3. Orthogonal / orthonormal systems and bases

Our main goal here is to study orthogonal (and orthonormal) bases in Hilbert spaces. They play the rôle somewhat similar to the rôle of bases (linear, i.e. of Hamel) in linear spaces. However, they are much more related to the analytic (the norm...) and geometric (the orthogonality...) structure of Hilbert space.

3.1. Orthogonal systems and sets

Let $\mathcal{A}$ be a set (we shall treat it as a set of "indices") and let $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ be an indexed system*) of vectors in a unitary space X. And let $S \subset X$.

Definition

- $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ is an orthogonal system**) iff $\forall_{\substack{\alpha,\beta\in\mathcal{A}\\ \alpha\neq\beta}}\ x_\alpha \perp x_\beta$;
- S is an orthogonal set**) iff $\forall_{\substack{x,y\in S\\ x\neq y}}\ x \perp y$.
- $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ is an orthonormal***) system iff it is an orthogonal system, and $\forall_{\alpha\in\mathcal{A}}\ \|x_\alpha\| = 1$.
- S is an orthonormal***) set iff S is an orthogonal set and $\forall_{x\in S}\ \|x\| = 1$.

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*) i.e. $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ is a function from $\mathcal{A}$ to $\mathcal{H}$, from the formal point of view.

**) Pl.: układ / zbiór ortogonalny. Some people assume "extra" that $\forall_{\alpha\in\mathcal{A}}\ x_\alpha \neq 0$ ($\forall_{x\in S}\ x\neq 0$, for orth. set), but we DON'T! — see also the definition of n-d. orth. system/set, p. HS-47.

***) Pl.: ortonormalny.

We shall use often the abbreviation o.s. for orthogonal system, and o.set.
In some parts of subsection 3 we shall use rather orthogonal systems than sets, because of some notation convenience. Note however, that having an orthogonal set S we can "canonically" get an orthogonal system $\{x_s\}_{s\in S}$ just defining $x_s := s$ for $s \in S$. But the "inverse", i.e., taking $S := \{x_\alpha : \alpha \in \mathcal{A}\}$ for an o.s. $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ can be sometimes not a good idea: if there is "a lot of" zero-vectors x_α among all x_α-s (which can be important for us), we get "only one" $0 \in S$...
Thus, for $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ - an o.s. denote:

$$\operatorname{supp}\{x_\alpha\}_{\alpha\in\mathcal{A}} := \{\alpha\in\mathcal{A} : x_\alpha \neq 0\}$$

(the support*) of $\{x_\alpha\}_{\alpha\in\mathcal{A}}$).

◆ „Orthogonal series"

We study first the series with orthogonal terms. – The result below would be crucial for our further considerations.

Fact ("On orthogonal sequences")

Let $\{x_n\}_{n\in\mathbb{N}_1}$ be an o.s. in a Hilbert space $\mathcal{H}$.
Then:

(1) $\sum_{n=1}^{+\infty} x_n$ is convergent iff $\sum_{n=1}^{+\infty} \|x_n\|^2 < +\infty$

(2) If $\sum_{n=1}^{+\infty} x_n$ is convergent, then

$$\Big\|\sum_{n=1}^{+\infty} x_n\Big\|^2 = \sum_{n=1}^{+\infty} \|x_n\|^2, \qquad \text{(GPF)}$$

and $\sum_{n=1}^{+\infty} x_n$ is unconditionally convergent, i.e., for any bijection (permutation) p of $\mathbb{N}_1$ the series $\sum_{n=1}^{+\infty} x_{p(n)}$ is convergent, and

$$\sum_{n=1}^{+\infty} x_{p(n)} = \sum_{n=1}^{+\infty} x_n \quad \blacksquare$$

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*) pol.: nośnik.

Remarks

1. The part (1) + the first part of (2) are sometimes jointly called "generalized / infinite Pythagorean formula" (GPF).

2. As we shall see below, if we assume that $\{x_n\}_{n\in\mathbb{N}_1}$ is an o.s. in a unitary space X only (instead of the Hilbert $\mathcal{H}$), then we have "$\Longrightarrow$" in (1) and the first part of (2) is also true.

