb{R})$ and the RHS — of the complex normed space $B(X, \mathbb{C})$.

(iii) If $\varphi \in X^{*_{\mathbb{R}}}$ then φ is continuous, and thus $\varphi_{\mathbb{C}}$ also, by (2) from (ii). And $\varphi_{\mathbb{C}} \in X^* \Rightarrow \varphi \in X^{*_{\mathbb{R}}}$, since $\varphi = \operatorname{Re} \varphi_{\mathbb{C}}$. Now, for $\varphi \in X^{*_{\mathbb{R}}}$ we have

$$\forall_{x \in X} \quad |\varphi(x)| = |(\operatorname{Re} \varphi_{\mathbb{C}})(x)| = |\operatorname{Re}(\varphi_{\mathbb{C}}(x))| \leq |\varphi_{\mathbb{C}}(x)| \leq$$
$$\leq \|\varphi_{\mathbb{C}}\| \cdot \|x\| \quad , \quad \text{so} \quad \|\varphi\| \leq \|\varphi_{\mathbb{C}}\|.$$

But note, that for any $z \in \mathbb{C}$ there exists $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that $|z| = \alpha \cdot z$ *), hence for any $x \in X$ we can choose such $\alpha_x \in \mathbb{C}$ with $|\alpha_x| = 1$, that

$$0 \leq |\varphi_{\mathbb{C}}(x)| = \alpha_x \varphi_{\mathbb{C}}(x) = \varphi_{\mathbb{C}}(\alpha_x \cdot x) = |\operatorname{Re}(\varphi_{\mathbb{C}}(\alpha_x \cdot x))|,$$

because $\varphi_{\mathbb{C}}(\alpha_x \cdot x) \in [0; +\infty)$. Thus:

$$|\varphi_{\mathbb{C}}(x)| = |(\operatorname{Re} \varphi_{\mathbb{C}})(\alpha_x \cdot x)| = |\varphi(\alpha_x \cdot x)| \leq \|\varphi\| \cdot \|\alpha_x \cdot x\| = \|\varphi\| \cdot \|x\|$$

— this gives $\|\varphi_{\mathbb{C}}\| \leq \|\varphi\|$, and finally $\|\varphi_{\mathbb{C}}\| = \|\varphi\|$.

(iv) If $\psi \in X^{\#}$, then by (i) and (ii) $\psi = (\operatorname{Re} \psi)_{\mathbb{C}}$, $(\operatorname{Re} \psi)_{\mathbb{C}} \in X^{\#_{\mathbb{R}}}$, so (iv) follows from (iii). ///

Exercise

Let $X = \mathbb{C}$ **) and consider $\varphi \in X^{*_{\mathbb{R}}}$ given by $\varphi(x+iy) := ax + by$ for any $x, y \in \mathbb{R}$, where a, b are fixed real numbers. Find the formula for $\varphi_{\mathbb{C}}(z)$ for $z \in \mathbb{C}$. — It should have

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*) Note, that the idea used below is realy deep and nontrivial... — try to prove that $\|\varphi_{\mathbb{C}}\| \leq C\|\varphi\|$ with $C = 1$ without the use of such a trick... Yes, it is easy to obtain $C = \sqrt{2}$... The previous parts of this proof were in fact easy to guess...

**) With $\|\cdot\| = |\cdot|$, as usual.

the form $\varphi_{\mathbb{C}}(z) = w_{a,b} \cdot z$, for some $w_{a,b} \in \mathbb{C}$ because $\varphi_{\mathbb{C}} \in X^* = \mathbb{C}^*$, so find $w_{a,b} \in \mathbb{C}$ in the explicit form...
Compute "manually" $\|\varphi\|$ and $\|\varphi_{\mathbb{C}}\|$ and check that they are equal. $\longrightarrow$ $\triangle$

The Continuous Hahn-Banach Theorem

Our first "more concrete" - continuous result concerns the extensions of continuous linear functionals. Here $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

$\boxed{\text{Theorem}}$ ("The Continuous Hahn-Banach Theorem")

Suppose that $Y \underset{lin}{\subseteq} X$, where X is a normed space and that $\varphi \in Y^*$. Then there exists $\tilde{\varphi} \in X^*$ such that $\tilde{\varphi}_{|Y} = \varphi$ and $\|\tilde{\varphi}\| = \|\varphi\|$.

$\boxed{\text{Proof}}$

Let $C := \|\varphi\|$ and consider $p : X \longrightarrow \mathbb{R}$ given by $p(x) = C\|x\|$, for $x \in X$. Obviously p is a B. f., because it is a seminorm (and a norm, if $C \neq 0$) – see Example 4 p. LF-4.

Let $\varphi_1 := \mathrm{Re}\,\varphi$ (in the $\mathbb{K}=\mathbb{R}$ case this means that $\varphi_1 = \varphi$). We have

$$\forall_{x \in X} \quad \varphi_1(x) \leqslant |\varphi(x)| \leqslant C\,\|x\| = p(x)$$

so φ_1 is dominated by p, and by The Abstract H.-B. Th. there exists $\tilde{\varphi}_1 \in X^{\#_{\mathbb{R}}}$ such that

$\tilde{\varphi}_1|_Y = \varphi_1$ and $\tilde{\varphi}_1 \leqslant p$. Moreover, for $x \in X$

$$\tilde{\varphi}_1(x) = -\tilde{\varphi}_1(-x) \geqslant -p(-x) = -p(x), \text{ so } -\tilde{\varphi}_1(x) \leqslant p(x)$$

and $|\tilde{\varphi}_1(x)| = \pm\,\tilde{\varphi}_1(x) \leqslant p(x) = C\,\|x\|$. This means that

$\tilde{\varphi}_1 \in X^{*_{\mathbb{R}}}$, and $\|\tilde{\varphi}_1\| \leqslant C = \|\varphi\|$

But $\tilde{\varphi}_1|_Y = \varphi_1$, so $\|\tilde{\varphi}_1\| \geqslant \|\varphi_1\|$ ($\rightarrow \Delta$ – it is almost obvious from the def. of the operator norm), but $\|\varphi_1\| = \|\varphi\|$ by Fact "On complex compl...." p. (iv) (for $\mathbb{K}=\mathbb{C}$, and by $\varphi_1 = \varphi$ for $\mathbb{K}=\mathbb{R}$), hence finally $\|\tilde{\varphi}_1\| = \|\varphi\|$.

This finishes the proof for $\mathbb{K}=\mathbb{R}$. If $\mathbb{K}=\mathbb{C}$, then using Fact "On complex compl...." again, we get:

$$\tilde{\varphi} := (\tilde{\varphi}_1)_{\mathbb{C}} \in X^*, \ \mathrm{Re}(\tilde{\varphi}|_Y) = (\mathrm{Re}\,\tilde{\varphi})|_Y = (\mathrm{Re}\,((\tilde{\varphi}_1)_{\mathbb{C}}))|_Y =$$

$$= \tilde{\varphi}_1|_Y = \varphi_1 = \mathrm{Re}\,\varphi,$$

hence also $\tilde{\varphi}|_Y = (\mathrm{Re}(\tilde{\varphi}|_Y))_{\mathbb{C}} = (\mathrm{Re}\,\varphi)_{\mathbb{C}} = \varphi$. Moreover $\|\tilde{\varphi}\| = \|\mathrm{Re}\,\tilde{\varphi}\| = \|\tilde{\varphi}_1\| = \|\varphi\|$.

(iii)

Note that (above), in the case $\mathbb{K} = \mathbb{R}$ we just used the Abstract version of H.-B. Theorem and the property $p(-x) = p(x)$ of this particular B. f. to get directly the assertion. And in the $\mathbb{C}$ - case the idea was to extend first the $\operatorname{Re}\varphi$ only, and then, to make the complex complement of this extension. — We got the proper $\mathbb{C}$ - linear and continuous extension by this two-steps procedure, thanks to the Fact "On complex compl."

◆ Some other continuous functionals of a special kind

Theorem 1 ("A continuous funct. for a fixed vector")

If X is a normed space and $0 \neq x \in X$, then there exists $\varphi \in X^*$ such that $\|\varphi\| = 1$ and

$$\varphi(x) = \|x\|.$$

Proof

Consider $Y := \{\lambda x : \lambda \in \mathbb{K}\}$ and $\varphi_0 : X \to \mathbb{K}$ given by $\varphi_0(\lambda x) := \lambda \|x\|$ (note, that $y \in Y$ can be expressed in the form λx uniquely, since $x \neq 0$) for any $\lambda \in \mathbb{K}$. We have $\varphi_0 \in X^*$ and $\|\varphi_0\| = 1$ ($\Rightarrow \triangle \ldots$) and $\varphi_0(x) = \|x\|$. Now we use The Continuous H.-B. Th. and we extend φ_0 to $\varphi \in X^*$. ▢

Corollary ("On separating points by X^*")

If $X \neq \{0\}$ – a normed space, then $X^* \neq \{0\}$; moreover, for any $x, y \in X$, $x \neq y$ there exists *) $\varphi \in X^*$ such that $\varphi(x) \neq \varphi(y)$.

Proof

It suffices to use Th. 1 for the vector $x - y$ – then the appropriate $\varphi \in X^*$ satisfies $\varphi(x-y) = \|x-y\| \neq 0$, so $\varphi(x) - \varphi(y) = \varphi(x-y) \neq 0$, i.e $\varphi(x) \neq \varphi(y)$. ▢

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*) This second part says that X^* separates points of X.

Note that this result gives the answer "YES" for our Question from page LF-2!

Now we present a generalisation of Th. 1.

Theorem 2 ("A continuous funct for a fixed vector and subspace")

If X is a normed space, $Y \underset{lin}{\subset} X$ and $x \in X \setminus \overline{Y}$, then there exists $\varphi \in X^*$ such that $\|\varphi\| = 1$ and

$$\varphi_{|Y} = 0 \ , \qquad \varphi(x) = \operatorname{dist}(x, Y).$$

Proof

Consider $\tilde{X} := X/\overline{Y}$ and $\tilde{x} := [x]$. We have $\tilde{x} \neq 0$, since $x \notin \overline{Y}$. So, by Th. 1, there exists $\tilde{\varphi} \in \tilde{X}^*$ such that $\|\tilde{\varphi}\| = 1$ and $\tilde{\varphi}(\tilde{x}) = \|\tilde{x}\|$.

Now, define $\varphi := \tilde{\varphi} \circ \pi$ (π - the quotient map $\pi(x) = [x]$). By properties of the quotient space (see Section I ...) we easily check ($\longrightarrow \triangle$) that φ satisfies the assertion.

$\square$

This result gives us a convenient density / linear density criterion (compare with "A linear density criterion" p. HS-32)

$\boxed{\text{Corollary 1}}$ ("A functional linear density criterion")

Let X be a normed space and $M \subset X$. Then M is linearly dense iff $M^{\perp *} = \{0\}$,

where
$$M^{\perp *} := \{\varphi \in X^* : \forall_{x \in M}\ \varphi(x) = 0\}.^{*)}$$

$\boxed{\text{Proof}}$

"$\Longrightarrow$" is obvious ($\operatorname{Ker}\varphi$ is a closed subspace for $\varphi \in X^*$ ...).

"$\Longleftarrow$" Suppose that $M^{\perp *} = \{0\}$, but $Y := \overline{\operatorname{lin} M} \neq X$. Then take $x \in X \setminus Y$ and construct $\varphi \in X^*$ for x and Y, as in Th. 2. Then $M \subset Y \subset \operatorname{Ker}\varphi$ but $\varphi(x) = \operatorname{dist}(x, Y) \neq 0$ ($\overline{Y} = Y \not\ni x$), so $\varphi \in M^{\perp *}$ and $\varphi \neq 0$ – a contradiction. $\square$

---

*) So $M^{\perp *} = \{\varphi \in X^* : M \subset \operatorname{Ker}\varphi\}$, but the "defining" formula for $M^{\perp *}$ was more similar to $M^{\perp}$ for the Hilbert (or unitary) space notation, where "$\varphi(x)$" is replaced by (φ, x) and X^* by X. Some people (including phicisists...) prefer even the notation $\langle \varphi, x \rangle$ or (φ, x) for $\varphi(x)$ in the general case of $\varphi \in X^*$ and $x \in X$...

$\boxed{\text{Corollary 2}}$

If X is a normed space and X^* is separable, then X is also separable.

$\boxed{\text{Proof}}$

Let $\{\varphi_n : n \in \mathbb{N}\}$ be a dense subset*) of X^*. If $X \neq \{0\}$ then for any $n \in \mathbb{N}$ choose $x_n \in S_X(0,1)$ such that

$|\varphi_n(x_n)| \geqslant \frac{\|\varphi_n\|}{2}$ (if $\varphi_n \neq 0$, then this is possible by the def. of $\| \ \|$ in X^*, because $\frac{\|\varphi_n\|}{2} < \|\varphi_n\|$, and if $\varphi_n = 0$ then any $x_n \in S_X(0,1)$ is OK).

Consider now $M := \{x_n : n \in \mathbb{N}\}$. Suppose that $\varphi \in M^{\perp *}$, then choose such a sequence $\{k_n\}_{n \geqslant 1}$ of elements of $\mathbb{N}$ that $\varphi_{k_n} \longrightarrow \varphi$ in X^*. We have $\varphi(x_{k_n}) = 0$ and $\|x_{k_n}\| = 1$ for any n, so

$$\underbrace{\|\varphi_{k_n} - \varphi\|}_{\downarrow \, 0} \geqslant |(\varphi_{k_n} - \varphi)(x_{k_n})| = |\varphi_{k_n}(x_{k_n})| \geqslant \frac{\|\varphi_{k_n}\|}{2} \geqslant 0.$$

Thus $\|\varphi_{k_n}\| \longrightarrow 0$, but also $\|\varphi_{k_n}\| \longrightarrow \|\varphi\|$, so $\varphi = 0$. This means that M is linearly dense, by "A funct. lin. density criterion".

*) This is possible also when the dense subset should be finite...

But M is at most countable, so X is separable by Fact "Separability from linear density" (p. PB-63). And if $X=\{0\}$, then it is also separable...

