V Operator Theory - some basic selected topics

Subsections

1. Bounded operators and completness

This part contains the most basic results concernig the consequences of completeness of norm spaces for linear operators. We prove here 4 classical theorems on this topic: the Open Mapping thm., the Inverse Mapping thm., the Closed Graph thm and the Banach-Steinhaus thm. In some sense, they are all consequences of the classical metric theory Baire theorem.

1.1 Open bounded operators and the Inverse Mapping thm.

The "open function", "closed function" *) notions are topological spaces notions and we use the first one also with "map", "operator" in place of "function", i.e., $A : X \to Y$ is **open** iff $A(U)$ is open set for any open set $U \subset X$.

Recall that we met open operators before, e.g., speaking on quotient spaces (the canonical quotient map π is open — see p. PB-52)

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*) Be careful! "Closed operator" will be used in this subsection (see 1.2) in a (completely...) different sense!

The Open Mapping theorem

The famous result below shows the tight connection of two properties for bounded operators between Banach spaces: of the surjectivity and of being open map.

Theorem (The Open Mapping theorem, ["The Banach-Schauder thm."])

If X and Y are Banach spaces, $A \in B(X,Y)$ and $A(X)=Y$, then A is an open map.

Proof

Suppose that $U \subset X$, U – open, and $y_0 \in A(U)$, so $y_0 = A(x_0)$ for some $x_0 \in U$. We have to prove that $K_Y(y_0, r) \subset A(U)$ for some $r>0$. Let $U' := U - x_0$, so $U = x_0 + U'$, $A(U) = A(x_0 + U') = y_0 + A(U')$ – hence it suffices to prove that $K_Y(0,r) \subset A(U')$. But U' is open and $0 \in U'$, so $K_X(0,s) \subset U'$ for some $s>0$. Thus, we shall prove that there exists $r>0$ satisfying

$$K_Y(0,r) \subset A(K_X(0,s)). \tag{1}$$

We shall do it using 3 steps:

Step 1: $\forall_{t>0}\ \mathrm{Int}\left(\overline{A(K_X(0,t))}\right) \neq \emptyset$. (2)

To prove it, observe that $X = \bigcup_{n\in\mathbb{N}_1} K_X(0,n)$, hence $Y = A(X) = A\left(\bigcup_{n\in\mathbb{N}_1} K_X(0,n)\right) = \bigcup_{n\in\mathbb{N}_1} A(K_X(0,n)) \subset \bigcup_{n\in\mathbb{N}_1} \overline{A(K_X(0,n))} \;(\subset Y)$ – so

$Y = \bigcup_{n \in \mathbb{N}_1} \overline{A(K_X(0,n))}$. But Y is a Banach space – – in particular it is a complete metric space! And $\operatorname{Int} Y = Y \neq \emptyset$. Thus, by the Baire Theorem there exists $n_0 \in \mathbb{N}_1$ such that $\emptyset \neq \operatorname{Int} \overline{A(K_X(0,n_0))} = \operatorname{Int}\left(\frac{n_0}{t} \cdot \overline{A(K_X(0,t))}\right) = \frac{n_0}{t} \operatorname{Int}\left(\overline{A(K_X(0,t))}\right)$, if $t > 0$. This proves (2).

Note, that we used above the fact that for any nonzero scalar λ the operation "$v \rightsquigarrow \lambda v$" is a homeomorphism in both spaces X and Y, so the λ-multiplication commutes with "Int" and "$\overline{}$" operations and with A (by linearity...). We shall denote such argumentation by "λ-hom" later.

Step 2 $$\exists_{\delta_0 > 0} \; K_Y(0,\delta_0) \subset \overline{A(K_X(0,1))}. \quad (3)$$

By (2) $K_Y(y,\delta_1) \subset \overline{A(K_X(0,1))}$ for some $y \in Y$ and $\delta_1 > 0$, but $y = Ax$ for some $x \in X$, so $Ax + K_Y(0,\delta_1) \subset \overline{A(K_X(0,1))}$. The above is equivalent to

$$K_Y(0,\delta_1) \subset A(-x) + \overline{A(K_X(0,1))} = \overline{(A(-x) + A(K_X(0,1)))}$$
$$= \overline{A(-x + K_X(0,1))} = \overline{A(K_X(-x,1))} \subset \overline{A(K_X(0, \|x\|+1))} =$$
$$= (\|x\|+1) \; \overline{A(K_X(0,1))},$$

i.e, $K_Y\left(0, \frac{\delta_1}{\|x\|+1}\right) \subset \overline{A(K_X(0,1))}$, thus (3) holds.

