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The continuous and the residual spectrum

When $\{0\}\neq X$ is a Banach space and $A\in B(X)$, then it is slightly simpler to study some "invertibility questions" related to the spectrum of A.

Generally, we know that "usually" $\sigma_p(A) \subsetneq \sigma(A)$ for $\dim X = +\infty$. So, the natural question is

what is the difference $\sigma(A)\setminus\sigma_p(A)$?

There exists a quite "popular" decomposition of this difference onto the two disjoint sets $\sigma_c(A)$, $\sigma_r(A)$, called the continuous spectrum and the residual spectrum of A, respectively and defined as follows:

$$\sigma_c(A) := \{\lambda\in\mathbb{K} : \operatorname{Ker}(A-\lambda I)=\{0\} \text{ and } \overline{\operatorname{Ran}(A-\lambda I)}=X, \text{ but } \operatorname{Ran}(A-\lambda I)\neq X\},$$

$$\sigma_p(A) := \{\lambda\in\mathbb{K} : \operatorname{Ker}(A-\lambda I)=\{0\} \text{ and } \overline{\operatorname{Ran}(A-\lambda I)}\neq X\}.$$

Obviously (but $\longrightarrow$ ⚠ ...) we have

$$\sigma(A) = \sigma_p(A)\cup\sigma_c(A)\cup\sigma_r(A),$$ and $\sigma_p(A)$, $\sigma_c(A)$, $\sigma_r(A)$ are disjoint.

You must be carefull with the above terminology! For the operators acting in Hilbert spaces (especially for some normal ones - see sub-subsection 4...) another "decomposition" is popular.

This is a "decomposition" onto the so-called absolutely continuous and singular continuous spectra of the "non-point" part of the spectrum. However this is not a "decomposition" in the previous meaning (the two parts can be not disjoint and $\sigma_p(A)$ schould be considered with the closure to get the whole $\sigma(A)$ after "summing"...)

3.3 Spectrum of compact operators

Theorem ("On the spectrum of compact operator", "The Riesz-Schauder Theorem")

Let $\{0\} \neq X$ be a Banach space and $K \in \mathcal{C}(X)$. Then:

(1) If $\dim X = +\infty$, then $0 \in \sigma(K)$.

(2) If $\lambda \in \sigma(K) \setminus \{0\}$, then $\lambda \in \sigma_p(K)$ and $\dim \operatorname{Ker}(K - \lambda I) < +\infty$.

(3) $\forall_{\varepsilon > 0}$ $\sigma(K) \cap \{\lambda \in \mathbb{C} : |\lambda| > \varepsilon\}$ is a finite set; in particular, if $\lambda \in \mathbb{C}$ is an accumulation point (limit point) of $\sigma_p(K)$, then $\lambda = 0$.

Proof

(1) Suppose, that $\dim X = +\infty$. If $0 \notin \sigma(K)$, then $\operatorname{Ran} K = X$, hence by thm "on properties of $\mathcal{C}(X,Y)$" part (iv) (p. OTST-21) $\operatorname{Ran} K$ is not closed, but X is closed! - Thus $0 \in \sigma(K)$.

(2) Suppose, that $\lambda \in \sigma(K) \setminus \{0\}$. If $\lambda \notin \sigma_p(K)$, then $\operatorname{Ker}(K - \lambda I) = \{0\}$. But $K - \lambda I = -\lambda(I - \frac{1}{\lambda}K)$ and $-\frac{1}{\lambda}K$ is also compact, $\operatorname{Ker}(K - \lambda I) = \operatorname{Ker}(I - \frac{1}{\lambda}K) = \{0\}$. Hence, by

Thm "On compact pert..." p. OTST-31, part 3 we have $\operatorname{codim} \operatorname{Ran}(I - \frac{1}{\lambda}K) = 0 \ (= \dim\{0\})$. Hence (by the def. of codim) $\operatorname{Ran}(I - \frac{1}{\lambda}K) = X$, but $\operatorname{Ran}(K - \lambda I) = \operatorname{Ran}(I - \frac{1}{\lambda}K) = X$. Thus $K - \lambda I$ is invertible, and $\lambda \in \rho(K)$, which contradicts $\lambda \in \sigma(K)$. Therefore $\lambda \in \sigma_p(K)$. Now, the eigenspace $\operatorname{Ker}(K - \lambda I) = \operatorname{Ker}(I - \frac{1}{\lambda}K)$, hence $\dim \operatorname{Ker}(K - \lambda I) < +\infty$ by theorem "On compact..." p. OTST-31, part 1.