$\boxed{}$

$\boxed{\text{Remark}}$

There is no equivalence in Coroll. 2; e.g. $(\ell^1(\mathbb{N}))^*$ is isometric to $\ell^\infty(\mathbb{N})$ ($\longrightarrow$ Exercises...), so $\ell^1(\mathbb{N})^*$ is not separable, but $\ell^1(\mathbb{N})$ is...

1.3. Convex sets and separation theorems

The main goal of this subsection is to prove the so-called separation theorems for convex sets ("CSST"). They are formulated typically in a rather strange form, so one could think that they are very abstract/complicated/difficult... But in fact, they can be presented in a quite clear geometric way – so we start from some geometry.

Some geometry: hyperplanes, half-spaces and convex sets

Suppose that X is a linear space – as usual $\mathbb{K}$ can be both $\mathbb{R}$ or $\mathbb{C}$, but we shall use here only its $\mathbb{R}$-linear structure. Suppose also, that there is a fixed topology $\mathcal{T}$ in X – usually, we shall assume that X is a normed space, so $\mathcal{T}$ is just the norm topology then. But the norm structure will be not very important here... So, let $X^{*\mathbb{R}}$ denote just the space of $\mathbb{R}$-linear (and real) functionals which are continuous with respect to $\mathcal{T}$ in X. We shall use functionals from $X^{*\mathbb{R}}$ and not from X^* (in case of $\mathbb{K}=\mathbb{C}$) *), because we would like to define the hyperplanes in X in the typical geometric sens of "shifted linear spaces of the "geometric co-dim $= 1$", where "dim" is understood in the usual – "real", not "complex" sense...

So, mimicking the finite dimensional way of defining hyperplanes by the use of one real "linear" non-degenated equation of the form (in $\mathbb{R}^d$) $a_1x_1 + \dots + a_dx_d = c$ we define similarly:

*) X^* is defined similarly but now we need $\mathbb{K}$-linearity.

Definition

$P \subset X$ is a hyperplane (in X with the fixed fixed $\mathcal{J}$) iff there exists $\varphi \in X^{*\mathbb{R}} \setminus \{0\}$ and $g \in \mathbb{R}$ such that

$$P = \{x \in X: \ \varphi(x) = g\}$$

Similarly, $H \subset X$ is an open (closed) half-space iff $\exists_{\substack{\varphi \in X^{*\mathbb{R}} \setminus \{0\} \\ g \in \mathbb{R}}}$

$$H = \{x \in X: \ \varphi(x) \underset{(\leq)}{<} g\}.$$

Observe that we could define both kind of half-spaces equivalently by

$$H = \{x \in X: \ g \underset{(\leq)}{<} \varphi(x)\}$$

for some $\varphi \in X^{*\mathbb{R}}$, $g \in \mathbb{R}$, because $-\varphi \in X^{*\mathbb{R}}$ when $\varphi \in X^{*\mathbb{R}}$

Note also that the names open/closed for the two types of half-spaces are related rather not to the topological properties of H, but just to the strict or weak kind of inequality ($<$ or $\leq$) in the definition.*) The same can be expressed slighty shorter:

open half-space has the form:

$$\varphi^{-1}((-\infty; g)) \text{ or } \varphi^{-1}((g; +\infty))$$

and closed half-space has the form:

$$\varphi^{-1}((-\infty; g]) \text{ or } \varphi^{-1}([g; +\infty))$$

for some $g \in \mathbb{R}$ and $\varphi \in X^{*\mathbb{R}} \setminus \{0\}$.

---

*) Nevertheless, each open half-space is open, and closed half-space - - closed with respect to $\mathcal{J}$, because φ is continuous...

One easily guesses that two half-spaces / hyperplanes are called parallel if the functional φ which "defines" them can be choosen the same (φ and g are not uniquelly defined, but only up to a multiplication by a non-zero real constant) and they can differ only by the value of "g" then.

It is not very unified what the (general notion "Polygon" means, but I think that everybody agrees to call $F \subset \mathbb{R}^2$ a convex closed Polygon as follows:

"it is a finite intersection of closed half-spaces in $\mathbb{R}^2$"

+ may be, to be more strict (and to not tolerate "unbounded polygons, e.g. the half-space itself) we should add: "which is bounded" (next, we could also exclude the case of empty or 1 point sets or some 1-dimensional intervals ect...). The similar definition can be used also for larger dimensions of X.

The above suggests to think about those "finite intersection" - what shall we get living only "intersection" - without "finite"?

- Obviously, any intersection of closed sets (see *) on LF-26) is a closed set and any intersection of convex sets is convex. - So such a set is surely a closed convex set. But:

$$\boxed{\text{Can we get any closed convex set this way ??}}$$

- we shall answer this question soon*), to show the strength of the CSST-s...

*) See Theorem "On closed convex sets representation" p. LF-34

Separation of sets

In accordance with the notions of hyperplane and of half-space we introduce here two notions of separation of subsets of X by hyperplanes.

Definition

Let $A, B \subset X$, then

- A is separated from B by a hyperplane iff

$$\exists_{\substack{\varphi \in X^{*\mathbb{R}} \setminus \{0\} \\ g \in \mathbb{R}}} \ \forall_{\substack{a \in A \\ b \in B}} \quad \varphi(a) < g \leq \varphi(b) \qquad \text{(sep)}$$

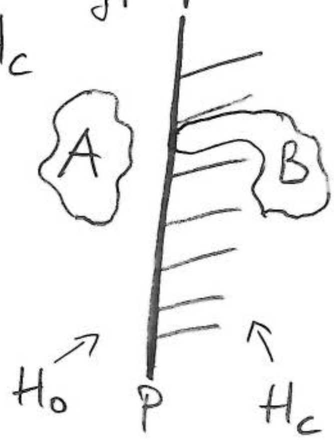
- A and B are strongly separated by hyperplanes iff

$$\exists_{\substack{\varphi \in X^{*\mathbb{R}} \setminus \{0\} \\ g_1, g_2 \in \mathbb{R}}} \ \forall_{\substack{a \in A \\ b \in B}} \quad \varphi(a) < g_1 < g_2 < \varphi(b) \qquad \text{(s-sep)}$$

Observations

1) The first notion means geometrically that we can find an open half-space H_0 related to a hyperplane P and the "opposite side closed half-space H_c

such that $A \subset H_0$, $B \subset H_c$.



The condition (sep) can be written also as:

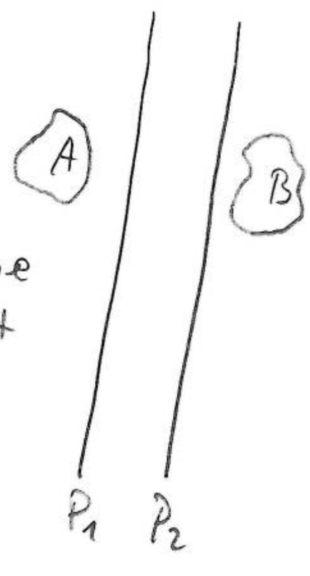
$$\exists_{\substack{\varphi \in X^{*\mathbb{R}} \setminus \{0\} \\ g \in \mathbb{R}}} \quad \varphi(A) < g \leqslant \varphi(B).$$

(with the standard meaning of $<$ and $\leqslant$ used as a relation between numbers / subsets from / of $\mathbb{R}$)

2) Similarly (s-sep) can be rewritten as:

$$\exists_{\substack{\varphi \in X^{*\mathbb{R}} \setminus \{0\} \\ g_1, g_2 \in \mathbb{R}}} \quad \varphi(A) < g_1 < g_2 < \varphi(B).$$

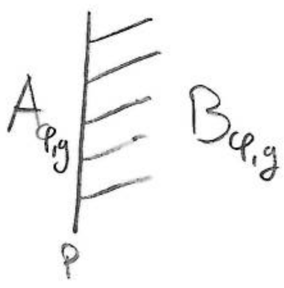
And the geometrical meaning is that we can separate A and B "inserting" two parallel but distinct hyperplans "between" them.



3) The first notion is not a symmetric relation! The half-plane for B can be closed, but for A - it has to be open! The example below shows that it is really not symmetric.
And the strong separation is obviously a symmetric relation.

Example

For $\varphi \in X^{*\mathbb{R}}$, $\varphi \neq 0$ and $g \in \mathbb{R}$ consider $A_{\varphi,g} := \varphi^{-1}((-\infty; g))$ and $B_{\varphi,g} := \varphi^{-1}([g; +\infty))$. Then:



- $A_{\varphi,g}$ is separated from $B_{\varphi,g}$ by a hyperplane (–using just the same φ and g ...)
- $B_{\varphi,g}$ is <u>not</u> separated from $B_{\varphi,g}$ by a hyperplane

– it seem obvious "from the picture", but it needs a proof $\longrightarrow \Delta$

- $B_{\varphi,g}$ and $A_{\varphi,g}$ <u>are not</u> strongly sep. by hyp. (it easily follows from the previous statement) ■

Now, having those separation notions, we are ready to formulate the separation theorem announced earlier.

Theorem ("CSST = Convex Sets Separation Theorem(s)")

Suppose, that X is a normed space (*) and $C_1, C_2 \subset X$ are two nonempty, disjoint convex sets.

1. If C_1 is open, then C_1 is separated from C_2 by a hyperplane.
2. If C_1 is compact and C_2 – closed, then C_1 and C_2 are strongly separated by hyperplans. ■

(*) The normed space assumption is not necessary here – we shall weaken it soon – see p. LF-31½ and

The Proof – p. LF –

The proof is placed later in this subsection. – First we shall make some remarks on CSST, define some generalizations of the norm topologies for 1. and 2. of CSST and prove the Thm "On closed convex sets representation" related to the question asked on p. LF – 27.

Let us recall (see Fact "On complex complement..." p. LF-14/15) that if $\mathbb{K} = \mathbb{C}$, then the condition $\varphi \in X^{*\mathbb{R}} \setminus \{0\}$ means exactly that there exists $\psi \in X^* \setminus \{0\}$ such that $\varphi = \mathrm{Re}\,\psi$, so the separation in CSST is just the separation "by real parts of ψ from $X^* \setminus \{0\}$", i.e. by hyperplans given by those real parts.

Topological vector spaces and locally convex spaces

As we promised, we now "generalize" CSST onto more general spaces. Suppose that X – a linear space and $\mathcal{T}$ – a topology in X.

Definition

- $(X, \mathcal{T})$ is a topological vector space iff $\mathcal{T}$ is Hausdorff*) topology and $\mathcal{T}$ is such that

*) see p. LF-32

the (linear) operations $\cdot : \mathbb{K} \times X \longrightarrow X$ and $+ : X \times X \longrightarrow X$ are continuous.

- (X, τ) is a locally convex space iff it is a topological vector space and each open neighbourhood of 0 contains an open convex neigh. of 0.

In particular – the normed space is obviously locally convex!

Remark

In CSST 1. the normed space X can be replaced by a topological vector space and in 2. it can be replaced by a locally convex space.

Proof $\rightarrow$ see Proof of CSST ... [L]

Recall now, that The Continuous Hahn-Banach Theorem was formulated and proved only for normed spaces – and the norm was really important there. Using this theorem we have also proved several important results: "A continuous functional for a fixed vector", "On separating points by X^*", "On a cont. funct. for a fixed vector and subsp.", "On a funct. linear density criterion".

---

*) – from p. LF-31: in fact it suffices to assume that it is T1-topology (every 1 point set is a closed set), and the Hausdorff property follows from this and from the second part of conditions of the definition.

It is interesting, that CSST (+ the Remark above...) allows to prove several analogs of those results also for locally convex spaces.

We formulate those analogic results here without proofs (they are easy...), treating this as an exercise (ces...).

Corollary 1

X^* separates points of a locally convex space X.

Corollary 2

If X is locally convex, $Y \underset{lin}{\subset} X$ and $\varphi \in Y^*$, then there exists $\tilde{\varphi} \in X^*$ such that $\tilde{\varphi}_{|Y} = \varphi$.

Corollary 3

If X is locally convex, $Y \underset{lin}{\subset} X$ and $x_0 \notin \overline{Y}$ $(x_0 \in X)$, then there exists $\varphi \in X^*$ such that $\varphi_{|Y} = 0$ and $\varphi(x_0) = 1$.

Corollary 4

"A functional linear density criterion" p. LF-22 with X - normed replaced by X - loc. convex is also true.

Exercise Prove Corol. 1., 2., 3. 4. $\longrightarrow$ △.

An equivalent characterisation of closed convex sets

Now we come back to the considarations from p. LF-27 and we give the a precise answer for the question posed there for all locally convex spaces X.

Theorem ("On closed convex sets representation")

If X is locally convex, $X \neq \{0\}$ and $C \subset X$ then C is a closed convex set iff C is an intersection of a family *) of some closed half-spaces in X.

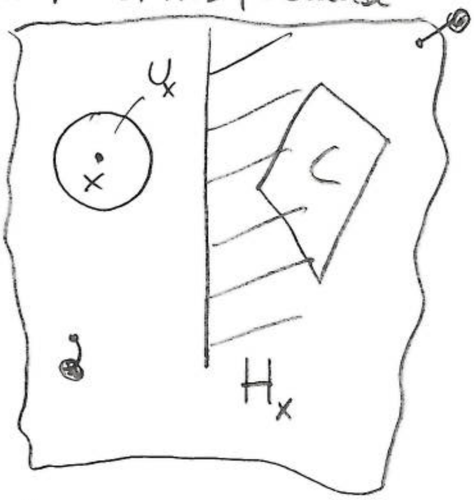
Proof

For $C = X$ – see *) above (below...). If $C = \emptyset$, then by Corol. 1 p. LF-33 there exists a nonzero $\varphi \in X^*$, so .... ⟶ △ (please, finish this $\emptyset$-case ...). Suppose, that $\emptyset \neq C \neq X$. Denote $\mathcal{H}_C := \{H \subset X : H$ is a closed half-space in X and $C \subset H\}$. Let $\tilde{C} := \bigcap_{H \in \mathcal{H}_C} H \; (= \bigcap \mathcal{H}_C)$.