Proof (of Fact)

(1) Since both $\mathcal{H}$ and $\mathbb{R}$ are complete spaces, thus the convergences of both series in (1) are equivalent to the appropriate Cauchy conditions. But by the Pythagorean formula, if $n \geq m$ then

$$\Big\|\sum_{k=m}^{n} x_k\Big\|^2 = \sum_{k=m}^{n} \|x_k\|^2,$$

so those Cauchy conditions are eqivalent one to the other).

(2) Suppose, that $\sum_{n=1}^{+\infty} x_n$ is convergent. Then, again by Pyth. form.,

$$\Big\|\sum_{n=1}^{\infty} x_n\Big\|^2 = \lim_{n\to+\infty} \Big\|\sum_{k=1}^{n} x_k\Big\|^2 = \lim_{n\to+\infty} \sum_{k=1}^{n} \|x_k\|^2 = \sum_{n=1}^{+\infty} \|x_n\|^2,$$

so (GPF) holds. Now, by (1) and the fact that the scalar series $\sum_{n=1}^{+\infty} \|x_n\|^2$ is absolutely convergent (and so – also unconditionally)

We get the convergence of $\sum_{n=1}^{+\infty} x_{p(n)}$ for any permutation $p: \mathbb{N}_1 \longrightarrow \mathbb{N}_1$. Fix now the permutation p and denote for $n \in \mathbb{N}_1$

$$L(n) := \{ p(j) : \ 1 \leq j \leq n \}.$$

Using the fact, that p is onto $\mathbb{N}_1$ let us choose for any $k \in \mathbb{N}_1$ such $n(k) \in \mathbb{N}_1$ that $\{1, \dots, k\} \subset L(n(k))$. We have $k \leq \# L(n(k)) = n(k)$ – by the fact that p is injective. Hence

$$\lim_{k \to +\infty} n(k) = +\infty. \qquad (2)$$

Denote $S_k := \sum_{j=1}^{k} x_j$, $\tilde{S}_k := \sum_{j=1}^{k} x_{p(j)}$ and let $S := \sum_{j=1}^{+\infty} x_j$, $\tilde{S} := \sum_{j=1}^{+\infty} x_{p(j)}$.

Now, using once again the Pythagorean formula and the injectivity of p we get

$$\| \tilde{S}_{n(k)} - S_k \|^2 = \Big\| \sum_{m \in L(n(k)) \setminus \{1, \dots, k\}} x_m \Big\|^2 = \sum_{m \in L(n(k)) \setminus \{1, \dots, k\}} \| x_m \|^2 \leq$$

$$\leq \sum_{m=k+1}^{+\infty} \| x_m \|^2 \xrightarrow[k \to +\infty]{} 0, \text{ by (1).}$$

Hence, $\left(\tilde{S}_{n(k)} - S_k \right) \xrightarrow[k \to +\infty]{} 0$, but by (2) also

$$\left(\tilde{S}_{n(k)} - S_k \right) \xrightarrow[k \to +\infty]{} \tilde{S} - S,$$

i.e., $\tilde{S} = S$. $\boxed{\text{k}}$

$\boxed{\text{HS-40}}$

Summing of numbers

We shall need soon the notion of the sum $\sum_{\alpha\in A} f(\alpha)$ for some functions $f: A \to \mathbb{C}$ and <u>any</u> set A of "indices". The definition is obvious for any finite A. When $f \geq 0$, then it can be defined for <u>arbitrary A</u>. One of the possibilities is just to take then:

$$\sum_{\alpha\in A} f(\alpha) := \sup\Big\{\sum_{\alpha\in A'} f(\alpha) : A' \text{ is a finite subset of } A\Big\}.$$

So the sum is a nonnegative number or $+\infty$ then. Next this definition can be extended onto any $f: A \to \mathbb{C}$ satisfying $\sum_{\alpha\in A} |f(\alpha)| < +\infty$ – we do it just analogically as in the abstract integral theory (decomposing first f "onto $\mathrm{Re} f$, $\mathrm{Im} f$" and then decomposing real functions f "onto f_+, f_-" being non-negative functions).

The second way is to define directly $\sum_{\alpha\in A} f(\alpha)$ just as $\int_A f \, d\#$, where $\#$ is the counting measure on A for any $\#$-integrable function f. But it is easy to see that both ways lead to <u>the same result</u> ($\longrightarrow \triangle$), i.e.: for any $f: A \to \mathbb{C}$ $\sum_{\alpha\in A} |f(\alpha)|$ obtained by the first way and $\int_A |f| \, d\#$ <u>are equal</u> and when they are finite, then also both ways of defining <u>$\sum_{\alpha\in A} f(\alpha)$ give the same value</u>!

