A1. Additive measures and integration of bounded functions

A1.1 Additive measure and its variation

Warning! We use a traditional terminology, which is somewhat strange...
Additive real/complex measure is a more general (and not more particular...) notion than real/complex measure, which is again more general than finite measure known from Measure Theory. I.e., each measure which is finite is a real and complex measure (and it is ≥ 0 function of set which is σ-additive), each real/complex measure is additive real/complex measure (and it is σ-additive). So:

„additive measure" $\neq$ „measure" + „additive"

„real/complex measure" $\neq$ „measure" + „real/complex"

More precisely:
let $\mathfrak{M}$ be an algebra* of subsets of Ω and, as usual, let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We can consider the set (a linear

---

*) i.e.: $\emptyset \in \mathfrak{M}$, $\forall_{\omega \in \mathfrak{M}}\ \Omega \setminus \omega \in \mathfrak{M}$, $\forall_{\omega, \omega' \in \mathfrak{M}}\ \omega \cup \omega' \in \mathfrak{M}$. In particular each σ-algebra is an algebra.

space) $\ell(\mathfrak{M})$ of all functions from $\mathfrak{M}$ to $\mathbb{K}$, and its subspace $\ell^{\infty}(\mathfrak{M})$ of bounded functions *)

Let $\mu \in \ell(\mathfrak{M})$.

$\boxed{\text{Definition}}$

- μ is $\underline{\mathbb{K}\text{-additive measure}}$ **) iff $\forall_{\omega,\omega' \in \mathfrak{M}} \left(\omega \cap \omega' = \emptyset \Rightarrow \mu(\omega \cup \omega') = \right.$ $\left. = \mu(\omega) + \mu(\omega') \right).$

- if $\mathfrak{M}$ is (moreover) a σ-algebra, then μ is $\underline{\mathbb{K}\text{-}\sigma\text{-additive measure}}$ **), which we abbreviate to $\underline{\mathbb{K}\text{-measure}}$ **), iff for any sequence $\{\omega_n\}_{n \geq 1}$ of sets from $\mathfrak{M}$ $\left(\forall_{i,j \geq 1} (i \neq j \Rightarrow \omega_i \cap \omega_j = \emptyset) \right) \Rightarrow \mu\left(\bigcup_{n \geq 1} \omega_n\right) =$ $= \sum_{n=1}^{+\infty} \mu(\omega_n)$ ***).

$\boxed{\text{Remarks}}$

1. Be carefull, because for both kinds μ as above $\mu(\omega) \in \mathbb{K}$ for $\underline{\text{any}}$ $\omega \in \mathfrak{M}$, so we can't have „$\mu(\omega) = +\infty$". It means, that if μ is a $\underline{\text{measure}}$ in a classical Measure Theory sense, then

---

*) But be carefull! – Distinguish them from $\ell(\Omega)$ and $\ell^{\infty}(\Omega)$!

**) We say $\underline{\text{real/complex additive measure}}$ for $\mathbb{K} = \mathbb{R}/\mathbb{C}$, and similarly with $\underline{\mathbb{K}/\text{real/complex } \sigma\text{-additive measure}}$ $=$ $= \underline{\mathbb{K}/\text{real/complex measure}}$!

***) In particular the scalar serie $\sum_{n=1}^{+\infty} \mu(\omega_n)$ is convergent!

μ is a $\mathbb{K}$-measure (or $\mathbb{K}$-additive measure) iff μ is finite! E.g.,

the Lebesgue measure on $\mathbb{R}$ is not a real/complex measure! But on $[0;1]$ - it is (with $\mathfrak{M}$ being, e.g., the Borel σ-alg.).

2. If μ is an ($\mathbb{K}$-)additive measure, but it is not ≥ 0, then the „property"

$$\forall_{\omega \in \mathfrak{M}} \quad \mu(\omega) \leq \mu(\Omega)$$

is generally not true (also for $\mathbb{K} = \mathbb{R}$ ...); but $\mu(\emptyset) = 0$ is true ...

3. If μ is a $\mathbb{K}$-measure, then it is a $\mathbb{K}$-additive measure (for $\mathfrak{M}$ - σ-algebra)*).

Denote by

$$\ell_{add}(\mathfrak{M}), \quad \ell_{\sigma\text{-}add}(\mathfrak{M})$$

the sets of all $\mathbb{K}$-additive measures, and $\mathbb{K}$-measures, resp.