We used above "λ-hom" and the similar argument related to the "$v \rightsquigarrow v + x_0$"-homeomorphism for any fixed x_0.

$\boxed{\text{Step 3}}$ $$\overline{A(K_X(0,1))} \subset A(K_X(0,2)). \tag{4}$$

Let $y \in \overline{A(K_X(0,1))}$ – we shall prove that $y \in A(K_X(0,2))$ by constructing the appropriate $x \in K_X(0,2)$ such that $y = Ax$.
Let δ_0 be "as in (3)" and choose $x_1 \in K_X(0,1)$ such that $\|y - Ax_1\| < \frac{\delta_0}{2}$ (we can do it by the definition of closure...)

We shall recursively construct $\{x_n\}_{n\geq 1}$ in X such that
$$\|x_n\| < \frac{1}{2^{n-1}} \quad \text{and} \quad \Big\|y - A\Big(\sum_{j=1}^{n} x_j\Big)\Big\| < \frac{\delta_0}{2^n}. \tag{5}$$
– The construction for $n=1$ is just made. If x_n is constructed, then by (3) and "λ-hom" again:
$$y - A\Big(\sum_{j=1}^{n} x_j\Big) \in K_Y\Big(0, \frac{\delta_0}{2^n}\Big) \subset \overline{A(K_X(0, \tfrac{1}{2^n}))}$$
so, similarly as before, we choose $x_{n+1} \in K_X(0, \frac{1}{2^n})$ such that
$$\Big\|y - A\Big(\sum_{j=1}^{n+1} x_j\Big)\Big\| = \Big\|\Big(y - A\Big(\sum_{j=1}^{n} x_j\Big)\Big) - Ax_{n+1}\Big\| < \frac{\delta_0}{2^{n+1}},$$
which finishes the construction of $\{x_n\}_{n\geq 1}$.
But $\sum_{n=1}^{+\infty} \|x_n\| < \sum_{n=1}^{+\infty} \frac{1}{2^{n-1}} = 2$, so $\sum_{n=1}^{+\infty} x_n$ is convergent to some $x \in X$ by the completness of X, and $\|x\| < 2$. Moreover $A\Big(\sum_{j=1}^{n} x_j\Big) \to Ax$ and $\Big\|y - A\Big(\sum_{j=1}^{n} x_j\Big)\Big\| \to 0$, by the continuity of A and by (5), so on the other hand, $\Big\|y - A\Big(\sum_{j=1}^{n} x_j\Big)\Big\| \to \|y - Ax\|$, i.e., $y = Ax$ and Step 3 is finished.

Now, we get easily (1) by (3) and (4) with the use of "λ-hom":
$$K_Y(0, \tfrac{1}{2}s\delta_0) = \tfrac{1}{2}s\, K_Y(0, \delta_0) \subset \overline{A(K_X(0, \tfrac{1}{2}s))} \subset A(K_X(0,s)).$$
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Remarks

1. Observe, that the "inverse" is also true (even without completeness):
If X, Y - norm spaces and $A \in B(X,Y)$ is open, then
$A(X) = Y$ (→ △).

2. The completeness of both X and Y was very important for the proof. And the both of them are important for the theorem itself! — Find some appropriate two examples! (→ △)

3. ...but sometimes we can get open operator without the completeness... (see, e.g., the open map property for the quotient map π [p. PB-52] or for a nonzero X^{*}-functional [see p. LF-38 - Lemma 4])

4. There exists also generalisations of this thm. for different type of linear spaces than normed spaces — e.g. for Fréchet spaces.

◆ The continuity of inverse mappings

The following result is a simple corollary from the previous thm. *)

Theorem ("The (Banach) Inverse mapping thm")

Suppose that X, Y are Banach spaces, $A \in B(X,Y)$ and $\operatorname{Ker} A = \{0\}$, $A(X) = Y$. Then $A^{-1} \in B(Y,X)$. **)

Proof

Obviously $A^{-1} \in \mathcal{L}(Y,X)$. Inverse image by A^{-1} is the image by A and A is an open map, by the Open mapping thm., so A^{-1} is continuous. ☐

Remark $A^{-1} \in B(Y,X)$ means that $\|A^{-1}y\| \leq C\|y\|$ for any $y \in Y$ with some constant $C \in \mathbb{R}_+$, so for any $x \in X$ $c\|x\| \leq \|Ax\|$ with $c = \frac{1}{C}$

Corollary

If X is a linear space and $\|\cdot\|_1$, $\|\cdot\|_2$ are two norms in X such that:

(i) both $(X, \|\cdot\|_1)$, $(X, \|\cdot\|_2)$ are Banach spaces,

(ii) $\exists_{C \in \mathbb{R}} \ \forall_{x \in X} \ \|x\|_1 \leq C\|x\|_2$,

then $\|\cdot\|_1 \equiv \|\cdot\|_2$ (the norms are equivalent).