We shall use later the following result, which can be easily proved ($\longrightarrow \triangle$) by standard application of Monotone or Dominated Convergence theorems for integrals.

$\boxed{\text{Fact}}$ ("On scalar sums")

If $f: A \to \mathbb{C}$ and $\gamma: \mathbb{N}_1 \to A$ is injective, and $\mathrm{supp} f \subset \gamma(\mathbb{N}_1)$, then:

(a) $\displaystyle\sum_{\alpha\in A} |f(\alpha)| = \sum_{n=1}^{+\infty} |f(\gamma(n))|$,

(b) if $\displaystyle\sum_{\alpha\in A} |f(\alpha)| < +\infty$, then $\displaystyle\sum_{\alpha\in A} f(\alpha) = \sum_{n=1}^{+\infty} f(\gamma(n))$.

Bessel's Lemma

Lemma ("The Bessel lemma")

Suppose that $\{x_\alpha\}_{\alpha\in A}$ is an o.s. in a unitary space X and $x \in X$. Then:

(i) ("the Bessel inequality") if $\tilde{x}_\alpha := \begin{cases} 0 & \text{if } x_\alpha = 0 \\ x_\alpha/\|x_\alpha\| & \text{if } x_\alpha \neq 0 \end{cases}$ for any $\alpha \in A$, then

$$\sum_{\alpha\in A} |(x, \tilde{x}_\alpha)|^2 \overset{*)}{\leq} \|x\|^2, \qquad (1)$$

(ii) $\{\alpha \in A : (x, x_\alpha) \neq 0\}$ is at most countable.

Proof

(i) By the definition of a finite sum of non-negative numbers, to get (1) it is sufficient to prove

$$\sum_{\alpha\in A'} |(x, \tilde{x}_\alpha)|^2 \leq \|x\|^2 \qquad (1')$$

for any finite subset A' of A. So, fix such A' and consider $\mathcal{H} := \operatorname{lin}(\{\tilde{x}_\alpha : \alpha \in A'\} \cup \{x\})$ and
$Y := \operatorname{lin}\{\tilde{x}_\alpha : \alpha \in A'\}$.

Since the dimension of $\mathcal{H}$ is finite, $\mathcal{H}$ is a Hilbert space (a subspace of X) and Y is its closed linear subspace.

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*) Note that the LHS of (1) is the sum of a non-negative function just recalled. And it is equal also to $\sum_{\alpha\in\tilde{A}} \frac{|(x, x_\alpha)|^2}{\|x_\alpha\|^2}$ for $\tilde{A} := \operatorname{supp}\{x_\alpha\}_{\alpha\in A}$.

HS-45

Let $y := \sum_{\alpha\in A'} (x, \tilde{x}_\alpha)\tilde{x}_\alpha$. We have $y \in Y$ and for any $\beta \in A'$, by ortogonality of $\{\tilde{x}_\alpha\}_{\alpha\in A}$ we also have

$$(x-y, \tilde{x}_\beta) = (x, \tilde{x}_\beta) - \sum_{\alpha\in A'} (x, \tilde{x}_\alpha)(\tilde{x}_\alpha, \tilde{x}_\beta) =$$

$$= (x, \tilde{x}_\beta) - (x, \tilde{x}_\beta)(\tilde{x}_\beta, \tilde{x}_\beta),$$

but: $\tilde{x}_\beta = 0$ or $(\tilde{x}_\beta, \tilde{x}_\beta) = \|\tilde{x}_\beta\|^2 = 1$, hence

$$(x-y, \tilde{x}_\beta) = 0.$$

Thus, by the definition of Y we get $x-y \perp Y$, so $y = P_Y x$ (where P_Y denotes the orthogonal projection onto Y in $\mathcal{H}$) by Theorem "On orthogonal projection". Therefore $\|y\| = \|P_Y x\| \leq \|x\|$ by the same theorem, and $\|y\|^2 = \sum_{\alpha\in A'} \|(x, \tilde{x}_\alpha)\tilde{x}_\alpha\|^2 = \sum_{\alpha\in A'} |(x, \tilde{x}_\alpha)|^2$ by the Pythagorean formula, so (1') holds.