---

*) We make the following extra agreement: the intersection of the empty family is equall to X; without it, we should exclude the case of $C = X$ here...

Let $x \in X \setminus C$ (we know, that such x exists, because $C \subset X$, $C \neq X$). Observe, that $x \in V := X \setminus C$, and V is open (C was closed), so there exists U_x – a convex neigh. of x such that $U_x \subset V$ (— to get it it suffices to take a convex open neigh. U of 0 contained in $V - \{x\}$ and then $U_x := \{x\} + U$ – open, convex and $U_x \subset V$). In particular we have $U_x \cap C = \emptyset$ and $U_x, C \neq \emptyset$. Thus, by Remark p. LF-32 we can use CSST p. 1 – denote the appropriate closed half-space by H_x – we have $H_x \supset C$ and $H_x \cap U_x = \emptyset$. In particular $H_x \in \mathcal{H}_C$ (so $\mathcal{H}_C \neq \emptyset$). We have $C \subset \tilde{C}$, since $C \subset H$ for each $H \in \mathcal{H}_C$. But also, when $x \in X \setminus C$, then $H_x \in \mathcal{H}_C$ and $x \notin H_x \supset \tilde{C}$, so $x \notin \tilde{C}$ i.e, $x \in X \setminus \tilde{C}$. This means that $X \setminus C \subset X \setminus \tilde{C}$, so $\tilde{C} \subset C$. Thus $C = \tilde{C}$!



Note here the rôle of the topology $\mathcal{T}$ for the both sides of "$\Longleftrightarrow$": "C is <u>closed</u> convex" – <u>closed</u> in $\mathcal{T}$-sense, and "... of some closed <u>half-space</u>" – the topology $\mathcal{T}$ is used in the <u>h.-s.</u> definition: we use the continuous (in $\mathcal{T}$-sense) functionals in it!

The proof of CSST

We shall prove CSST here in the version extended by Remark p. LF-32.

Of course, we shall not use any results obtained on pp. LF-31 – 35 which were proved using CSST... But we need several topological lemmas first.

Lemma 1

Suppose that $A \in \mathcal{L}(X, Y)$, where X is a topological vector space (with the topology $\mathcal{T}$) and Y is a normed space (e.g. $Y = \mathbb{K}$).

Then TFCAE:

(i) A is continuous

(ii) $\exists_{\substack{U \in \mathcal{T} \\ U \neq \emptyset}}$ $A(U)$ is bounded in Y.

Proof

(i) $\Rightarrow$ (ii) $U := A^{-1}(K(0,1))$. (ii) $\Rightarrow$ (i) By linearity it suffices to prove the continuity of A at 0, so we have to prove, that for any $\varepsilon > 0$ there exist U_ε – an open neigh. of 0 such that $A(U_\varepsilon) \subset K(0, \varepsilon)$. But "shifting" $U \in \mathcal{T}$ from (ii) to 0, we get U_0 – open neigh. of 0 such that $A(U_0) \subset K(0, M)$ for some $M \in \mathbb{R}$ ... $\to \triangle$ (please finish it!)

VCT

Lemma 2

If X is a topological vector space and $U \subset X$ is an open neigh. of 0, then U is absorbing.

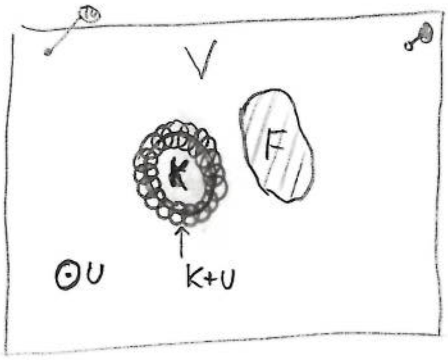
Proof $\to \triangle$ (Hint: use the continuity of $\cdot : \mathbb{K} \times X \to X$ ...)

G

$\boxed{\text{Lemma 3}}$

Let X – topological vector space, $K, F \subset X$, K – compact, F – closed. Then there exits U – an open neigh. of 0 such that $(K+U) \cap F = \emptyset$; moreover U can be chosen convex, if X is locally convex.

$\boxed{\text{Proof}}$



If $K = \emptyset$ then any open (and convex for l.c.sp.) neigh of 0 is OK, since $\emptyset + U = \emptyset$. Suppose, that $K \neq \emptyset$

Let $V := X \setminus F$ and for any $x \in V$ denote $V_x := V - \{x\}$ – this is an open neigh. of 0, since V is open and $0 = x - x \in V_x$. Now, using the fact that $+ : X \times X \longrightarrow X$ is continuous and $0 + 0 = 0$ *) we can choose an open neigh. U_x of 0 such that $U_x + U_x \subset V_x$, and in the case of X – loc. convex U_x can be chosen convex. In particular $K \subset V$, because $K \cap F = \emptyset$, so $\{U_x + \{x\}\}_{x \in K}$ is an open cover of K. Let us choose its finite subcover $\{U_{x_j} + \{x_j\}\}_{j=1,\dots n}$ and define $U := \bigcap_{i=1}^{n} U_{x_i}$ (– note that $K \neq \emptyset$, so the subcover is not empty, i.e., $n \geqslant 1$). U is a neigh. of 0, convex for the l.c. space case. Moreover $K + U \subset \left(\bigcup_{j=1}^{n} (\{x_j\} + U_{x_j})\right) + U =$

$$= \bigcup_{j=1}^{n} \left(\{x_j\} + (U_{x_j} + U)\right) \subset \bigcup_{j=1}^{n} \left(\{x_j\} + (U_{x_j} + U_{x_j})\right) \subset \bigcup_{j=1}^{n} \left(\{x_j\} + V_{x_j}\right) =$$

$= \bigcup_{j=1}^{n} V = V = X \setminus F$, so $(K+U) \cap F = \emptyset$. $\square$

---

*) More precisely: the above continuity of "+" guarantte the existance of U'_x, U''_x – two open neigh. of 0, such that $U'_x + U''_x \subset V_x$, and then, choosing any open $U_x \ni x$, such that $U_x \subset U'_x \cap U''_x$ (note that $0 \in U'_x \cap U''_x$ – open) we get $U_x + U_x \subset U'_x + U''_x \subset V_x$.

Lemma 4

If X - topological vector space, $0 \neq \psi \in X^{*_{\mathbb{R}}}$ and $U \subset X$, U - open, then $\psi(U)$ - open (in $\mathbb{R}$).

Proof

Let $x_0 \in X \setminus \operatorname{Ker}\psi$, i.e. $\psi(x_0) \neq 0$. Let V be an open neigh. of 0 - we shall prove, that $\psi(V)$ contains an open interval $(-\varepsilon; \varepsilon)$ for some $\varepsilon > 0$.

We have $0 \cdot x_0 = 0 \in V$, so there exists $\delta > 0$ that $(-\delta; \delta) \cdot x_0 \subset V$ (because $\mathbb{R} \ni t \rightsquigarrow t x_0 \in X$ is continuous...) so $\psi((-\delta; \delta) \cdot x_0) \subset \psi(V)$, but $\psi((-\delta; \delta) \cdot x_0) =$

$= (-\delta\psi(x_0); \delta\psi(x_0))$ or $(\delta\psi(x_0); -\delta\psi(x_0))$, so $\varepsilon := \pm\delta\psi(x_0)$ where $+/-$ is chosen contrary to the sign of $\psi(x_0)$.

Now, if $U = \emptyset$, then $\psi(U) = \emptyset$ - open.. If $U \neq \emptyset$ then let $z_0 \in \psi(U)$ and consider $V := U - \{x_0\}$, where x_0 is such that $x_0 \in U$, $\psi(x_0) = z_0$. Then $\psi(U) = \{z_0\} + \psi(V)$ and V is an open neigh. of 0... - so $\psi(U)$ contains an open neigh. of z_0 (in $\mathbb{R}$)!

$\boxed{4}$

$\boxed{\text{Proof of CSST}}$

1. For X – topological vector space
2. For X – locally convex space

1. Suppose that C_1, C_2 – nonempty, disjoint, convex in X and C_1 – open. Let $x_0 := x_{02} - x_{01}$, where x_{01}, x_{02} are fixed elements of C_1, C_2, resp. ($C_i \neq \phi \ldots$).

The general idea of the proof is to construct a "convenient" convex absorbing set C and a "simple" $\mathbb{R}$-linear functional φ on an $\mathbb{R}$-subspace Y of X, such that:

(a) φ is dominated by the Minkowski functional μ_C

– and for its extension $\tilde{\varphi}$ onto X obtained via Abstr. H-B Th.:

(b) the domination by μ_C gives the continuity

(c) the domination by μ_C can be "reconstructed" as the separation of C_1 from C_2 "by $\tilde{\varphi}$"

First, let's think a little about (c). Observe, that (c) would be satisfied iff

$$\forall_{\substack{x_1 \in C_1 \\ x_2 \in C_2}} \quad \tilde{\varphi}(x_1) < g \leqslant \tilde{\varphi}(x_2) \qquad (1)$$

for some $g \in \mathbb{R}$, so in particular

$$\forall_{\substack{x_1 \in C_1 \\ x_2 \in C_2}} \quad \tilde{\varphi}(x_1 - x_2) = \tilde{\varphi}(x_1) - \tilde{\varphi}(x_2) < 0. \qquad (1')$$

If we would like to get (1') using the domination of $\tilde{\varphi}$ by μ_C we could estimate

$$\tilde{\varphi}(x_1-x_2) \le \mu_C(x_1-x_2),$$

but the RHS could not be < 0, since the Minkowski functional is non-negative ... So, we must be more clever... But try to start "from the RHS" – for which $x \in X$ $\mu_C(x)$ can be easily estimated from the above? Obviously, we have (see Fact "On the Minkows. f.")

$$\forall_{x \in C} \ \mu_C(x) \le 1, \qquad (2)$$

but if C would be aso an open set we would get even

$$\forall_{x \in C} \ \mu_C(x) < 1, \qquad (2')$$

by the definition of μ_C ($\longrightarrow \triangle$). So, we shall rather apply $\tilde{\varphi}$ and μ_C to some elements x of C to get (1'), and not directly to $x_1 - x_2$ for $x_i \in C_i$. Try then to "shift" $x_1 - x_2$ by x_0 (defined at the begining of the proof) – it has two benefits:

(i) $$(x_{10} - x_{20}) + x_0 = 0,$$

so the set $C := \{(x_1-x_2)+x_0 : x_1 \in C_1, x_2 \in C_2\}$ contains 0, and it has "chance" to be absorbing, contrary to $\{x_1 - x_2 : x_1 \in C_1, x_2 \in C_2\}$ (why <u>no</u> chance? $\longrightarrow \triangle$)

(ii) with C as above, if C is open, convex and absorbing, then:

$$\forall_{\substack{x_1 \in C_1 \\ x_2 \in C_2}} \quad \tilde{\varphi}(x_1 - x_2 + x_0) \leqslant \mu_C(x_1 - x_2 + x_0) < 1 \qquad (3)$$

by $(2')$.

But $\tilde{\varphi}(x_1 - x_2 + x_0) = \tilde{\varphi}(x_1 - x_2) + \tilde{\varphi}(x_0)$, so $\overbrace{\text{we}}^{\text{by }(3)}$ would also get $(1')$, if e.g. $\tilde{\varphi}(x_0) = 1$! But to have $\tilde{\varphi}(x_0) = 1$ we can just define such φ first that $x_0 \in Y$ (– the domain of φ) and $\varphi(x_0) := 1$. So, the simplest choice for such φ is:

$$Y := \operatorname{lin}_{\mathbb{R}}\{x_0\}$$

– note that $x_0 \neq 0$, because $x_0 = x_{02} - x_{01}$, and $C_1 \cap C_2 = \emptyset$; so any $x \in Y$ has a unique form $x = tx_0$ for some $t \in \mathbb{R}$ and we can set:

$$\varphi(tx_0) := t, \qquad t \in \mathbb{R}.$$

In particular we get $\varphi(x_0) = 1$ then. Thus, we "guessed" φ and C – reccall now what we need to get the result... :

1° C is open, convex, absorbing

2° (a)

3° (b)

4° to get (c) from $(1')$ (which would be guarantee by 1° and 2° and the above considerations).

Let us start from $1°$. Note that $C = (C_1 - C_2) + \{x_0\}$ so C is convex, since convexity is invariant under the algebraic operations $+$ and $-$ on sets in linear space ($\to \triangle$). Moreover C is open, because C_1 is open:

$$C = \bigcup_{x_2 \in C_2} C_1 + \{(-x_2 + x_0)\}$$

and $C_1 + \{x\}$ is open for any $x \in X$.

Now C is also absorbing by Lemma 2, because $0 \in C$ by (i).

To get $2°$ observe that $x_0 \notin C$, since if $x_0 = C$, then $x_1 = x_2$ for some $x_1 \in C_1$, $x_2 \in C_2$, but $C_1 \cap C_2 = \phi$. Thus, $\mu_C(x_0) \geqslant 1$ (e.g. by the def. of μ_C). So, we have

$$\forall_{t \geqslant 0} \quad \varphi(tx_0) = t \leq t\mu_C(x_0) = \mu_C(tx_0)$$

and for $t < 0$ $\quad \varphi(tx_0) = t < 0 \leqslant \mu_C(tx_0)$, so $2°$ holds.

We shall now prove $3° = (b)$. Observe that

$$\forall_{x \in C} \quad \tilde{\varphi}(x) \leq \mu_C(x) \leq 1 \quad \text{and} \quad \tilde{\varphi}(-x) \geqslant -1,$$

hence $\forall_{x \in C \cap (-C)}$ $\quad -1 \leq \tilde{\varphi}(x) \leq 1$. This means, that $\tilde{\varphi}$ is bounded on $C \cap (-C)$, but C and $-C$ are open, and $0 \in C$ and $0 \in (-C)$, thus $C \cap (-C)$ is an open neigh. of 0, so $\tilde{\varphi}$ is continuous, by Lemma 1. It suffices now to prove (1) for some $g \in \mathbb{R}$ by the use of (1'). But $\tilde{\varphi}(C_1), \tilde{\varphi}(C_2)$ are convex, because $\tilde{\varphi}$ is linear and C_j - convex, so they are nonempty intervals and $\tilde{\varphi}(C_1) < \tilde{\varphi}(C_2)$ by (1'). Let $g := \sup \tilde{\varphi}(C_1)$.