Proof

Consider $A := I$. If we treat it as map from X with $\|\ \|_2$ onto X with $\|\ \|_1$, then (ii) means, that $A = I$ is continuous. So $A^{-1} = I$ is a continuous linear map from X with $\|\ \|_1$ onto X with $\|\ \|_2$ – i.e. we have the both inequalities for "$\equiv$". ☐

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*) And, in fact, it seems to be more frequently used in practice...

**) So, A is a linear isomorphi-sm of X onto Y!

1.2 The closedness of the operator and its continuity

Now we shall obtain some very important / convenient result on so-called "closed operators", however those "closed" is <u>not</u> the analog of "open" from 1.1...

Fact

Let X, Y be two metric spaces and $f: X \to Y$ - a continuous function. Then the graph of f ($\Gamma(f) := \{(x,y) \in X \times Y : y = f(x)\}$ *)) is a closed subset of $X \times Y$ (with the product topology/metric)

Proof

Denote by ρ the both metrics - in X and in Y, and suppose that $(x_n, y_n) \in \Gamma(f)$ and $(x_n, y_n) \to (x, y) \in X \times Y$ - hence $x_n \to x$ and $y_n \to y$. But $y_n = f(x_n)$ and $f(x_n) \to f(x)$, by continuity, so $y_n \to f(x)$ and thus $f(x) = y$, i.e. $(x, y) \in \Gamma(f)$. So $\Gamma(f)$ is closed.

Definition

If $A \in \mathcal{L}(X, Y)$ where X, Y are normed (or topological vector) spaces then <u>A is closed</u> iff $\Gamma(A)$ is closed in $X \times Y$. **))

Note, that the inversion of the above fact is generally not true! ($\longrightarrow$). But for linear operators and Banach spaces...

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*) Observe, that formally $f = \Gamma(f)$...

**) Recall that this is completely different notion than the "closed function" in topological terminology sense...

Theorem ("The Closed Graph thm.")

If X and Y are Banach spaces and $A \in \mathcal{L}(X,Y)$ is a closed operator, then $A \in B(X,Y)$.

Proof

Observe, that $\Gamma(A) \underset{lin}{\subset} X \times Y$, because A is linear ($\rightarrow \triangle$).
Moreover $X \times Y$ is a Banach space, because X and Y are, and $\Gamma(A)$ is a closed subspace of $X \times Y$. Hence $\Gamma(A)$ is a Banach space (with the product norm from $X \times Y$).
We have: $P_1 : \Gamma(A) \longrightarrow X$ and $P_2 : \Gamma(A) \longrightarrow Y$ given by $P_1(x,y) = x$, $P_2(x,y) = y$ for $(x,y) \in \Gamma(A)$ are both continuous linear operators (by the def. of the product norm). But if $(x,y) \in \operatorname{Ker} P_1$, then $x = P_1(x,y) = 0$ so $y = Ax = A0 = 0$ and $(x,y) = 0$. This means that $\operatorname{Ker} P_1 = \{0\}$. Obviously $P_1(\Gamma(A)) = X$, so P_1 is an invertible linear map between two Banach spaces – so P_1^{-1} is continuous. But $Ax = P_2((x, Ax)) = P_2(P_1^{-1}x)$ for any $x \in X$, i.e. $A = P_2 \circ P_1^{-1}$ – it is a composition of two continuous maps.

Why we claimed, that this result is so convenient?

Observe, that when we want to prove that an operator is continuous, then "by definition" we should check whether the assumption:

$$x_n \to x \qquad (1)$$

results in:

$$Ax_n \to Ax. \qquad (2)$$

So, we should obtain (2), using only (1)! In particular we do not have any prior information, that $\{Ax_n\}$ has any limit...