(3)

Suppose now, that for some $\varepsilon > 0$ $\{\lambda \in \mathbb{C} : |\lambda| > \varepsilon\} \cap \sigma(K)$ is infinite. By (2) there exists a sequence $\{\tilde{\lambda}_n\}_{n \geq 1}$ of different eigenvalues of K such that $|\tilde{\lambda}_n| > \varepsilon$, hence (since $|\sigma(K)| \leq \|K\|$) there exists a convergent sequence $\{\lambda_n\}_{n \geq 1}$ of it denote $\lambda_0 := \lim_{n \to +\infty} \lambda_n$ - we have $|\lambda_0| \geq \varepsilon > 0$ and $\forall_{n,m \geq 1}\ \lambda_n \neq \lambda_m$. By fact "On eigenvectors independence", choosing any eigenvector $x_n \in X$ for K and λ_n we get $\{x_n\}_{n \geq 1}$ - a linearly independent system in X (note, that the infinitness of the system is not any problem here, by the definition of linear independence/dependence...). So, define

$$X_n := \operatorname{lin}\{x_1, \dots x_n\}, \quad n \geq 1$$

- by the linear independence we have:

$$\forall_{n \geq 1}\ X_{n+1} \supsetneq X_n.$$

We shall now construct a certain sequence $\{y_n\}_{n \geq 1}$ in $S(0,1)$

such that $y_n \in X_n$ for any $n \geq 1$ and which "gives a contradiction with the compactness of K"...
Its definition is recursive:

$$y_1 := \frac{x_1}{\|x_1\|},$$

so, $y_1 \in X_1$ and $\|y_1\| = 1$. Suppose that $y_1, \dots, y_n \in S(0,1)$ and $y_j \in X_j$ for $j = 1, \dots n$. Since $X_{n+1} \supsetneq X_n$, using the Riesz Lemma (see PB- ...) we can choose $y_{n+1} \in X_{n+1}$ such that $y_{n+1} \in S(0,1)$ and $\mathrm{dist}(y_{n+1}, X_n) \geq \frac{1}{2}$.

So, the construction is ready and, moreover,

$$\forall_{n \geq 1} \ \mathrm{dist}(y_{n+1}, X_n) \geq \frac{1}{2}. \qquad (1)$$

To get the mentioned contradiction consider a "rescaled" sequence $\{\frac{1}{\lambda_n} y_n\}_{n \geq 1}$ - it is bounded, because $|\lambda_n| > \varepsilon$ and $y_n \in S(0,1)$ for any $n \geq 1$. We shall prove that $\{u_n\}_{n\geq 1} = \{K(\frac{1}{\lambda_n} y_n)\}_{n \geq 1}$ has no convergent subsequence. A "chance" for it comes from the fact that:

(i) the above is true for $\{y_n\}_{n \geq 1}$ in place of $\{u_n\}_{n \geq 1}$, by (1);

(ii) u_n is "a small perturbation" of y_n in a sense...

- we have $u_n = y_n + r_n$,

where $r_n = K(\frac{1}{\lambda_n} y_n) - y_n$. Using $y_n \in X_n = \mathrm{lin}\{x_1, \dots, x_n\}$ for some $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ we have $y_n = \sum_{j=1}^{n} \alpha_j x_j$, and

$$r_n = \frac{1}{\lambda_n} K\Big(\sum_{j=1}^{n} \alpha_j x_j\Big) - \sum_{j=1}^{n} \alpha_j x_j = \sum_{j=1}^{n} \alpha_j \Big(\frac{\lambda_j}{\lambda_n} - 1\Big) x_j = \sum_{j=1}^{n-1} \alpha_j \Big(\frac{\lambda_j}{\lambda_n} - 1\Big) x_j \in X_{n-1},$$

with $X_0 := \{0\}$.

So, the above "smallness" is in the sense: $\forall_{n\geq 1} r_n \in X_{n-1}$.

— This is really a smallness, because (now) by (1), for any $m \leq n$

$\| u_{n+1} - u_m \| = \| y_{n+1} + r_{n+1} - y_m - r_m \| = \| y_{n+1} - (y_m - r_{n+1} - r_m) \|$

$\geq \frac{1}{2}$, since $(y_m - r_{n+1} - r_m) \in X_n$ $(r_m \in X_{m-1} \subset X_{n-1} \subset X_n$,

$y_m \in X_m \subset X_n$, $r_{n+1} \in X_n$. But the above means, that

$\{u_n\}_{n\geq 1}$ does not possess any Cauchy subsequence! $\boxed{//}$