Then $g \in \mathbb{R}$ ($\tilde{\varphi}(C_1)$ is bounded by any element of $\tilde{\varphi}(C_2)$) and $g \notin \tilde{\varphi}(C_1)$, because $\tilde{\varphi}(C_1)$ is an open interval — — this follows from the fact that C_1 is open and $\tilde{\varphi}$ is a non-zero continuous functional — see Lemma 4.

Finally $\tilde{\varphi}(C_1) < g \leq \tilde{\varphi}(C_2)$ and the proof of 1. part is finished.

2. Suppose that C_1 is compact, C_2 — closed and again $C_1 \cap C_2 = \emptyset$, C_j - nonempty and convex for $i = 1, 2$. And X is loc. conv. Using Lemma 3 we get $(C_1 + U) \cap C_2 = \emptyset$ for some U - a convex open neigh. of 0. Thus $C_1 + U$ is open and convex and we can use the part 1. for $C_1 + U$ and C_2 — Let $\psi \in X^{*\mathbb{R}}$ and $g \in \mathbb{R}$ be such that $\psi(C_1 + U) < g \leq \psi(C_2)$. But $0 \in U$, so $C_1 \subset C_1 + U$, hence $\psi(C_1) < g$. Let $g' = \max_{x \in C_1} \psi(x)$ — by compactness of C_1 the max exists (The Weierstrass th.) and $g' \in \psi(C_1)$, hence $g' < g$. Finally, taking any $g_1 < g_2$ such that $g_1, g_2 \in (g', g)$ we get

$$\psi(C_1) < g_1 < g_2 < \psi(C_2).$$

$\boxed{//}$

LF-43

2. Weak topologies and reflexibility

We consider here only some basics of the notions from the above title...

2.1 Weak topologies

Reminder of some topological notions

We recall here the definition of the topology induced in a set by a family of functions on it. *)

Consider a set X and a topological space (Z, τ_Z), and let $\mathcal{F}$ be a family (set) of some functions $f : X \longrightarrow Z$.

$\boxed{\text{Definition}}$

The topology in X induced by $\mathcal{F}$, denoted by $\tau(X, \mathcal{F})$, is the smallest topology**) $\mathcal{T}$ in X, such that $\forall_{f \in \mathcal{F}}$ f is continuous (in $\mathcal{T} - \mathcal{T}_Z$ sense).

---

*) We shall need here only the case with the functions into a one space, so we restrict ourselves to it.

**) One can easily check that the intersection of all the topologies as above is the smallest one – note, that the intersection of a set of topologies is still a topology...

The following two results from Topology I are well-known (and moreover – easy to prove – try, if you didn't know them ...)

$\boxed{\text{Fact 1}}$

The subset $SB_{\mathcal{F}}$ of $\tau(X, \mathcal{F})$ given by

$$SB_{\mathcal{F}} := \{ f^{-1}(U) : f \in \mathcal{F},\ U \in \tau_Z \}$$

is a sub-base of $\tau(X, \mathcal{F})$, i.e.,

$$B_{\mathcal{F}} := \{ V_1 \cap \ldots \cap V_k : k \in \mathbb{N},\ \forall_{i \in \{1,\ldots,k\}}\ V_i \in SB_{\mathcal{F}} \}$$

is a base of $\tau(X, \mathcal{F})$.

$\boxed{\text{Fact 2}}$

Let Y be any topological space and $G : Y \to X$.
Then TFCAE:

(i) G is continuous as a function from Y to X with $\tau(X, \mathcal{F})$

(ii) $\forall_{f \in \mathcal{F}}$ $f \circ G$ is continuous (from Y to Z with τ_Z).

Weak topology in X, weak topologies in X^*, the bidual space X^{**} for a normed space X

Suppose now, that X is a normed space.
Recall that X^* – the dual space for X is in particular a set of some functions from X to $\mathbb{K}$ – so, we can use X^* as $\mathcal{F}$ in the construction of $\tau(X, \mathcal{F})$ in X. The topology $\tau(X, X^*)$ is called the weak topology in X.

The name "weak" comes from the fact, that it is "weaker", i.e. "smaller" in the $\subset$ - sense, than the "original" norm topology in X (because each $\varphi \in X^*$ is continuous in the norm topology in X ⟨by the definition of X^*⟩, so the smallest $\tau(X, X^*)$ should be contained in it...)

We can obviously construct such weak topology in any normed space X – in particular – also in X^*, which is a norm space as well....
So we shall obtain the weak topology in X^* this way which is equall to $\tau(X^*, (X^*)^*)$.
The space $(X^*)^*$ seems to be a complicated object... – we call it the bidual space (to X)

and we denote it

$$\boxed{X^{**}}$$

for short. Note, that X^{**} is also a normed space. But generally the "usual" weak topol. in X^* is not very important / popular in a sense... – it is sometimes "too small" (too weak), since $X^{**} = (X^*)^*$ can be "too large".

So we shall find now a convenient subset (and a subspace ...) of X^{**}.

Let $x \in X$ and consider the following $\Psi_x : X^* \longrightarrow \mathbb{K}$:

$$\Psi_x(\varphi) := \varphi(x) \quad \text{for} \quad \varphi \in X^*.$$

Observation

$\forall_{x \in X} \quad \Psi_x \in X^{**}$ and $\|\Psi_x\| \leq \|x\|$.

Proof

Obviously $\Psi_x \in (X^*)^{\#}$, moreover for any $\varphi \in X^*$ we have $|\Psi_x(\varphi)| = |\varphi(x)| \leq \|\varphi\| \cdot \|x\| = \|x\| \cdot \|\varphi\|$ – hence $\Psi_x \in B(X^*, \mathbb{K})$, i.e. $\Psi_x \in X^{**}$, and $\|\Psi_x\| \leq \|x\|$. $\square$

Consider now the map: $X \ni x \rightsquigarrow \Psi_x \in X^{**}$
We denote it by κ (KAPPA) – it is sometimes called "canonical map" or "evaluation map".
It has some very good properties ...

Theorem ("On canonical $X \hookrightarrow X^{**}$ embedding")

The map $\kappa : X \longrightarrow X^*$ is a linear isometry onto its image $\kappa(X)$. In particular $\|\Psi_x\| = \|x\|$ for any $x \in X$.

Proof

We start from "the end". For $x \in X$ we have $\Psi_x = 0$ if $x = 0$, so $\|\Psi_0\| = 0 = \|0\|$; if $x \neq 0$ then consider φ_x given by Thm "A continuous functional for a fixed vector" (p. LF-19) – recall $\varphi_x \in X^*$, $\|\varphi_x\| = 1$ and $\varphi_x(x) = \|x\|$. Thus:

$|\Psi_x(\varphi_x)| = |\varphi_x(x)| = \|x\|$ and $\varphi_x \in \overline{K}_{X^*}(0,1)$, so $\|\Psi_x\| \geqslant \|x\|$, but we have also "$\leq$" by Observation !, So $\|\Psi_x\| = \|x\|$. The linearity of κ is obvious... ($\longrightarrow \triangle$), hence κ is a linear isometry onto $\kappa(X)$. $\square$

The above theorem means, that $\varkappa(X)$ is always (for any normed space X) a linear subspace of X^{**} and moreover it is isometric (linearly isometric...) to X! I.e., $\varkappa(X)$ and X can be "identified" from the point of view of the normed space structure.

Now we define the second - "more popular" "weak" topology in X^*: it is $\tau(X^*, \varkappa(X))$ and we call it weak* topology (in X^*). Because of the above "identification" $\varkappa(X) \longleftrightarrow X$ we denote it rather by somewhat informal symbol

$$\tau(X^*, X)$$

instead of $\tau(X^*, \varkappa(X))$ - the more formal, but longer one ...

Fact

All the "weak" topologies: $\tau(X, X^*)$, $\tau(X^*, X^{**})$ and $\tau(X^*, X)$ are some locally convex topologies (in particular they are Hausdorff and topol. vector topologies in X, X^*, X^* respectively).

Proof $\longrightarrow \triangle$ (it is easy... - use the Reminder and the point separation by functionals...)

$\boxed{\text{Ш}}$

2.2. Reflexive spaces

It is a very short sub-subsection ...

Definition

A norm space X is reflexive iff $\varkappa(X) = X^{**}$.

Note, that $X^{**} = (X^*)^*$, hence it is a Banach space, not only a norm space. By theorem "On canonical ... emb." if X is reflexive, then X and X^{**} are isometric spaces, so X have to be Banach space too.

Exercise / Examples $\longrightarrow$ $\triangle$

1. X is a normed finite dim. space $\Longrightarrow$ X – reflexive.
2. $\ell^p(\mathbb{N})$ is reflexive for any $p \in (1; +\infty)$.
3. $\ell^1(\mathbb{N})$ and $\ell^\infty(\mathbb{N})$ are not reflexive.

3. Representations of some X^* spaces

We shall identify here the exact form of any functional $\varphi \in X^*$ for 3 cases of X: for each Hilbert space, for each $C(K)$ space, with any compact topological space K, and for $L^p(\Omega, \mu)$ spaces for $p \in [1; +\infty)$ with σ-finite measures μ. The first two cases are traditionally associated with the name of Riesz.

3.1. Hilbert space functionals

◆ The abstract Hilbert space representation theorem

Let $\mathcal{H}$ be a Hilbert space. For $y \in \mathcal{H}$ consider $\eta_y : \mathcal{H} \longrightarrow \mathbb{K}$ given by

$$\eta_y(x) := (x, y), \qquad x \in \mathcal{H}$$

Obviously $\eta_y \in \mathcal{H}^\#$, and moreover, by Schwarz inequality,

$$|\eta_y(x)| = |(x,y)| \leq \|y\| \cdot \|x\|, \qquad x \in \mathcal{H},$$

so $\eta_y \in \mathcal{H}^*$ and $\|\eta_y\| \leq \|y\|$. But if $y = 0$ then $\eta_y = 0$, and if $y \neq 0$ then $\tilde{y} := \frac{y}{\|y\|}$ is such that $\|\tilde{y}\| = 1$ and $|\eta_y(\tilde{y})| = \frac{1}{\|y\|}(y, y) = \|y\|$. Hence

$$\forall_{y \in \mathcal{H}} \quad \|\eta_y\| = \|y\|. \tag{1}$$

Observe, that $\varphi = \eta_y$ is an element of $\mathcal{H}^*$ which has the following two properties:

(a) $(\operatorname{Ker} \varphi)^\perp = (\{y\}^\perp)^\perp = \operatorname{lin}\{y\}$

(b) $\varphi(y) = (y, y) = \|y\|^2$

(see Th. "On orth. decomposition" p. HS-30 for (a)).

We shall prove now, that the functionals η_y fill exactly $\mathcal{H}^*$.

$\boxed{\text{Theorem}}$ ("The Hilbert space Riesz representation theorem")

Each $\varphi \in \mathcal{H}^*$ has a form η_y for some $y \in \mathcal{H}$, if $\mathcal{H}$ is a Hilbert space. Moreover $h : \mathcal{H} \to \mathcal{H}^*$ given by

$$h(y) := \eta_y, \quad y \in \mathcal{H}$$

is an conjugate-linear *) isometry of $\mathcal{H}$ onto $\mathcal{H}^*$.

$\boxed{\text{Proof}}$

Observe, that the second part follows immediately from the first, by (1) and by the sesquilinearity of scalar product. To prove the first part, consider some $\varphi \in \mathcal{H}^*$. If $\varphi = 0$, then

---

*) Hence, it is linear if $\mathbb{K} = \mathbb{R}$. See p. HS-2.

$\varphi = \eta_0$ (i.e. η_y for $y=0$). Suppose that $\varphi \neq 0$.
Thus $\operatorname{Ker}\varphi$ is a closed subspace of $\mathcal{H}$, and $\operatorname{Ker}\varphi \subsetneq \mathcal{H}$.
Therefore $(\operatorname{Ker}\varphi)^\perp \neq \{0\}$, by Thm. "On orth. decomp.",
so let $0 \neq y_0 \in (\operatorname{Ker}\varphi)^\perp$. Now the observation (a) p. 52
suggests that we schould search the appropriate $y \in \mathcal{H}$
as a vector of the form

$$y = \lambda_0 y_0 \, , \quad \lambda_0 \in \mathbb{K}\setminus\{0\} \tag{2}$$

for some $\lambda_0 \in \mathbb{K}$. We shall guess the value of λ_0 trying
to "satisfy the observation (b)". If (2) holds then:

$$\varphi(y) = \|y\|^2 \iff \varphi(\lambda_0 y_0) = \|\lambda_0 y_0\|^2 \iff \lambda_0 \varphi(y_0) = \lambda_0 \bar{\lambda}_0 \|y_0\|^2$$
$$\iff \varphi(y_0) = \bar{\lambda}_0 \|y_0\|^2 \iff \overline{\frac{\varphi(y_0)}{\|y_0\|^2}} = \lambda_0 .$$

So, let us define y by (2) with $\lambda_0 := \overline{\frac{\varphi(y_0)}{\|y_0\|^2}}$. Since $\varphi \in \mathcal{H}^\# \setminus \{0\}$
and $y_0 \in \mathcal{H} \setminus \operatorname{Ker}\varphi$ (because $y_0 \in (\operatorname{Ker}\varphi)^\perp$ and $y_0 \neq 0$), thus
by Fact "On kernel of a functional" (p. OF-55) for
any $x \in \mathcal{H}$ $\quad x = \lambda y_0 + z$ for some $\lambda \in \mathbb{K}$ and $z \in \operatorname{Ker}\varphi$.
So we have $\varphi(x) = \lambda\varphi(y_0)$ and $\eta_y(x) = (x, y) =$
$= (\lambda y_0 + z, \lambda_0 y_0) = (\lambda y_0, \lambda_0 y_0) = \lambda \bar{\lambda}_0 \|y_0\|^2 = \lambda\varphi(y_0) = \varphi(x)$,
i.e., $\varphi = \eta_y$. $\boxed{\text{Q.E.D.}}$

The assertion of the above Riesz thm. seems somewhat inconvenient from the "linear theory" point of view, because of this conjugate-linearity... If we had a linear isometry of $\mathcal{H}^*$ and $\mathcal{H}$, than we would be able to get immediately the result below (which is true without this linearity, but with less immediate proof..).