And when we want to use the closed graph thm. to prove the continuity of A our situation is much better. We assume $(x_n, Ax_n) \longrightarrow (x, y)$, which means that $x_n \to x$ <u>and</u> $Ax_n \longrightarrow y$, so istread of (1) only, we have the "extra" assumption:

$$Ax_n \to y. \qquad (1')$$

And the only thing we have to prove is:

$$Ax = y. \qquad (3)$$

And note, that in this case proving (2) is enough, because (2) + (1') give (3) obviously. So, with the closed graph thm. we have "double" assumption ((1)+(1') instead of (1) only) and the same assertion (2) to prove !

Let us study a simple example of such argumentation, to understand the advantage of this method better.

Example / **Fact**

Let $A: C(K) \to C(K)$ (K - compact), A - linear. Suppose that for any $\{f_n\}_{n\geq 1}$ in $C(K)$, if $f_n \rightrightarrows 0$, then $Af_n \to 0$ *). Then $A \in B(C(K))$.

Proof

Of course, by linearity, if $f_n \rightrightarrows f$ for $f \in C(K)$, then $Af_n \to Af$. Suppose now, that $f_n \rightrightarrows f$ and $Af_n \rightrightarrows g$, then we have $Af_n \to Af$ and $Af_n \rightrightarrows g$ at the same time, so $Af = g$. Hence A is continuous, by the closed graph thm. $\boxed{\text{c}}$

Note, at the end, that the assumption of completness of both spaces X and Y is essential.

Exercise $\to$ $\triangle$

Find two examples showing that the completeness of X only is not enough, as wel as the completeness of Y only.

1.3. "Pointwise boudedness" and "uniform boudedness" of some operators families.

The next result has essentially different form than the previous two (3). And its proof presented here is "independent".

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*) Here we use "$\rightrightarrows$" for uniform convergence (i.e. the convergence in the norm in $C(K)$) and "$\to$" for the point(wise) conv.

$\boxed{\text{Theorem}}$ ("The Banach-Steinhaus Theorem")

Let X be a Banach space and Y – a normed space and let $\{T_\alpha\}_{\alpha\in\mathcal{A}}$ be a family of $B(X,Y)$ operators. If for all $x \in X$ there exists $M \ (= M(x)) \in \mathbb{R}$ such that

$$\forall_{\alpha\in\mathcal{A}} \ \|T_\alpha x\| \leq M, \ ^{*)} \qquad (1)$$

then $\{T_\alpha\}_{\alpha\in\mathcal{A}}$ is bounded in $B(X,Y)$, i.e. there exists $C\in\mathbb{R}$ such that

$$\forall_{\alpha\in\mathcal{A}} \ \|T_\alpha\| \leq C. \ ^{**)}$$

$\boxed{\text{Proof}}$

Consider $X_n := \{x\in X : \forall_{\alpha\in\mathcal{A}} \|T_\alpha x\| \leqslant n\}$ for $n\geqslant 1$. If $x \in X$, then choose M as in (1) and let $n \geqslant M$. Then $\forall_{\alpha\in\mathcal{A}} \|T_\alpha x\| \leq M \leq n$, so $x\in X_n$. This means, that $\bigcup_{n\geqslant 1} X_n = X$. Moreover $X_n = \overline{X_n}$ (– it is an intersection of closed sets, since T_α are continuous). By the Baire thm, for some $n_0 \geqslant 1$ $\operatorname{Int} X_{n_0} \neq \emptyset$. Choose $x_0\in X$ and $r>0$ such that $K_X(x_0,r) \subset X_{n_0}$. So, if $x\in K_X(x_0,r)$ then $\forall_{\alpha\in\mathcal{A}} \|T_\alpha x\| \leq n_0$. Now take $v \in X$ with $\|v\| \leq 1$. Then $v = \frac{2}{r}\left[\left(\frac{r}{2}v + x_0\right) - x_0\right]$ and $\frac{r}{2}v + x_0 \in K_X(x_0,r)$, so

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*) This condition could be called "point(wise) bdd:ness (the whole sentence...).

**) And this – the "uniform" one...

$$\forall_{\alpha\in A}\quad \|T_\alpha v\| = \frac{2}{r}\left\|T_\alpha\left(\frac{r}{2}v + x_0\right) - T_\alpha x_0\right\| \leq \frac{2}{r}(n_0 + M_0),$$

where M_0 is as in (1) for x_0. Hence, we can take $C := \frac{2}{r}(n_0 + M_0)$. $\boxed{L}$

We shall show now two consequences of this theorem. The first is related to the problem of continuity of the <u>point</u> limit of continuous function. Obviously – there is no continuity of such limit, "in general". But in some special situations, when the functions being terms of the sequence belong to some special classes of functions, the point convergence suffices to get the continuity of the limit (e.g. for the class of convex functions on open interval, ... $\to \triangle$). Also the class of linear operators on Banach space has such property!