We obviously have

$$\ell_{add}(\mathfrak{M}), \ell_{\sigma\text{-}add}(\mathfrak{M}) \underset{lin}{\subset} \ell(\mathfrak{M}).$$

We denote also

$$\ell^{\infty}_{add}(\mathfrak{M}) := \ell^{\infty}(\mathfrak{M}) \cap \ell_{add}(\mathfrak{M}), \qquad (1)$$

so

$$\ell^{\infty}_{add}(\mathfrak{M}) \underset{lin}{\subset} \ell^{\infty}(\mathfrak{M}).$$

Recall that $\ell^{\infty}(\mathfrak{M})$ is a Banach space with its standard $\|\ \|_\infty$-norm, so $\ell^{\infty}_{add}(\mathfrak{M})$ is a norm space in the norm subspace sense.

---

*) Because $\emptyset = \bigcup_{n \geq 1} \emptyset$, so $\mu(\emptyset) = \sum_{n=1}^{+\infty} \mu(\emptyset) \Rightarrow \mu(\emptyset) = 0$.

It can be somewhat strange (but it is true, and quite easy to prove ...), that we don't need to define a notation similar to (1) for the „σ-add" case!

Fact 1

If $\mathfrak{M}$ is a σ-algebra, then $\ell_{\sigma\text{-add}}^{(\mathfrak{M})} \subset \ell^{\infty}(\mathfrak{M})$.

Proof $\longrightarrow \triangle$ *) 4

So, generally we have two norm spaces:

$$\ell^{\infty}_{add}(\mathfrak{M}) \subset \ell^{\infty}(\mathfrak{M}),$$

and for $\mathfrak{M}$ being a σ-algebra – three of them:

$$\ell_{\sigma\text{-add}}(\mathfrak{M}) \subset \ell^{\infty}_{add}(\mathfrak{M}) \subset \ell^{\infty}(\mathfrak{M}).$$

Fact 2

$\ell^{\infty}_{add}(\mathfrak{M})$ is a closed subspace of $\ell^{\infty}(\mathfrak{M})$, and when $\mathfrak{M}$ is a σ-alg., then also $\ell_{\sigma\text{-add}}(\mathfrak{M})$ is closed. In particular, both ℓ^{∞}_{add}, $\ell_{\sigma\text{-add}}$ are Banach spaces (with $\| \ \|_{\infty}$ norm).

Proof

The part „for additive μ" is trivial, since the pointwise convergence follows from the convergence in $\| \ \|_{\infty}$.

---

*) Hint: for a sequence $\{\omega_n\}_{n \geq 1}$ of sets from $\mathfrak{M}$ try to find „an appropriate" sequence $\{\tilde{\omega}_n\}_{n \geq 1}$ of disjoint $\mathfrak{M}$-sets...

But the proof of „σ-add" part is „almost the same" as for the proof that the uniform limit of continuous function is also continuous ... $\rightarrow \triangle$. $\boxed{4}$

Observe that for $\mu \in \ell(\mathfrak{M})$ its absolute value function $|\mu|$ (in the usual „pointwise" sense: $|\mu|(\omega) := |\mu(\omega)|$) is also in $\ell(\mathfrak{M})$. The same is true for $\ell^\infty(\mathfrak{M})$ – it is „$|\ |$-invariant", but it is NOT true for ℓ_{add}!

We shall define something which would be „a compromise" between being „$|\ |$ of μ" and possesing „the invariance property for ℓ_{add}" – it will be the variation-operation.

For any $\mu \in \ell(\mathfrak{M})$ denote by $\mathrm{var}\,\mu$ a new function from $\mathfrak{M}$ to $[0; +\infty]$ (so: not to $\mathbb{K}$ – it may happen...) called <u>variation of μ</u> and defined by

$$(\mathrm{var}\,\mu)(\omega) := \sup\left\{\sum_{j=1}^{n} |\mu(\omega_j)| : \{\omega_j\}_{j=1}^{n} \text{ is a } \mathfrak{M}\text{-d.d. of } \omega, n \in \mathbb{N}\right\}, \omega \in \mathfrak{M}. \quad (2)$$

Above we abbreviate: d.d. = disjoint decomposition, and <u>$\{\omega_j\}_{j=1}^{n}$ is a $\mathfrak{M}$-d.d. of ω</u> iff $\forall_{j=1,\dots n}\ \omega_j \in \mathfrak{M}$, $\bigcup_{j=1}^{n} \omega_j = \omega$, and $\forall_{i,j=1,\dots n} (i \neq j \Rightarrow \omega_i \cap \omega_j = \emptyset)$.

We similarly define a $\mathfrak{M}$-d.d. for infinite sequences $\{\omega_j\}_{j\geq 1}$ (however $\bigcup_{j=1}^{\infty} \omega_j$ can be not in $\mathfrak{M}$, when all ω_j-s are, if we do not assume, that $\mathfrak{M}$ is $\underline{\sigma}$-algebra ...).