Fact

If $\mathcal{H}$ is a Hilbert space, then $\mathcal{H}^*$ is also a Hilbert space, and the scalar product in $\mathcal{H}^*$ is given by the formula

$$(\eta_x, \eta_y)_{\mathcal{H}^*} := (y, x)_{\mathcal{H}} \quad \text{for } x, y \in \mathcal{H}. \qquad (1)$$

Proof

Observe, that by The H sp. Riesz th. the above formula defines $(\varphi, \psi)_{\mathcal{H}^*}$ for any $\varphi, \psi \in \mathcal{H}^*$. In fact (1) means that

$$(\varphi, \psi)_{\mathcal{H}^*} := (h^{-1}(\psi), h^{-1}(\varphi))_{\mathcal{H}}, \quad \varphi, \psi \in \mathcal{H}^*, \qquad (1')$$

so $(\ ,\ )_{\mathcal{H}^*}$ is first variable additive and $(\lambda\varphi, \psi)_{\mathcal{H}^*} = (h^{-1}(\psi), \bar{\lambda} h^{-1}(\varphi))_{\mathcal{H}} = \lambda (\varphi, \psi)_{\mathcal{H}^*}$ by conjugate linearity of h^{-1} (following by the same for h...) and of "$(x, \cdot)_{\mathcal{H}}$" for any x. The conjugate-symmetry is obvious, so it suffices to prove*) that $(\varphi, \varphi)_{\mathcal{H}^*} = \|\varphi\|^2_{\mathcal{H}^*}$ for any $\varphi \in \mathcal{H}^*$. But h^{-1} is an isometry, so

$$\|\varphi\|^2_{\mathcal{H}^*} = \|h^{-1}(\varphi)\|^2_{\mathcal{H}} = (h^{-1}(\varphi), h^{-1}(\varphi))_{\mathcal{H}} = (\varphi, \varphi)_{\mathcal{H}^*}. \qquad \boxed{\text{q}}$$

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*) $\mathcal{H}^*$ is a Banach space, so it is enough to check tha it is a Hilbert space it suffices that the norm is induced by a scalar product.

Corollary

Each Hilbert space is reflexive.

Proof $\longrightarrow$ (Hint: Let h_X be Riesz thm. conj.-linear isometry for the Hilbert space X and h_{X^*} - the analog one for X^*. Using the formula for the scalar product in X^* and X^{**}, check that $h_{X^*} \circ h_X = \varkappa$ - the canonical embedding of X into X^{**} - so it is <u>onto</u> ...)

$\boxed{4}$

Exercise / **Remark**

But does there exist <u>a linear</u> *) isometry of $\mathcal{H}$ onto $\mathcal{H}^*$ for any Hilbert space $\mathcal{H}$? — YES! Prove this finding it in any $\ell^2(\Omega)$ space (by the use of Riesz thm) first and then use thm. "On ℓ^2 representation of Hilbert space" p. HS-65. Try to use the found linear isometry to prove again the Corollary above ... . $\longrightarrow$

Measure Theory applications: Radon-Nikodym and Lebesgue theorems

The abstract Riesz representation result for Hilbert spaces was obtained quite easily, thank to our well-developed theory of Hilbert spaces from Section III. But it gives us a very strong practical tool, which can be applied to prove some "function existence results". We show here two such applications in "new" proofs of classical (but "difficult") measure theory results. One more application will be shown in sub-sub section 3.2 ...

*) Surely, it is an interesting problem for $\mathbb{K} = \mathbb{C}$ only... Recall also that "linear isometry = unitary transformation" for $\mathcal{H}$ - Hilbert space...

Definition

Let $(\Omega, \mathfrak{M}, \mu)$ be a measure space and let ν be a measure or a complex measure (see p. APP-2) on $\mathfrak{M}$. Then ν is absolutely continuous with respect to μ iff $\forall_{\omega \in \mathfrak{M}} \; \mu(\omega) = 0 \Rightarrow \nu(\omega) = 0$. We denote this relation shortly by

$$\nu \ll \mu .$$

Theorem

(The Radon – Nikodym Theorem)

Let $(\Omega, \mathfrak{M}, \mu)$ be a measure space with a σ-finite measure μ and let ν be

(i) a σ-finite measure on $\mathfrak{M}$

or

(ii) a complex measure *) on $\mathfrak{M}$.

If $\nu \ll \mu$ then there exists a measurable function $g : \Omega \to \mathbb{C}$ such that $\nu = g\,d\mu$ **) and:

---

*) Recall that each real measure is also a complex measure and each finite measure (non-negative...) is a real measure, but complex measure is <u>NOT a measure</u> "usually"...

**) Recall that $\nu = g\,d\mu$ iff $\forall_{\omega \in \mathfrak{M}} \; \nu(\omega) = \int_\omega g\,d\mu$ ("ν has the density g with respect to μ"; the name "derivative" or "the Radon–Nikodym derivative" is also used for g)

- for (i): g is nonnegative and a $\sigma - L^1(\mu)$ *) function
- for (ii): $g \in \tilde{\mathcal{L}}^1(\Omega, \mu)$.

$\boxed{\text{Proof}}$

We start from the "most important" (from the point of view of the application of The Hilbert space Riesz repr. th.) <u>case (iii)</u> ***): μ and ν are both finite measures (so both are $\geqslant 0$ – i.e. measures, in particular). In this case we define $\varrho := \mu + \nu$ and we consider the <u>real</u> "seminormed" space $\tilde{\mathcal{L}}^2 := \tilde{\mathcal{L}}^2(\Omega, \varrho)$ together with the real Hilbert space $\mathcal{H} := L^2(\Omega, \varrho)$. ****) For any $f \in \tilde{\mathcal{L}}^2$ we have by Hölder ineq.:

$$\int_\Omega |f|\, d\nu = \int_\Omega |f| \cdot 1\, d\nu \leq \Big(\int_\Omega |f|^2 d\nu\Big)^{\frac12} \big(\nu(\Omega)\big)^{\frac12} \leq C \cdot |||f|||_2 , \qquad (3)$$

with $C = (\nu(\Omega))^{\frac12}$ and $|||\ |||_2$ – the seminorm in $\tilde{\mathcal{L}}^2$ (we used $0 \leq \nu \leq \varrho$). So, f is ν-integrable for any $f \in \tilde{\mathcal{L}}^2$ and thus we can define $\tilde{\varphi} : \tilde{\mathcal{L}}^2 \to \mathbb{R}$ by $\tilde{\varphi}(f) := \int_\Omega f\, d\nu$ and $\tilde{\varphi} \in (\tilde{\mathcal{L}}^2)^\#$.
Moreover, by (3) we have $|\tilde{\varphi}(f)| \leq \int_\Omega |f|\, d\nu \leq C\, |||f|||_2$, so

---

*) i.e. there exists a sequence $\{\Omega_n\}_{n\geqslant 1}$ of sets from $\mathfrak{M}$ such that $\Omega = \bigcup_{n \geqslant 1} \Omega_n$ and $\forall_{n \geqslant 1} \int_{\Omega_n} |g|\, d\mu < +\infty$. Moreover, (here) we can get $\exists_{\tilde{C} \in \mathbb{R}} \forall_{n \geqslant n}\, \mu(\Omega_n) \leqslant \tilde{C}$.

***) The idea of the proof of case (iii) comes from J. von Neumann.

****) i.e. ("real") in the sense that $\tilde{\mathcal{L}}^2$ consists of <u>real</u> functions...

$\tilde{\varphi}$ is a seminorm-bounded linear functional (see p. OF-16). Also, if $|||f|||_2 = 0$ then $f=0$ ϱ-a.e., so also $f=0$ ν-a.e (because $0 \leq \nu \leq \varrho$, again). Thus, by Fact "On bounded operator from seminorm-bounded" (p. OF-17) there exists $\varphi \in \mathcal{H}^*$ given by the formula $\varphi([f]) \overset{*)}{:=} \tilde{\varphi}(f)$ for any $f \in \tilde{L}^2$.
This is the central point of the proof — we use The Hilb. sp. Riesz repr. th. and we get: there exists $y \in \mathcal{H}$ such that

$$\forall_{x \in \mathcal{H}} \quad \varphi(x) = (x, y).$$

Let us choose some $g_0 \in \tilde{L}^2$ such that $[g_0] = y$. So we have

$$\forall_{f \in \tilde{L}^2} \quad \int_\Omega f \, d\nu = \tilde{\varphi}(f) = \varphi([f]) = ([f], [g_0]) = \int_\Omega f \cdot g_0 \, d\varrho =$$

(note, that g_0 is real, since we used real spaces here)

$$= \int_\Omega f \cdot g_0 \, d\nu + \int_\Omega f \cdot g_0 \, d\mu .$$

This gives

$$\forall_{f \in \tilde{L}^2} \quad \int_\Omega f(1-g_0) \, d\nu = \int_\Omega f \cdot g_0 \, d\mu . \tag{4}$$

The measure ϱ is finite (since both ν and μ are), so the above formula can be used in particular for any $f = \chi_A$, where $A \in \mathfrak{M}$ $\left(\int_\Omega |\chi_A|^2 d\varrho \leq \varrho(\Omega)\right)$.

---

*) $[\cdot]$ is with respect to $\tilde{L}^2(\Omega, \varrho)$ in the whole proof here.

Note, that we have not used our main assumption "$\nu \ll \mu$" yet. We shall use it soon. Denote:

$$A_0 := \{t \in \Omega : g_0(t) < 0\}, \quad A_1 := \{t \in \Omega : g_0(t) \geqslant 1\}.$$

If $f_j = \chi_{A_j}$ for $j = 1, 2$, then by (4) we have for $j=0$

$$0 \leq \int_{A_0} (1 - g_0(t))\, d\nu(t) = \int_{A_0} g_0(t)\, \mu(t) \leq 0,$$

because $1 - g_0|_{A_0} > 1 > 0$, $g_0|_{A_0} < 0$. Thus $\int_{A_0}(1-g_0)\,d\nu =$
$= \int_{A_0} g_0\, d\mu = 0$ and so $\mu(A_0) = \nu(A_0) = 0$ again, by the above (strong) inequalities. Now, similarly, for $j=1$ we get:

$$0 \geqslant \int_{A_1} (1 - g_0(t))\, d\nu(t) = \int_{A_1} g_0(t) \mu(t) \geqslant 0$$

– here we have $1 - g_0|_{A_1} \leq 0$ ("$\leq$" only, and we are not sure, whether "$<$" holds ...) and $g_0|_{A_1} > 0$ – so, again, $\int_{A_1} g_0\, d\mu = 0$ and using this last "$>$", we get $\mu(A_1) = 0$. So, also $\nu(A_1) = 0$, by the abs. cont. of ν with respect to μ ! *)

Finally: $\rho(A_j) = 0$ for both $j = 0, 1$ and hence $\rho(A_0 \cup A_1) = 0$ and $g_0(t) \in [0;1)$ for $t \in \Omega \setminus (A_0 \cup A_1)$. So we define $\tilde{g}_0 : \Omega \to [0;1)$ by

$$\tilde{g}_0(t) := \begin{cases} g_0(t) & \text{for } t \in \Omega \setminus (A_0 \cup A_1) \\ 0 & \text{for } t \in A_0 \cup A_1, \end{cases}$$

and $g_1 := 1 - \tilde{g}_0$. So $\tilde{g}_0$ and g_1 are measurable, $g_1 > 0$, and $g_0(t) = \tilde{g}_0(t)$, $g_1(t) = 1 - g_0(t)$ for ν and μ a.e. $t \in \Omega$. We shall now guess g such, that $\nu = g\,d\mu$. Note that (4) can be now rewritten to

---

*) Note, that ~~without~~ using this abs. cont assumption we can get $\nu(\tilde{A}_1) = 0$ for $\tilde{A}_1 := \{t \in \Omega : g_0(t) > 1\} \subset A_1$, using the method as before. Also, for $D :=$ $\{t \in \Omega : g_0(t) = 1\}$ we get by (4) $0 = \int_D 0\, d\nu = \int_D 1\, d\mu = \mu(D)$ ... – see p. LF-63.

$$\forall_{f \in \tilde{L}^2} \quad \int_\Omega f g_1 \, d\nu = \int_\Omega f \tilde{g}_0 \, d\mu . \qquad (4')$$

This suggests to define $g := \tilde{g}_0 \cdot g_1^{-1}$. *) Fix some $\omega \in \mathfrak{M}$ and let $h = \chi_\omega$. Consider $\omega_n := \{t \in \omega : g_1(t) > \frac{1}{n}\}$ for $n \geq 1$ and let $h_n := \chi_{\omega_n}$. We have $\omega_{n+1} \supset \omega_n$ for $n \geq 1$ and $\bigcup_{n \geq 1} \omega_n = \omega$, because $g_1 > 0$. So $h_n \nearrow h$, $h_n g \nearrow hg$ and $h_n, h_n g \geq 0$. Therefore by Monotone Convergence Theorem we obtain

$$\int_\Omega h_n \, d\nu \longrightarrow \int_\Omega h \, d\nu \quad \text{and} \quad \int_\Omega h_n \cdot g \, d\mu \longrightarrow \int_\Omega hg \, d\mu . \quad (5)$$

Observe, that defining $f_n := h_n \cdot g_1^{-1}$ we get

$$f_n(t) = \begin{cases} g_1^{-1}(t) & \text{for } g_1(t) > \frac{1}{n} \\ 0 & \text{for } g_1(t) \leq \frac{1}{n} \end{cases}, \text{ so } \forall_{t \in \Omega} \; 0 \leq f_n(t) < n$$

– this means, that $f_n \in \tilde{L}^2$, because ϱ is a finite measure. So, we can use (4') for it, and by (5):

$$\begin{array}{ccccccc} \int_\Omega h_n d\nu & = & \int_\Omega f_n g_1 d\nu = \int_\Omega f_n \tilde{g}_0 \, d\mu & = & \int_\Omega h_n \cdot g \, d\mu \\ \downarrow & & & & \downarrow \\ \nu(\omega) = \int_\Omega h \, d\nu & & & & \int_\Omega h \cdot g \, d\mu = \int_\omega g \, d\mu , \end{array}$$

i.e. $\nu(\omega) = \int_\omega g \, d\mu$, which means that $\nu = g \, d\mu$. Moreover, for $\omega = \Omega$ we get $\int_\Omega g \, d\mu = \nu(\Omega) < +\infty$, so $g \in \tilde{L}_1(\Omega, \mu)$.