(ii) Note first, that $\{\alpha\in A : (x, x_\alpha) \neq 0\} = \{\alpha \in A : (x, \tilde{x}_\alpha) \neq 0\} = \bigcup_{n\in\mathbb{N}_1} A_n$, where $A_n := \{\alpha\in A : |(x, \tilde{x}_\alpha)| \geq \frac{1}{n}\}$. So it sufices to prove that each A_n is finite. Suppose, that it is infinite for some $n\in\mathbb{N}_1$. Then there exists an injective sequence $\{\alpha_k\}_{k\in\mathbb{N}_1}$ of elements of A_n, so $\{x_{\alpha_k}\}_{k\in\mathbb{N}_1}$ is also an o.s. in X. Thus, using (i) for this o.s., we get:

$$\|x\|^2 \geq \sum_{k\in\mathbb{N}_1} |(x, \tilde{x}_{\alpha_k})|^2 \geq \sum_{k\in\mathbb{N}_1} \frac{1}{n^2} = +\infty \quad \text{– a contradiction!}$$

$\boxed{\mathcal{U}}$

HS-46

HS3.1-7

Quasi-countable o.s. – the sum

The Bessel lemma gives in particular the information that for any o.s. $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ and any $x\in X$ the scalar product $(x,\tilde{x}_\alpha)$ is non-zero for at most countable number of α-s. As we shall see soon, it is convenient in our theory to study o.s.-s of the form $\{(x,\tilde{x}_\alpha)\tilde{x}_\alpha\}_{\alpha\in\mathcal{A}}$ $\left(\text{or } \left\{\frac{(x,x_\alpha)}{\|x_\alpha\|^2}x_\alpha\right\}_{\alpha\in\tilde{\mathcal{A}}}\right.$ where $\tilde{\mathcal{A}}:=$ $\left.\operatorname{supp}\{x_\alpha\}_{\alpha\in\mathcal{A}}\right)$, which have only countable number of non-zero vectors by the above fact. Such systems will be called "quasi-countable" here. More precisely:

Definition

O.s. $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ is quasi-countable *) if $\operatorname{supp}\{x_\alpha\}_{\alpha\in\mathcal{A}}$ is at most countable set.
We abbreviate it: $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ is q.c.o. or we write $\{x_\alpha\}_{\alpha\in\mathcal{A}}\in QCO$ (or $QCO(X)$, $QCO(\mathcal{A},X)$ also).

We shall need a precise mathematical notion of the sum of all vectors of some q.c.o. systems. To obtain it we shall define first the convergence of sums (and divergence) of such systems, and having it we shall define the notion of the sum of any convergent q.c.o. system. For the case, when $\tilde{\mathcal{A}}:=\operatorname{supp}\{x_\alpha\}_{\alpha\in\mathcal{A}}$

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*) pol.: quasi-przeliczalny

is finite this is easy. And when $\tilde{A}$ is countable, then we shall "number" the elements of $\tilde{A}$ by natural numbers and then we shall "sum" all the terms using the notion of the sum of the serie and the results of Fact "on orthogonal sequences".

So consider the set $\mathrm{Num}\ (= \mathrm{Num}(\{x_\alpha\}_{\alpha\in A})) :=$
$= \{ \gamma : \mathbb{N}_1 \longrightarrow \tilde{A} : \gamma \text{ is a bijection} \}$; any $\gamma \in \mathrm{Num}$ is called here a numbering for $\{x_\alpha\}_{\alpha\in A}$.

Definition

Let $\{x_\alpha\}_{\alpha\in A}$ be a q.c.o.system. If $\tilde{A}$ is finite, then the sum of $\{x_\alpha\}_{\alpha\in A}$ is always convergent and

$$\sum_{\alpha\in A}' x_\alpha := \sum_{\alpha\in\tilde{A}} x_\alpha .$$

If $\tilde{A}$ is countable, then the sum of $\{x_\alpha\}_{\alpha\in A}$ is convergent iff for some $\gamma \in \mathrm{Num}$

$$\sum_{n=1}^{+\infty} \| x_{\gamma(n)} \|^2 < +\infty$$

and then

$$\sum_{\alpha\in A} x_\alpha := \sum_{n=1}^{+\infty} x_{\gamma(n)} .$$

The vector $\sum_{\alpha\in A} x_\alpha$ is called the sum of $\{x_\alpha\}_{\alpha\in A}$, and divergence is (of course) the opposite to convergence.