$\boxed{\text{Fact}}$ (On the boundedness of the strong*) limit)

Suppose that X is a Banach space, Y is a norm space, and $A_n \in B(X,Y)$ for any $n \geq n_0$. If A is a function from X into Y such that

$$\forall_{x\in X}\quad A_n x \xrightarrow{Y} A(x), \tag{2}$$

then $A \in B(X,Y)$.

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*) For the case of linear operators the point(wise) limit is also named "<u>strong limit</u>".

Proof

Observe, that by (2) $A \in \mathcal{L}(X,Y)$, because all A_n are linear. By (2) also the family of operators $\{A_n\}_{n\in\mathbb{N}}$ satisfies the assumptions of the B.-S. thm. (since any convergent sequence $\{A_n x\}_{n\geq n_0}$ in Y is bounded...).

Hence $\forall_{n\geq n_0} \|A_n\| \leq C$ for some $C \in \mathbb{R}_+$.

Now: $\forall_{x\in X} \|Ax\| = \|\lim_{n\to+\infty} A_n x\| = \lim_{n\to+\infty} \|A_n x\| \leq C\|x\|$ – i.e., $A \in B(X,Y)$. $\boxed{\text{//}}$

The second consequence is in fact an extension of the B.-S. theorem.

Corollary

If X is Banach and Y – a norm space and $\{T_\alpha\}_{\alpha\in\mathcal{A}}$ – a family of bounded operators from X into Y, then TFCAE:

(i) $\exists_{C\in\mathbb{R}_+} \forall_{\alpha\in\mathcal{A}} \|T_\alpha\| \leq C$;

(ii) $\forall_{x\in X} \exists_{C\in\mathbb{R}_+} \forall_{\alpha\in\mathcal{A}} \|T_\alpha x\| \leq C$;

(iii) $\forall_{\substack{x\in X\\ \varphi\in Y^*}} \exists_{C\in\mathbb{R}_+} \forall_{\alpha\in\mathcal{A}} |\varphi(T_\alpha x)| \leq C$.

Proof

(i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious (because ... $\to \Delta$).
(ii) $\Rightarrow$ (i) – this is the B.-S. thm. assertion! So, it suffices to prove (iii) $\Rightarrow$ (ii).

Suppose that (iii) holds and consider the following new family of operators:

$\{\Phi_{(x,\alpha)}\}_{(x,\alpha)\in X\times A}$, where

$$\Phi_{(x,\alpha)} : Y^* \longrightarrow \mathbb{K}, \quad \Phi_{(x,\alpha)}(\varphi) := \varphi(T_\alpha x)$$

for $x \in X, \alpha \in A, \varphi \in Y^*$. Observe, that $\Phi_{(x,\alpha)} \in (Y^*)^* (= Y^{**})$ because $\Phi_{(x,\alpha)} = \varkappa(T_\alpha x)$, where $\varkappa : Y \longrightarrow Y^{**}$ is the canonical embedding $Y \hookrightarrow Y^{**}$ (see p. LF 47–48). Moreover (iii) means, that for any fixed $x_0 \in X$ the family $\{\Phi_{(x_0,\alpha)}\}_{\alpha\in A}$ satisfies the conditions of the B-S. thm. (note, that Y^* is always a Banach space). Thus there exists $C = C_{x_0} \in \mathbb{R}_+$ such that $\forall_{\alpha\in A} \|\Phi_{(x_0,\alpha)}\| = \|\varkappa(T_\alpha x_0)\| \leqslant C_{x_0}$. But by thm. "On canonical $X \hookrightarrow X^*$ embedding" (here "$Y \hookrightarrow Y^*$ rather...") p. LF-48 $\|\varkappa(T_\alpha x_0)\| = \|T_\alpha x_0\|$. Therefore (ii) holds! $\boxed{(i)}$

Note that here also, as in the previous theorems the completeness of X is crucial (find an example $\longrightarrow$ $\triangle$). However, we do not need*) the completeness of Y! Why?... —Observe, that we can always take $\tilde{Y}$ being a completion of Y and any $T \in B(X,Y)$ is then also in $T \in B(X, \tilde{Y})$ automatically. And the value of $\|Tx\|$ and of $\|T\|$ remains the same then!

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*) and we don't assume...