This finishes the proof for the case (iii) (treated as a both – "subcase" of (i) and of (ii)). The rest of the proof is purely a "measure-theory proof" –

---

*) We can guess this by taking informally "$f := h \cdot g_1^{-1}$" in (4') for $h = \chi_\omega$ with such ω, that $f \in \tilde{L}^2$ (– this is informall, since we do not know this apriori for all $\omega \in \mathfrak{M}$ ...)

We prove several next steps – special cases**) to get finally (i) and (iii), using the proved case (iii).

case (iv) (= "(i) ∩ (ii)"): ν is a finite measure and μ is a σ-finite measure.
We choose a countable $\mathfrak{M}$-measurable disjoint resolution $\{\Omega_n\}_{n\in\mathbb{N}}$ of Ω, such that $\mu(\Omega_n) < +\infty$ for any $n \in \mathbb{N}$. Now, for any $n\in\mathbb{N}$ we use the case (iii) for the pair of measures ν_n, μ_n on $(\Omega_n, \mathfrak{M}_n)$, where $\nu_n, \mu_n, \mathfrak{M}_n$ are the restrictions of $\nu, \mu, \mathfrak{M}$ to Ω_n, respectively*). We obtain the densities $g_n : \Omega_n \to [0;+\infty)$ such that $g_n \in \tilde{L}^1(\Omega_n, \mu_n)$ and $\nu_n = g_n d\mu_n$. Defining g as the set-theory sum of $\{g_n : n \in \mathbb{N}\}$ (in the graph sense) we easily get the assertion for this case ($\longrightarrow \triangle$).
case (v): ν is a real measure and μ is a σ-finite measure. We decompose ν into its parts ν_+, ν_- from the Jordan decomposition of real measure (see Appendix). So ν_+, ν_- are finite measures and $\nu = \nu_+ - \nu_-$. Now for $\nu_{+/-}$ we use case (iv) and we get ($\longrightarrow \triangle$ ..) the assertion.
case (ii) (ν is a complex measure and μ is a σ-finite measure) Wy apply case (v) for $\mathrm{Re}\,\nu$ and $\mathrm{Im}\,\nu$ ($\longrightarrow \triangle$).

case (i) (both ν and μ are σ-finite measures)
The proof is similar to the proof of case (iv), but now we decompose Ω for "the ν purpose" and we use case (iv) (or we can decompose

---

*) i.e., $\mathfrak{M}_n := \{\omega \in \mathfrak{M} : \omega \subset \Omega_n\}$, $\nu_n := \nu|_{\mathfrak{M}_n}$, $\mu := \mu|_{\mathfrak{M}_n}$.

**) those "special cases" mean always: the full assertion of the theorem for some special (extra) assumptions.

"more thing" for both ν and μ and use case (iii)) $\longrightarrow \triangle$.

It is worth noting that our Hilbert space Riesz representation th. argumentation used to prove the case (iii) of the Radon-Nikodym th. can be also used to get another important measure theory result.

Theorem (The Lebesgue Decomposition Theorem)

Let $(\Omega, \mathfrak{M}, \mu)$ be a measure space with a σ-finite measure μ and let ν be

(i) a σ-finite measure on $\mathfrak{M}$

or

(ii) a complex measure on $\mathfrak{M}$.

Then there exist a unique pair (ν_{ac}, ν_s) of:

- measures for case (i)
- complex measures for case (ii)

such that: $\nu = \nu_{ac} + \nu_s$ and $\nu_{ac} \ll \mu$, and ν_s is singular with respect to μ. *)

Proof

The structure of the proof is analogic to the previous one, i.e. we consider the same extra cases (iii), (iv), (v), and the way of obtaining all the cases ((iv), (v), (i), (ii)) from case (iii) is also the same ($\longrightarrow \triangle$). So, it suffices to prove case (iii).

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*) Definition: ν is singular with respect to μ iff there exists $\tilde{\Omega} \in \mathfrak{M}$ such that $\forall_{\omega, \omega' \in \mathfrak{M}} \left[(\omega \subset \tilde{\Omega} \text{ and } \omega' \subset (\Omega \setminus \tilde{\Omega})) \Rightarrow (\mu(\omega) = \nu(\omega') = 0) \right]$. The above means that ν "lives" on a μ-zero measure set (note, that the "singularity relation" is symmetric...). If both ν and μ are measures, it obviously suffices that $\mu(\tilde{\Omega}) = \nu(\Omega \setminus \tilde{\Omega}) = 0$ in the above definition.

To do this we just repeat with no changes the 1-st part of proof of The Radon-Nikodym Th. case (iii) until we "reach" the sentence "So, also..." on the 3-rd page of the proof (we started to use the a.c. of ν there, which is not our assumption now...). Then we use the footnote *) to those sentence together with the definition of D. Now we define ν_{ac} and ν_s on $\mathfrak{M}$ by

$$\nu_s(\omega) := \nu(D \cap \omega), \quad \nu_{ac}(\omega) := \nu((\Omega \setminus D) \cap \omega)$$

for any $\omega \in \mathfrak{M}$. So, ν_s and ν_{ac} are measures / complex measures on $\mathfrak{M}$, respectively (for (i)/(ii) cases) and $\nu = \nu_{ac} + \nu$. Moreover, obviously $\nu_s(\Omega \setminus D) = \nu(\emptyset) = 0$ and $\mu(D) = 0$ (by the mentioned footnote *)) – so ν_s is singular w.r. to μ. It remains now to prove that $\nu_{ac} \ll \mu$, which can be easily obtained by (4) p. LF – with the use of $g_0, A_1, \tilde{A}_1, A_0$ (and D) constructed before... $\longrightarrow$. The unicity of the choice of ν_{ac} and ν_s follows simply from the definition of absolutely continuous and of singular measures / complex measures ($\longrightarrow$)

Example

Let μ be a complex measure on $(\Omega, \mathfrak{M})$ and consider $\mathrm{var}(\mu)$ – it is a finite measure on $(\Omega, \mathfrak{M})$ (see App), hence we can decompose μ with respect to $\mathrm{var}(\mu)$. So $\mu = \mu_{ac} + \mu_s$ where μ_{ac}, μ_s are complex measures abs. cont., singular with respect to $\mathrm{var}(\mu)$, respectively. But observe that by the definition of $\mathrm{var}(\mu)$ we have

$$0 \le |\mu(\omega)| \le (\mathrm{var}\,\mu)(\omega)$$

for any $\omega \in \mathfrak{M}$ ($\{\omega\}$ is 1-element decomposition of ω...). So $\mu(\omega) = 0$, if $(\mathrm{var}\,\mu)(\omega) = 0$. This means, that μ is absol. cont. with respect to $\mathrm{var}\,\mu$. But the zero (complex) measure is singular with resp. to $\mathrm{var}\,\mu$, obviously, and $\mu = \mu + 0$. Hence, by the unicity in The L. D. Th. we get $\mu_{ac} = \mu$, $\mu_s = 0$. Now we can use The R–N Th. for μ, and it gives us the existence of a density $g \in \tilde{L}^1(\Omega, \mathrm{var}\,\mu)$ such that $\mu = g\, d(\mathrm{var}\,\mu)$. The following nice and convenient result can be proved.

$\boxed{\text{Fact}}$ ("On the density with respect to the variation")

If μ is a complex measure on $(\Omega, \mathfrak{M})$, then there exists a measurable function $g : \Omega \to \{z \in \mathbb{C} : |z| = 1\}$ such that $\mu = g\, d(\mathrm{var}\,\mu)$.

$\boxed{\text{Proof}}$ $\longrightarrow$ $\triangle$

3.2. L^p-functionals

Let $\Omega \neq \emptyset$ and assume that $(\Omega, \mathfrak{M}, \mu)$ is a measure space.

In 3.1 we started our considerations on the exact form of each functional $\varphi \in \mathcal{H}^*$ by constructing some "natural" functionals η_y for any $y \in \mathcal{H}$. We shall proceed similarly for all the $\mathbb{L}^p := \mathbb{L}^p(\Omega, \mu)$ with $p \in [1; +\infty]$. – Now the role analogic to the role of η_y-s will be played by the functionals $\Psi_y \in (\mathbb{L}^p)^*$ defined in (3''') p. OF-62 for any $y \in \mathbb{L}^{(p^*)} := \mathbb{L}^{(p^*)}(\Omega, \mu)$. Recall here, that $p^* \in [1; +\infty]$ is given by $\frac{1}{p} + \frac{1}{p^*} = 1$ (with $\frac{1}{+\infty} = 0$) and that

$$\Psi_y(x) := \int_\Omega f \cdot g \, d\mu$$

for any f and g, such that $[f] = x \in \mathbb{L}^p$, $[g] = y \in \mathbb{L}^{(p^*)}$,

i.e.

$$\Psi_{[g]}([f]) = \int_\Omega f \cdot g \, d\mu \quad , \; [f] \in \mathbb{L}^p, \, [g] \in \mathbb{L}^{(p^*)}.$$

Recall also that $\|\Psi_y\| \leq \|y\|_{(p^*)}$ by Fact p. OF-62, where (! – be careful) we changed slightly the previous notation for the case $p^* = +\infty$ (i.e, $p=1$) and $\| \ \|_\infty$ means here $\| \cdot \|_{ess\,\infty}$ – i.e the standard norm i L^∞ space!

It turns out that the situation is much better and "almost as good" as for the Hilbert space case ...

$\boxed{\text{Theorem}}$ ("The L^p-space representation theorem")

Suppose that $p \in [1;+\infty]$ and, for the case $p=1$, that μ is σ-finite.

(a) The map $I_p : L^{(p^*)} \longrightarrow (L^p)^*$ defined by

$$I_p(y) := \Psi_y \ , \quad y \in L^{(p^*)}$$

is a linear isometric embedding of $L^{(p^*)}$ (onto its image in $(L^p)^*$).

(b) If $p \in [1;+\infty)$ then I_p is an isometry onto $(L^p)^*$.

We shall prove this theorem for some special case for (b) – we shal assume the σ-finitness for any $p \in [1;+\infty)$, not only for $p=1$ (this restriction to σ-finite measures is related to the fact, that we want to lean on the Radon-Nikodym th.). We shall also leave many technical details to be proved by the Readers...

$\boxed{\text{Proof}}$

(a) The linearity is obvious and we have also $\|\Psi_y\| \leq \|y\|_{p^*}$, so it suffices to prove that $\|\Psi_y\| \geqslant \|y\|_{p^*}$. This is related to the problem of "equality in Hölder inequality":

Fix g such that $y = [g]$ and for $p > 1$ define f by

$$f(t) := \begin{cases} \overline{\operatorname{sgn}(g(t))} \cdot |g(t)|^{p^*/p} & p \in (1; +\infty)\ (\ni p^*) \\ \overline{\operatorname{sgn}(g(t))} & p = +\infty\ (p^* = 1), \end{cases}$$

$t \in \Omega$,

where $\operatorname{sgn} z := \begin{cases} \frac{z}{|z|} & \text{for } z \neq 0 \\ 1 & \text{for } z = 0 \end{cases}, \quad z \in \mathbb{C}$

Observe that for any $z \in \mathbb{C}$

$$|\operatorname{sgn} z| = 1, \quad \operatorname{sgn} z \cdot \overline{\operatorname{sgn} z} = 1, \quad z = (\operatorname{sgn} z) \cdot |z|$$

Using this and tha fact that $p^* \cdot p = p^* + p$ one can easily check, that $[f] \in L^p$ and

$$\int_\Omega fg \, d\mu = \|[f]\|_p \cdot \|[g]\|_{p^*},$$

and $[f] \neq 0$ if $[g] \neq 0$. This proves that $\|\Psi_y\| \geq \|y\|_{p^*} \left(= \|[g]\|_{p^*}\right)$ and in fact $\|\Psi_y\| = \|y\|_{p^*}$ for $p > 1$. If $p = 1$ we assumed that μ is σ-finite. Let $c = \|y\|_{p^*} = \|[g]\|_\infty$. Hence for any $k \in \mathbb{N}_1$ there exists $\omega_k \in \mathfrak{M}$ satisfying:

$$0 < \mu(\omega_k) < +\infty, \quad \forall_{t \in \omega_k} \ |g(t)| \geq c - \frac{1}{k}$$

$(\longrightarrow \triangle 14)$. Suppose that $c > 0$ (if $c = 0$ then obviously $y = 0$ and $\Psi_y = 0$, i.e. $\|\Psi_y\| = \|y\| = 0$) then for $k \geq 1$ we can define

$f_k := \overline{\operatorname{sgn} g} \cdot \chi_{\omega_k}$. We have then $[f_k] \in L^1$ and $\|[f_k]\|_1 =$

$= \int_\Omega |f_k| d\mu = \mu(\omega_k) \neq 0$. Moreover $\Psi_y([f_k]) = \int_\Omega f_k g \, d\mu =$

$$= \int_{\omega_k} (\overline{\operatorname{sgn} g} \cdot \operatorname{sgn} g) |g| \, d\mu \geq \int_{\omega_k} (c - \tfrac{1}{k}) \, d\mu = (c - \tfrac{1}{k}) \cdot \mu(\omega_k) =$$

$= (c - \frac{1}{k}) \|[f_k]\|_1$. Hence $\forall_{k \geq 1}$ $\|\Psi_y\| \geq c - \frac{1}{k}$, so $\|\Psi_y\| \geq c$ – finally $\|\Psi_y\| = c = \|y\|_\infty$, and (a) is proved.