We shall also say: $\sum_{\alpha\in A} x_\alpha$ is convergent/divergent (instead of "the sum of"...) ●

Surely, the main problem is now whether this definition i correct. But we have

Remark 1

The definition of convergence (and divergence) and of the sum of a q.c.o. system is correct i.e. the values of $\sum_{n=1}^{+\infty} \| x_{\gamma(n)} \|^2$ and of $\sum_{n=1}^{+\infty} x_{\gamma(n)}$ do not depend of the choice of $\gamma \in \mathrm{Num}$ (the second – only for the convergence case. The both facts follows directly from Fact "on orthogonal sequences".

Remark 2

Note, that there is a possible risk of confusion related to the above two definitions. In the case of $\mathcal{A}$ - finite it is simpler – the symbol $\sum_{\alpha \in \mathcal{A}} x_\alpha$ has two meanings:

(1) the "usual" finite sum of vectors

(2) the above (p. HS-41) defined $\sum_{\alpha \in \mathcal{A}} x_\alpha := \sum_{\alpha \in \hat{\mathcal{A}}} x_\alpha$.

– Fortunately, both meanings give the same value (the difference is just finite sum of zeros...).

But the problem exists also for the case of $\mathcal{A} = \mathbb{N}_{n_0}$ – it is not so "strict", because the symbols

(i) $\sum_{n \in \mathbb{N}_{n_0}} x_n$ and (ii) $\sum_{n=n_0}^{+\infty} x_n$ do not coincide,

however they look very similar! – Does the both values coincide? – YES, but it should be proved...

The same question, also with positive answer, concerns the problem of the convergence \ divergence of (i) and (ii).

The proofs follow immediately from the simple fact below ($\longrightarrow$)

Lemma

Let X be a normed space and $\{x_n\}_{n \in \mathbb{N}_1}$ – a sequence in X with infinitely many non-zero terms. Let $k_n \in \mathbb{N}_1$ be the index of the n-th non-zero term of $\{x_n\}_{n\in\mathbb{N}_1}$ *) for any $n \in \mathbb{N}_1$.

Then $\sum_{n=1}^{+\infty} x_n$ is convergent iff $\sum_{n=1}^{+\infty} x_{k_n}$ is convergent, and if those series are convergent, then they converge to the same vector of X.

Proof $\longrightarrow$

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*) i.e., $\{k_n\}_{n\in\mathbb{N}_1}$ is strictly increasing and $m \in \mathbb{N}_1$ has the form k_n for some $n \in \mathbb{N}_1$ iff $x_m \neq 0$.

HS-44

Properties of sums of q.c.o. systems

We formulate here several properties of just defined notions. All of them are rather expected.

$\boxed{\text{Fact}}$ ("On q.c.o")

Suppose that $\{x_\alpha\}_{\alpha\in A}$ is a q.c.o. system in a Hilbert space $\mathcal{H}$. Then:

(i) $\sum_{\alpha\in A} x_\alpha$ is convergent iff $\sum_{\alpha\in A}\|x_\alpha\|^2<+\infty$;

if $\sum_{\alpha\in A} x_\alpha$ is convergent, then $\sum_{\alpha\in A} x_\alpha \in \overline{\text{lin}\{x_\alpha : \alpha\in A\}}$ and

$$\Big\|\sum_{\alpha\in A} x_\alpha\Big\|^2 = \sum_{\alpha\in A}\|x_\alpha\|^2 \quad {}^{*)}$$ ("more generalized" Pythagorean formula...)

(ii) if $\sum_{\alpha\in A} x_\alpha$ is convergent and $y\in\mathcal{H}$, then

(a) $\left(y,\sum_{\alpha\in A} x_\alpha\right)=\sum_{\alpha\in A}(y,x_\alpha)$ $^{*)}$,

(b) $\sum_{\alpha\in A}|(y,x_\alpha)|$ $^{*)}$ $\leqslant \|y\|\cdot\Big\|\sum_{\alpha\in A} x_\alpha\Big\|$

(c) $\sum_{\alpha\in\tilde{A}} x_\alpha$ is convergent for $\tilde{A}=\text{supp}\{x_\alpha\}_{\alpha\in A}$ and

$$\sum_{\alpha\in\tilde{A}} x_\alpha=\sum_{\alpha\in A} x_\alpha$$

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*) Note, that those 3 sums are just the sums of numbers – its sense has been recalled before (see eq. p. HS-47).