(b) This is the main (and the most difficult) part of the proof (and of the theorem...). First we assume extra, that $\mu(\Omega) < +\infty$. Then for any $\omega \in \mathfrak{M}$ $[\chi_\omega] \in L^p$, so if $\varphi \in (L^p)^*$, then we can define $\nu : \mathfrak{M} \longrightarrow \mathbb{K}$ by

$$\nu(\omega) = \varphi([\chi_\omega]), \quad \omega \in \mathfrak{M}. \qquad (1)$$

It can be easily checked (here $p < +\infty$ is important!) that ν is a complex measure $\longrightarrow \triangle$. Moreover, if $\mu(\omega) = 0$, then $[\chi_\omega] = 0$ in L^p, so $\nu(\omega) = 0$ by (1), because φ is linear. This means, that ν is absolutely continuous with respect to μ. Thus by Radon - Nikodym thm. there exists $g_0 \in \tilde{L}^1$ such that $\nu = g_0 d\mu$, i.e.

$$\forall_{\omega \in \mathfrak{M}} \quad \nu(\omega) = \int_\omega g_0 \, d\mu = \int_\Omega \chi_\omega g_0 \, d\mu. \qquad (2)$$

Our goal is to prove that $\varphi = \Psi_y$ for some $y \in L^{(p^*)}$, which means that there exists $g \in \widetilde{L} := \begin{cases} \widetilde{L}^{(p^*)} & \text{for } p^* < +\infty \ (p>1) \\ \mathcal{M}_b & \text{for } p^* = +\infty \ (p=1) \end{cases}$ such that

$\mathcal{M}_b = \mathcal{M}_b(\Omega, \mu)$

$$\varphi([f]) = \int_\Omega f g \, d\mu \quad \text{for any } f \in \widetilde{L}^p. \qquad (3)$$

We shall prove that this is true for $g = g_0$ – – so we should prove that $g_0 \in \widetilde{L}$ and that (3) holds. Note, that by (1) and (2) the equality from (3) (with $g = g_0$) is satisfied by any $f = \chi_\omega$, $\omega \in \mathfrak{M}$, and hence also for any simple function, by linearity. But recall, that each bounded measurable function f is a uniform limit of a sequence of simple functions $\{f_n\}_{n \geq 1}$, so we have

$$\varphi([f_n]) = \int_\Omega f_n g_0 \, d\mu \quad \text{for any } n \geq 1, \text{ and}$$

$[f_n] \longrightarrow [f]$ in L^p ($\int_\Omega |f_n - f|^p d\mu \to 0$, because μ is finite), so $\varphi([f_n]) \longrightarrow \varphi([f])$ by continuity. We also have

$\int_\Omega f_n g_0 \, d\mu \longrightarrow \int_\Omega f g_0 \, d\mu$ by Lebesgue dominated convergence thm, so

$$\varphi([f]) \overset{*)}{=} \int_\Omega f g_0 \, d\mu \quad \text{also for any } f \in \mathcal{M}_b. \qquad (3')$$

If $p^* < +\infty$ $(p > 1)$, then we have:

$$|g_0|^{p^*} = |g_0|^{p^*-1} \cdot |g_0| = |g_0|^{p^*-1} \cdot (\operatorname{sgn} g_0)^{-1} g_0.$$

For any $n \geq 1$ consider $\Omega_n := \{t \in \Omega : |g_0(t)| \leq n\}$.
Now applying (3') to $\tilde{f}_n := \chi_{\Omega_n} |g_0|^{p^*-1} (\operatorname{sgn} g_0)^{-1} \in \mathcal{M}_b$ we get

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*) But $[f]$ is here in the L^p sense here (despite the fact that $f \in \mathcal{M}_b$)

$$\int_{\Omega_n} |g_0|^{p^*} d\mu = \int_{\Omega} \tilde{f}_n \cdot g_0 \, d\mu = \varphi([\tilde{f}_n]) = |\varphi([\tilde{f}_n])| \leq$$

$$\leq \|\varphi\| \cdot \|[\tilde{f}_n]\|_p = \|\varphi\| \Big(\int_{\Omega_n} |g_0|^{(p^*-1)p}\Big)^{\frac{1}{p}} = \|\varphi\| \Big(\int_{\Omega_n} |g_0|^{p^*}\Big)^{\frac{1}{p}},$$

because $(p^*-1)p = p^*p - p = p^* + p - p = p^*$. So

$$\Big(\int_{\Omega_n} |g_0|^{p^*} d\mu\Big)^{1-\frac{1}{p}} \leq \|\varphi\|$$, and by $\frac{1}{p^*} + \frac{1}{p} = 1$

$$\int_{\Omega} |g_0|^{p^*} d\mu = \lim_{n \to +\infty} \int_{\Omega_n} |g_0|^{p^*} d\mu \leq \|\varphi\|^{p^*},$$ so $g_0 \in \tilde{L} = \tilde{L}^{p^*}$.

<u>If $p^* = +\infty$, i.e., $p = 1$</u>, then we can apply $(3')$ to any $f = \chi_\omega$ for $\omega \in \mathfrak{M}$ and

$$\Big|\int_{\omega} g_0 \, d\mu\Big| = |\varphi([\chi_\omega])| \leq \|\varphi\| \, \|[\chi_\omega]\|_1 = \|\varphi\| \mu(\omega),$$

which proves $(\longrightarrow \triangle)$ that $|g_0| \leq \|\varphi\|$ μ-a.e., so $\|g_0\|_\infty \leq \|\varphi\|$ and $g_0 \in \mathcal{M}_b$. Hence in both cases $g_0 \in \tilde{L}$, which means that we can consider $\Psi_{[g_0]} \in L^{(p^*)}$. And $(3')$ means that φ and $\Psi_{[g_0]}$ restricted to $X := \{[f] : f \in \mathcal{M}_b\}$ – so $\varphi = \Psi_{[g_0]}$, by the continuity of $\varphi, \Psi_{[g_0]}$, because X is dense in L^p. – So, the ~~proof~~ proof for the extra assumption $\mu(\Omega) < +\infty$ is finished.

Suppose now that μ is σ-finite "only".
Decompose Ω onto $\mathfrak{M}$-measurable disjoint sets G_n of finite measures, $n \geq 1$. Let $S_n := \bigcup_{k=1}^{n} G_k$ and consider the appropriate spaces $\tilde{\mathcal{L}}^p_n, \tilde{\mathcal{L}}_n, L^p_n, L^{(p^*)}_n$ for measure spaces $(G_n, \mathfrak{M}_n, \mu_n)$, where $\mathfrak{M}_n := \{\omega \in \mathfrak{M} : \omega \subset G_n\}$ and $\mu_n := \mu_{|\mathfrak{M}_n}$ ($\tilde{\mathcal{L}}_n$ is equal $\tilde{\mathcal{L}}^{(p^*)}$ if $p < +\infty$ and $\mathcal{M}_b$ if $p = +\infty$ for this measure spaces)
Denote also $i_n : \tilde{\mathcal{L}}^p_n \to \tilde{\mathcal{L}}^p \; (= \tilde{\mathcal{L}}^p(\Omega, \mathfrak{M}, \mu))$ – the operator of "extrapolation by 0" from G_n to Ω, i.e.

$$(i_n f)(t) = \begin{cases} f(t) & t \in G_n \\ 0 & t \notin G_n \end{cases} \quad \text{for } f \in \tilde{\mathcal{L}}^p_n.$$

Obviously $\|[i_n f]\|_p = \|[f]\|_p$ *) for any $f \in \tilde{\mathcal{L}}^p_n$. Hence, if $\varphi \in (L^p)^*$, then φ_n given by $\varphi_n([f]) = \varphi([i_n f])$ for $f \in \tilde{\mathcal{L}}^p_n$ is properly defined, and $\varphi_n \in (L^p_n)^*$, moreover

$$\forall_{n \geq 1} \quad \|\varphi_n\| \leq \|\varphi\|.$$

But each μ_n is a finite measure, thus we can choose $g_n \in \tilde{\mathcal{L}}_n$ for each n, such that $\varphi_n([f]) = \int_{\Omega_n} f g_n \, d\mu_n$ (by the previously proved special case). Now, define $g : \Omega \to \mathbb{C}$ by $g(t) := g_n(t)$ if $t \in G_n$. We have obviously $g \in \tilde{\mathcal{L}}^{p^*}$ if $p^* < +\infty$.
If $p^* = +\infty$ ($p = 1$), then we need $g \in \mathcal{M}_b$ – it is less obvious, but we shall prove it soon... (only the boundedness needs a proof).

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*) We use the same symbol $\| \; \|_p$ for L^p and for L^p_n.

Consider $u \in \tilde{L}^p$. Since $p \in [1; +\infty)$, we have $[u] = \sum_{n=1}^{+\infty} [\chi_{G_n} u]$, where the series is convergent in L^p. Thus

$$\varphi([u]) = \sum_{n=1}^{+\infty} \varphi([\chi_{G_n} u]) = \sum_{n=1}^{+\infty} \varphi([i_n(u_{|G_n})]) =$$

$$= \sum_{n=1}^{+\infty} \varphi_n([u_{|G_n}]) = \sum_{n=1}^{+\infty} \int_{G_n} (u_{|G_n}) \cdot g_n \, d\mu_n =$$

$$= \sum_{n=1}^{+\infty} \int_{G_n} u \cdot g \, d\mu . \qquad (4)$$

Observe, that if $p > 1$ ($p^* < +\infty$), then the RHS of (4) is convergent to $\int_\Omega u g \, d\mu = \Psi_{[g]}([u])$ by the Dominated Conv. th. (since $u \in \tilde{L}^p$, $g \in \tilde{L}^{(p^*)}$, so $ug \in \tilde{L}^1$ by the Hölder ineq.). So $\varphi = \Psi_{[g]}$ in this case. It remains to study $p = 1$ with $p^* = +\infty$. Fix $M \in \mathbb{R}_+$ and consider $A_M := \{t \in \Omega : |g(t)| > M\}$. If $\mu(A_M) > 0$, then also $\mu(A_M \cap S_N) > 0$ for some $N \geq 1$. Using (4) to $u = \chi_{(A_M \cap S_N)} \cdot (\operatorname{sgn} g)^{-1}$

$$\varphi([u]) = \sum_{n=1}^{N} \int_{G_n \cap A_M} |g| \, d\mu = \int_{S_N \cap A_M} |g| \, d\mu \geq \int_{S_N \cap A_M} M \, d\mu = M \mu(S_N \cap A_M)$$

but $\varphi([u]) = |\varphi([u])| \leq \|\varphi\| \cdot \int_{A_M \cap S_n} 1 \, d\mu = \|\varphi\| \cdot \mu(A_M \cap S_n)$ –

– hence $M \leq \|\varphi\|$ i.e. $\mu(A_M) = 0$ for $M > \|\varphi\|$, i.e. g is bounded μ-a.e. – this allows to finish the proof also in this case $\rightarrow \triangle$.

$\square$

Remark

In our proof (and also generally if $p=1$) we considered only σ-finite measures. Observe however, that our previous result (the Hilbert space Riesz repr. thm. p. LF-52) gives easily the proof for $p=2$ <u>without</u> this extra assumption ($\longrightarrow$).

Our L^p-space repr. th. gives also a reflexivity information.

Corrolary

For any measure space $(\Omega, \mathfrak{M}, \mu)$ and $p \in (1;+\infty)$ $L^p(\Omega,\mu)$ is reflexive.

Proof The fact that the canonical embedding $\varkappa : X \to X^{**}$ is <u>onto</u> $X^{**} = \left(L^p(\Omega,\mu)\right)^{**}$ is easy to check with the use of the isometry I_p (see also the hint for Hilbert space reflexibility proof).

The spaces $L^1(\Omega,\mu)$ and $L^\infty(\Omega,\mu)$ are "usually" not reflexive.

Example

$\ell^1(\mathbb{N})$ and $\ell^\infty(\mathbb{N})$ are not reflexiv ! $\longrightarrow$

3.3. C(K) functionals

Let K be a compact *) topological space with topology $\mathcal{T}$.

Some "obvious" continuous functionals and regularity

Consider the <u>Borel σ-algebra for K</u>: $\mathcal{B}(K) := \sigma(\mathcal{T})$, and let $\mu \in \ell^\infty_{add}(\mathcal{B}(K))$ (i.e. μ is an additive Borel $\mathbb{K}$-measure which is bounded – see Apendix A.1.).

Each $f \in C(K)$ is a Borel ($\mathcal{B}(K)$-measurable) and bounded function, i.e $C(K) \subset \mathcal{M}_b(K, \mathcal{B}(K))$. Hence, we can integrate functions from $C(K)$ with respect to μ (see p. APP–15/16). Let $S_\mu : C(K) \longrightarrow \mathbb{K}$ be given by

$$S_\mu(f) := \int_K f \, d\mu, \quad f \in C(K). \qquad (1)$$

Theorem "On integral" (p. APP–16) gives in particular, that $S_\mu \in (C(K))^*$ and that $\|S_\mu\| \leq \|\mu\|_{var} = (\operatorname{var}\mu)(K)$.

It may seem strange to you, but there <u>is no</u> equality above, in general, I.e., it may happen that $\|S_\mu\| < \|\mu\|_{var}$ and, moreover, that $S_\mu = 0$, but $\|\mu\|_{var} > 0$! **) – so the mapping $\mu \rightsquigarrow S_\mu$ is not "one-to-one" then, when we consider it on the whole $\ell^\infty_{add}(\mathcal{B}(K))$. It is also

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*) Recall that compact is Haussdorf in particular (in this Lecture/Script)

**) It can be particularly strange when you compare it with the "1-1" properties of mappings $y \rightsquigarrow I_y$, $y \rightsquigarrow \eta_y$ from two previous representation thm-s. But it is not easy to construct the appropriate example of μ...

interesting, that the analog mapping

$$\ell^{\infty}_{add}(B(K)) \ni \mu \mapsto \tilde{S}_{\mu} \in (\mathcal{M}_b(K, B(K)))^*$$

is one-to-one (and is an isometry), where $\tilde{S}_{\mu}$ is given as in (1), but for all $f \in \mathcal{M}_b(K, B(K))$ (see Fact "On the representation of $\mathcal{M}_b(\Omega, \mathfrak{M})$ functionals" p. LF-78).

But here, we restrict our attention to the $C(K)$ space...

Note, that the mentioned "not one-to-one paradox" for $\mu \mapsto S_{\mu}$ suggests, that some additive measures $\mu \in \ell^{\infty}_{add}(B(K))$ can be just omitted if we want to describe the functionals on $C(K)$. The first idea is to consider only $\mathbb{K}$-measures (i.e. $\mu \in \ell_{\sigma\text{-}add}(B(K))$) instead of all the additive and bounded ones. This means that we restrict the "large" class of all the additive (+bdd.) measures to the much smaller class of σ-additive $\mathbb{K}$-measures.

– But another "strange" example (also non-trivial ...) shows that the mapping $\ell_{\sigma\text{-}add}(B(K)) \ni \mu \mapsto S_{\mu} \in (C(K))^*$ can be still not one-to-one fore some "strange" compact spaces K...

So – there are +/– two ways: we can consides only some "better" spaces K or restrict again the class of "measures" – now the σ-additive ones. Fortunately, the appropriate restriction is quite natural and easy to define, so we can choose the second (more general) way...

Suppose that X is a topological space with topology $\mathcal{T}$ and $\mathfrak{M}$ is a σ-algebra for X such that $\mathfrak{M} \supset \mathcal{T}$ (i.e., $\mathfrak{M} \supset \mathcal{B}(X)$ - the Borel sets in X; e.g. $\mathfrak{M} = \mathcal{B}(X)$) and let μ be a measure or a complex measure on $\mathfrak{M}$.

$\boxed{\text{Definition}}$

μ is regular iff $\forall_{\substack{\varepsilon>0 \\ A\in\mathfrak{M}}} \exists_{\substack{G\text{ - open} \\ F\text{ - closed} \\ G\supset A\supset F}} \forall_{\substack{\omega\subset G\setminus F \\ \omega\in\mathfrak{M}}} |\mu(\omega)| < \varepsilon.$

$\boxed{\text{Remarks}}$

- Obviously, if μ is a measure, then the part "$\forall_{\omega\ldots} \cdots$" above can be just replaced by "$\mu(G\setminus F) = 0$"
- If μ is a complex measure, then μ is regular iff $\operatorname{var}\mu$ is regular $(\longrightarrow \triangle)$.

Denote by **$\ell_{\sigma\text{-add-reg}}(\mathfrak{M})$** the set (linear space and a Banach space ... $\longrightarrow \triangle$ - the norm subspace of $\ell_{\sigma\text{-add}}(\mathfrak{M})$) of all regular $\mathbb{K}$-measures on $\mathfrak{M}$.

A propos our first posible way (see the lower part of p. LF-75 ...) let us note the following result, showing that measures/compl. meas. are "often" regular.

Fact

If X is a metric space, then each Borel finite measure (and thus each Borel complex measure) in X is regular.

Proof $\longmapsto$ ćw $\square$

Note, that the above assumption (on X) holds e.g. for each compact <u>metric</u> space.

The main result of this sub-sub section – the Riesz thm. for $C(K)$ is based on a "delicate" result which says, that any (Borel) additive $\mathbb{K}$-measure can be replaced by a regular measure if we want to integrate only continuous functions on compact K...

Lemma ("On σ-additive regularisation of additive measure")

If $\tilde{\mu} \in l^{\infty}_{add}(\mathcal{B}(K))$ then there exists $\mu \in l_{\sigma\text{-add-reg}}(\mathcal{B}(K))$ such that $(\operatorname{Var}\mu)(K) = (\operatorname{Var}\tilde{\mu})(K)$ and

$$\forall_{f \in C(K)} \quad \int_K f\,d\mu = \int_K f\,d\tilde{\mu}. \qquad (2)$$

Proof (Sketch) Suppose that $\tilde{\mu} \geqslant 0$.

For any open $G \subset K$ we define $\check{\mu}(G) := \sup\{\tilde{\mu}(F) : F\text{-closed}, F \subset G\}$ and for any $A \in 2^K$ $\mu_{out}(A) := \inf\{\check{\mu}(G) : G\text{-open}, A \subset G\}$.

- Now we prove, that μ_{out} is an outer measure.
- Then, we use The Carathéodory thm. to constr.ct the measure – we prove that each open set satisfy the Carathéodory condition. This proves, that this measure can be defined on $\mathcal{B}(K)$ – this is the measure μ, we search.
- It can be proved, that μ:
  (a) is regular
  (b) satisfies $(\mathrm{var}\,\tilde{\mu})(K) = (\mathrm{var}\,\mu)(K)$
  (c) satisfies (2)

Unfortunately all the above "points" to prove are rather technically complicated... The general case, when $\tilde{\mu}$ is not $\geqslant 0$ is easy to obtain now by the decomposition of $\tilde{\mu}$ onto $\mathrm{Re}\,\tilde{\mu}$ and $\mathrm{Im}\,\tilde{\mu}$ (if $\mathbb{K}=\mathbb{C}$) and then, by the Jordan decomposition onto "$\geqslant 0$" parts (p. APP-17). $\boxed{\text{K}}$

Bounded measurable functions functionals

Before we formulate (and prove) our main result for $C(K)^*$, we shall study much easier problem of representation of all $\mathcal{M}_b(K, \mathcal{B}(K))$ functionals, which can be with no problems generalized for $\mathcal{M}_b(\Omega, \mathfrak{M})$ for any σ-algebra $(\mathfrak{M})$ for any Ω.

$\boxed{\text{Theorem}}$ ("On $(\mathcal{M}_b(\Omega, \mathfrak{M}))^*$ representation")

If $\mathfrak{M}$ is a σ-algebra of subsets of $\Omega \neq \emptyset$, then the map

$$l^{\infty}_{add}(\mathfrak{M}) \ni \tilde{\mu} \rightsquigarrow \tilde{S}_{\tilde{\mu}} \in (\mathcal{M}_b(\Omega, \mathfrak{M}))^*, \quad \tilde{S}_{\tilde{\mu}} f := \int_\Omega f \, d\tilde{\mu} \text{ for}$$

$f \in \mathcal{M}_b(\Omega, \mathfrak{M})$ is a linear isometry onto $(\mathcal{M}_b(\Omega, \mathfrak{M}))^*$.

Note, that $\tilde{S}_{\tilde{\mu}}$ defined above is a continuous linear functional by Thm. "On integral" (p. APP-16).

$\boxed{\text{Proof}}$

Let $\phi(\tilde{\mu}) := \tilde{S}_{\tilde{\mu}}$ for $\tilde{\mu} \in l^{\infty}_{add}(\mathfrak{M})$. Obviously ϕ is linear, moreover for any $\tilde{\mu}$ $\|\tilde{S}_{\tilde{\mu}}\| \leq (\operatorname{var}\tilde{\mu})(\Omega) = \|\tilde{\mu}\|$.
We shall prove that $\|\tilde{S}_{\tilde{\mu}}\| = \|\tilde{\mu}\|$. Let $\varepsilon > 0$ and choose a finite $\mathfrak{M}$-d.d. $\{\omega_j\}_{j=1,\dots n}$ of Ω such that
$\sum_{j=1}^{n} |\tilde{\mu}(\omega_j)| \geq (\operatorname{var}\tilde{\mu})(\Omega) - \varepsilon$. Define:

$$f := \sum_{j=1}^{n} \operatorname{sgn}(\tilde{\mu}(\omega_j))^{-1} \cdot \chi_{\omega_j}$$

then f is a simple function ($\|f\|_{sup} = 1$), so $f \in \mathcal{M}_b(\Omega, \mathfrak{M})$, and

$$|\tilde{S}_{\tilde{\mu}}(f)| = \left|\int_{\Omega} f \, d\tilde{\mu}\right| = \left|\sum_{j=1}^{n} \operatorname{sgn}(\tilde{\mu}(\omega_j))^{-1} \cdot \tilde{\mu}(\omega_j)\right| = \sum_{j=1}^{n} |\tilde{\mu}(\omega_j)| \geq$$
$$\geq (\operatorname{var}\tilde{\mu})(\Omega) - \varepsilon$$

So $\|\tilde{S}_{\tilde{\mu}}\| \geq (\operatorname{var}\tilde{\mu})(\Omega)$, i.e. "=" holds. Thus, it remains to prove, that ϕ is "onto". Let $\varphi \in (\mathcal{M}_b(\Omega, \mathfrak{M}))^*$. Observe that for any $\omega \in \mathfrak{M}$ its characteristic function $\chi_\omega \in \mathcal{M}_b(\Omega, \mathfrak{M})$. So define $\tilde{\mu} : \mathfrak{M} \longrightarrow \mathbb{K}$ by the formula

$$\tilde{\mu}(\omega) := \varphi(\chi_\omega), \quad \omega \in \mathfrak{M}.$$

Observe, that if ω_1, ω_2 are disjoint, then $\chi_{\omega_1 \cup \omega_2} = \chi_{\omega_1} + \chi_{\omega_2}$, so $\tilde{\mu}$ is additive, by the linearity of φ. It is also bounded, since $|\tilde{\mu}(\omega)| = |\varphi(\chi_\omega)| \leq \|\varphi\| \cdot \|\chi_\omega\|_{sup}$, but $\|\chi_\omega\|_{sup} = 1$ or 0.... So $\tilde{\mu} \in l^\infty_{add}(\mathcal{M})$. Now, observe that for any simple function f, by the linearity of φ and $\tilde{S}_{\tilde{\mu}}$, $\varphi(f) = \tilde{S}_{\tilde{\mu}}(f)$ holds. But this gives $\varphi = \tilde{S}_{\tilde{\mu}}$ by the continuity of φ and $\tilde{S}_{\tilde{\mu}}$ and by the fact that the set of simple functions is dense in $\mathcal{M}_b(\Omega, \mathcal{M})$.

$\boxed{\mu}$

The main Riesz theorem for $C(K)$

We are ready now to formulate and quickly prove our main result

$\boxed{\text{Theorem}}$ ("The Riesz representation Theorem for $C(K)$")

Let K be a compact space. For any $\varphi \in (C(K))^*$ there exists a unique Borel regular $\mathbb{K}$-measure*) μ on K, such that

$$\forall_{f \in C(K)} \quad \varphi(f) = \int_K f \, d\mu;$$

moreover $\|\varphi\| = \|\mu\|_{var}$. The map

$$l_{\sigma\text{-add-reg}}(\mathcal{B}(K)) \ni \mu \mapsto S_\mu \in (C(K))^*$$

is a linear isometry onto $(C(K))^*$.

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*) The choice of $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ here depends of the type of the (real or complex) functions which we consider writting $C(K)$.

$\boxed{\text{Proof}}$

The map described above is obviously a linear map, so it suffices to prove the first part.*) Let $\varphi \in (C(K))^*$. Observe that $B(K)$ is a $\underline{\underline{\sigma}}$-algebra and it contains all open sets, so $C(K) \subset \mathcal{M}_b := \mathcal{M}_b(K, B(K))$. Hence, we can use Hahn-Banach Thm. ("The Continuous H-B. Thm. – p. LF-17) and let $\tilde{\varphi} \in (\mathcal{M}_b)^*$ be an extension of φ onto $\mathcal{M}_b$ with $\|\tilde{\varphi}\| = \|\varphi\|$.

Now, by thm. "On $\mathcal{M}_b^*$ representation" let $\tilde{\mu}$ be the appropriate additive (bounded) measure determined for φ by thm "On $\mathcal{M}_b^*$ representation (p. LF-78). So we have in particular

$$\forall_{f \in C(K)} \quad \varphi(f) = \tilde{\varphi}(f) = \int_K f \, d\tilde{\mu}$$

and $\quad \|\tilde{\mu}\|_{var} = \|\tilde{\varphi}\|$.

The next step is to use the Lemma "On σ-additive regul...." p. LF-77: let $\mu \in l_{\sigma\text{-add-reg}}(B(K))$ be the regular $\mathbb{K}$-measure such that

$$\forall_{f \in C(K)} \quad \int_K f \, d\tilde{\mu} = \int_K f \, d\mu$$

and $(\operatorname{var} \tilde{\mu})(K) = (\operatorname{var} \mu)(K)$. This finishes our proof, because

$$\|\mu\|_{var} = (\operatorname{var} \mu)(K) = (\operatorname{var} \tilde{\mu})(K) = \|\tilde{\mu}\|_{var} = \|\tilde{\varphi}\| = \|\varphi\|. \qquad \boxed{\text{□}}$$

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*) Note, that the unicity follows from the isometricity...

Remark

The Riesz representation theorem can be used to better understand "the structure" and properties of $C(K)$ spaces, because the measure theory (and Lebesgue integration) is very well developped mathematical theory.
It can be used e.g. to see that "usually" (for which K? → ⚠) $C(K)$ is not reflexible (however it is not difficult to see it also without the use of the representation thm.).

Example / Exercise

Considering the set $\dot{\mathbb{N}} := \mathbb{N} \cup \{+\infty\}$ and defining there the "natural" topology of one point compactness of $\mathbb{N}$ (treated as a subspace of $\mathbb{R}$ with the standard topology) we can treat the space c of convergent sequences as $C(\dot{\mathbb{N}})$
This allows to prove easily the fact (already known from exercises-classes by the "manual" proof, probably ...) that $c^* \equiv \ell^1(\dot{\mathbb{N}}) \left(\equiv \ell^1(\mathbb{N})\right)$, and an explicit formula for isometry can be described. In particular c can not be reflexive... And so, also c_0 can't.