(iii) if $\mathcal{A} = \mathcal{A}' \cup \mathcal{A}''$ and $\mathcal{A}' \cap \mathcal{A}'' = \emptyset$, then

$\sum_{\alpha \in \mathcal{A}} x_\alpha$ is convergent iff $\sum_{\alpha \in \mathcal{A}'} x_\alpha$ and $\sum_{\alpha \in \mathcal{A}''} x_\alpha$ are convergent, and for $\sum_{\alpha \in \mathcal{A}} x_\alpha$ - convergent

$$\sum_{\alpha \in \mathcal{A}} x_\alpha = \sum_{\alpha \in \mathcal{A}'}' x_\alpha + \sum_{\alpha \in \mathcal{A}''}' x_\alpha \quad \text{and} \quad \sum_{\alpha \in \mathcal{A}'} x_\alpha \perp \sum_{\alpha \in \mathcal{A}''} x_\alpha .$$

Proof

If $\mathcal{A}$ is finite then the assertion is clear, except the point (ii) (b), which can be proved in a manner analogic (or simpler...) to the proof for infinite case. For the infinite $\mathcal{A}$ consider a numbering $\gamma : \mathbb{N}_1 \longrightarrow \tilde{\mathcal{A}}$. By Fact "on scalar sums" (a) we have $\sum_{n=1}^{+\infty}' \| x_{\gamma(n)} \|^2 = \sum_{\alpha \in \mathcal{A}} \| x_\alpha \|^2$, so the first part of (i) follows from the definition of the convergence of $\sum_{\alpha \in \mathcal{A}}' x_\alpha$. And the second part is obvious from the definition of $\sum_{\alpha \in \mathcal{A}} x_\alpha$: $\sum_{\alpha \in \mathcal{A}} x_\alpha = \lim_{n \to +\infty} \sum_{k=1}^{n} x_{\gamma(k)}$. Using also the continuity of $\| \ \|$ we get

$$\Big\| \sum_{\alpha \in \mathcal{A}} x_\alpha \Big\|^2 = \lim_{n \to \infty} \Big\| \sum_{k=1}^{n} x_{\gamma(k)} \Big\|^2 = \sum_{k=1}^{+\infty} \| x_{\gamma(k)} \|^2 = \sum_{\alpha \in \mathcal{A}} \| x_\alpha \|^2$$

by Pyth. formula and by the part just proved. So (i) is proved.

Suppose that $\sum_{\alpha \in A} x_\alpha$ is convergent, $y \in \mathcal{H}$.

By Hölder inequality for the measure $\#$ in A and for "$p=q=2$", defining $\{\tilde{x}_\alpha\}_{\alpha \in A}$ as in Bessels lemma, we get

$$\sum_{\alpha \in A} |(y, x_\alpha)| = \sum_{\alpha \in A} \|x_\alpha\| \, |(y, \tilde{x}_\alpha)| \leqslant \Big(\sum_{\alpha \in A} \|x_\alpha\|^2\Big)^{\frac{1}{2}} \Big(\sum_{\alpha \in A} |(y, \tilde{x}_\alpha)|^2\Big)^{\frac{1}{2}}.$$

Now the Bessel ineq. and the part (i) give (b).

To obtain (a) let us use Fact on scalar sums to the function $f : A \to \mathbb{C}$ given by $f(\alpha) := (y, x_\alpha)$, $\alpha \in A$.

By the definition of $\sum_{\alpha \in A} x_\alpha$ we have (using the continuity of $(\cdot,\cdot)$)

$$\Big(y, \sum_{\alpha \in A} x_\alpha\Big) = \Big(y, \sum_{n=1}^{+\infty} x_{\gamma(n)}\Big) = \sum_{n=1}^{+\infty} (y, x_{\gamma(n)}) \qquad (1)$$

and the serie on (the) RHS is absolutely convergent, because

$$\sum_{n=1}^{+\infty} |(y, x_{\gamma(n)})| = \sum_{n=1}^{+\infty} |f(\gamma(n))| = \sum_{\alpha \in A} |f(\alpha)| = \sum_{\alpha \in A} |(y, x_\alpha)| < +\infty \qquad (1')$$

by the part (a) of Fact of sc. sums, and by the part (b) (just proved) above. Now, using the part (b) of Fact of sc. sums we get also by (1) and (1')

$$\Big(y, \sum_{\alpha \in A} x_\alpha\Big) = \sum_{n=1}^{+\infty} (y, x_{\gamma(n)}) = \sum_{n=1}^{+\infty} f(\gamma(n)) = \sum_{\alpha \in A} f(\alpha) = \sum_{\alpha \in A} (y, x_\alpha).$$

So (a) is proved.

The convergence of $\sum_{\alpha \in \tilde{A}} x_\alpha$ is clear from (i), because (by $\tilde{A} \subset A$)

$$\sum_{\alpha \in \tilde{A}} \|x_\alpha\|^2 \leqslant \sum_{\alpha \in A} \|x_\alpha\|^2.$$ Denote $x := \sum_{\alpha \in A} x_\alpha$ and $\tilde{x} := \sum_{\alpha \in \tilde{A}} x_\alpha$.

Consider now arbitrary $y' \in \mathcal{H}$. By the part (ii)(a) and by some obvious properties of the #-integrals $\left(= \text{"}\sum_{\alpha \in \mathcal{A}} \text{scalar sums"}\right)$ we get:

$$(y', \tilde{x}) = \sum_{\alpha \in \tilde{\mathcal{A}}} (y', x_\alpha) = \sum_{\alpha \in \mathcal{A}} (y', x_\alpha) = (y', x).$$

Hence $\forall_{y' \in \mathcal{H}} \; (y', \tilde{x} - x) = 0$, i.e. $(\tilde{x} - x) \perp \mathcal{H}$, so $\tilde{x} - x = 0$, and (ii) is proved.

To prove (iii) observe first, that

$$\sum_{\alpha \in \mathcal{A}'} \|x_\alpha\|^2 + \sum_{\alpha \in \mathcal{A}''} \|x_\alpha\|^2 = \sum_{\alpha \in \mathcal{A}} \|x\|^2$$

by the well known properties of integrals of non-negative functions. This gives the "iff" part of (iii). So when $\sum_{\alpha \in \mathcal{A}} x_\alpha$ is convergent then defining $x := \sum_{\alpha \in \mathcal{A}} x_\alpha$,

$x' := \sum_{\alpha \in \mathcal{A}'} x_\alpha$ and $x'' := \sum_{\alpha \in \mathcal{A}''} x_\alpha$ we get $x' \in \overline{\operatorname{lin}\{x_\alpha : \alpha \in \mathcal{A}'\}}$ and $x'' \in \overline{\operatorname{lin}\{x_\alpha : \alpha \in \mathcal{A}''\}}$ by the part (i), so $x' \perp x''$ because $\{x_\alpha\}_{\alpha \in \mathcal{A}}$ is an O.S. and $\mathcal{A}', \mathcal{A}''$ are disjoint (we use also the continuity of $(\cdot,\cdot)$). To prove that $x = x' + x''$ it suffices to prove that $\forall_{y \in \mathcal{H}} \; (y, x) = (y, x' + x'')$ (see the argumentation from the proof of (ii)), but this follows again from (ii)(a) formula and "the set-additivity" of scalar function #-integrals with fixed integrable function f used in the proof of (ii)(a):

$$\sum_{\alpha \in \mathcal{A}} f(\alpha) = \sum_{\alpha \in \mathcal{A}'} f(\alpha) + \sum_{\alpha \in \mathcal{A}''} f(\alpha). \quad \blacksquare$$

Sums of q.c.o systems and orthogonal projections

We shall now find some formulae for orthogonal projections using the notion of the sum of q.c.o.

Theorem ("The orthogonal projection formula")

Suppose that $\{x_\alpha\}_{\alpha\in A}$ is an o.s. in a Hilbert space $\mathcal{H}$. Let

$$Y := \overline{\operatorname{lin}\{x_\alpha : \alpha\in A\}}$$

and denote $\tilde{x}_\alpha := \begin{cases} 0 & \text{if } x_\alpha = 0, \\ \frac{x_\alpha}{\|x_\alpha\|} & \text{if } x_\alpha \neq 0. \end{cases}$

Then for any $x\in\mathcal{H}$ $\{(x,\tilde{x}_\alpha)\cdot\tilde{x}_\alpha\}_{\alpha\in A} \in QCO$ and

$$P_Y x = \sum_{\alpha\in A}{}' (x,\tilde{x}_\alpha)\tilde{x}_\alpha ; \qquad (2)$$

in particular the sum $\sum'_{\alpha\in A}(x,\tilde{x}_\alpha)\tilde{x}_\alpha$ is convergent and

$$\Big\|\sum_{\alpha\in A}{}'(x,\tilde{x}_\alpha)\tilde{x}_\alpha\Big\|^2 = \sum_{\alpha\in A}{}' |(x,\tilde{x}_\alpha)|^2 \le \|x\|^2. \qquad (3)$$

Proof

By Bessel lemma we get $\{(x,\tilde{x}_\alpha)\cdot\tilde{x}_\alpha\}_{\alpha\in A} \in QCO$ by its part (ii). So, we can use the Fact p. HS-48*) for it, and we get the convergence of $\sum'_{\alpha\in A}(x,\tilde{x}_\alpha)\tilde{x}_\alpha$, because $\sum'_{\alpha\in A}\|(x,\tilde{x}_\alpha)\tilde{x}_\alpha\|^2 = \sum'_{\alpha\in A}|(x,\tilde{x}_\alpha)|^2 \le \|x\|^2$, by the Bessel inequality; moreover we get also (3) by the part (ii)*) of the above Fact. To prove (2) observe first, that the RHS of (2) belongs to Y, by the definition of the sum of o.s., because

*) or p. HS3.1-12, Fact "on q.c.o." part (i)

$\tilde{x}_\alpha \in Y$ for any $\alpha \in A$. So, by Theorem "On orth. projection", to prove (2) is suffices to check, that

$$z := \Big(x - \sum_{\alpha \in A} (x, \tilde{x}_\alpha)\tilde{x}_\alpha \Big) \in Y^\perp. \qquad (4)$$

But if $\beta \in A$, then by part (i) of the Fact p. HS-48 *) we have

$$(\tilde{x}_\beta, z) = (\tilde{x}_\beta, x) - \Big(\tilde{x}_\beta, \sum_{\alpha \in A} (x, \tilde{x}_\alpha)\tilde{x}_\alpha\Big) = (\tilde{x}_\beta, x) - \sum_{\alpha \in A} \big(\tilde{x}_\beta, (x, \tilde{x}_\alpha)\tilde{x}_\alpha\big) =$$

$$= (\tilde{x}_\beta, x) - \sum_{\alpha \in A} \overline{(x, \tilde{x}_\alpha)}(\tilde{x}_\beta, \tilde{x}_\alpha) = (\tilde{x}_\beta, x) - (\tilde{x}_\beta, x)\|\tilde{x}_\beta\|^2$$

(we use the ortogonality of $\{\tilde{x}_\alpha\}_{\alpha \in A}$), hence

$$(\tilde{x}_\beta, z) = \begin{Bmatrix} 0 - 0 & \text{for } x_\beta = 0 \\ (\tilde{x}_\beta, x) - (\tilde{x}_\beta, x)\cdot 1 & \text{for } x_\beta \neq 0 \end{Bmatrix} = 0.$$

This means, that $\forall_{\beta \in A}\ x_\beta \perp z$ i.e. $z \in \{x_\alpha : \alpha \in A\}^\perp$.

But $Y^\perp = \{x_\alpha : \alpha \in A\}^\perp$, by Remark p. HS-21, so (4) holds. $\boxed{\text{///}}$

$\boxed{\text{Corollary}}$ ("The orthogonal projection formula II")

If $\{x_\alpha\}_{\alpha \in A}$ is an ortho<u>normal</u> system in a Hilbert space $\mathcal{H}$ and $Y = \overline{\operatorname{lin}\{x_\alpha : \alpha \in A\}}$, then for any $x \in \mathcal{H}$ the system $\{(x, x_\alpha)x_\alpha\}$ is a quasi-countable o.s., the sum $\sum_{\alpha \in A} (x, x_\alpha)x_\alpha$ is convergent and

$$P_Y x = \sum_{\alpha \in A} (x, x_\alpha)x_\alpha, \qquad \|P_Y x\|^2 = \sum_{\alpha \in A} |(x, x_\alpha)|^2 \leq \|x\|^2. \qquad (5)$$

$\boxed{\text{Proof}}$ We use the previous theorem, so the assertion is obvious by $\tilde{x}_\alpha = x_\alpha$. $\boxed{\text{///}}$

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*) or p. HS3.1-12, Fact "on q.c.o." part (ii)