The operation var is defined for any $\mu \in \ell(\mathfrak{M})$, however it will be an important tool for us mainly when $\mu \in \ell_{add}(\mathfrak{M})$ (especially when $\mu \in \ell^\infty_{add}(\mathfrak{M})$).

As we can see below, our goal concerning the „compromise properties" of the operation var is reached.

$\boxed{\text{Fact}}$ („On variation")

Suppose that $\mu \in \ell(\mathfrak{M})$. Then:

1.
$$\forall_{\substack{\omega, \omega' \in \mathfrak{M} \\ \omega \subset \omega'}} \quad |\mu(\omega)| \leq (\operatorname{var}\mu)(\omega) \leq (\operatorname{var}\mu)(\omega')$$

2. if $\{\omega_n\}_{n \geq 1}$ is a $\mathfrak{M}$-d.d. of $\omega \in \mathfrak{M}$, then
$$\sum_{n=1}^{+\infty} |\mu(\omega_n)| \leq (\operatorname{var}\mu)(\omega); \quad {}^{*)}$$

3. if $\mu(\emptyset) \neq 0$, then $\forall_{\omega \in \mathfrak{M}} \ (\operatorname{var}\mu)(\omega) = +\infty$;

4. $\operatorname{var}\mu$ is super-additive, i.e.,
$$\forall_{A,B \in \mathfrak{M}} \ \big(A \cap B = \emptyset \Rightarrow (\operatorname{var}\mu)(A \cup B) \geq (\operatorname{var}\mu)(A) + (\operatorname{var}\mu)(B)\big)$$

5.
$$\forall_{\omega \in \mathfrak{M}} \quad \sup\{|\mu(\tilde{\omega})| : \tilde{\omega} \in \mathfrak{M}, \tilde{\omega} \subset \omega\} \leq (\operatorname{var}\mu)(\omega);$$

6. if $\mu \in \ell_{add}(\mathfrak{M})$, then $\operatorname{var}\mu$ is additive${}^{**)}$, i.e.,
$$\forall_{A,B \in \mathfrak{M}} \ \big(A \cap B = \emptyset \Rightarrow (\operatorname{var}\mu)(A \cup B) = (\operatorname{var}\mu)(A) + (\operatorname{var}\mu)(B)\big);$$

7. if $\mathfrak{M}$ is a σ-algebra and $\mu \in \ell_{\sigma\text{-add}}(\mathfrak{M})$, then $\operatorname{var}\mu$ is σ-additive, ${}^{***)}$ i.e is a measure on $\mathfrak{M}$ (in the classical sense)

---

*) So, for $\omega \in \mathfrak{M}$ we could replace in (2) the finite $\mathfrak{M}$-d.d.-s $\{\omega_j\}_{j=1,\dots,n}$ by the infinite $\{\omega_j\}_{j \geq 1}$ ones.

**) which <u>doesn't mean</u>, that $\operatorname{var}\mu \in \ell_{add}(\mathfrak{M})$, since it could reach the $+\infty$ value...

***) i.e., $\sum_{n=1}^{+\infty} (\operatorname{var}\mu)(\omega_n) = (\operatorname{var}\mu)(\omega)$ for any ω and $\{\omega_n\}_{n \in \mathbb{N}}$ being $\mathfrak{M}$-d.d. of ω — here the $+\infty$ values of $(\operatorname{var}\mu)$ are also possible „a priori", but see later (

8. $\forall_{\omega \in \mathfrak{M}} \quad (\mathrm{var}\,\mu)(\omega) \leq C_{\mathbb{K}} \cdot \sup\{|\mu(\tilde{\omega})| : \tilde{\omega} \in \mathfrak{M},\ \tilde{\omega} \subset \omega\}$,

where $C_{\mathbb{K}} = \begin{cases} 2 & \text{for } \mathbb{K} = \mathbb{R} \\ 4 & \text{for } \mathbb{K} = \mathbb{C} \end{cases}$;

if $\omega \in l_{add}(\mathfrak{M})$, then

9. if $\mu \in l_{add}(\mathfrak{M})$, then $\mu \in l^{\infty}_{add}(\mathfrak{M})$ iff $\mathrm{var}(\mu)(\Omega) < +\infty$.

Proof

1. If $\omega \in \mathfrak{M}$ then (ω) - a $\mathfrak{M}$-d.d. of ω of „the length 1", so $|\mu(\omega)| \leq (\mathrm{var}\,\mu)(\omega)$. If $\omega \subset \omega'$ and also $\omega' \in \mathfrak{M}$ then for any $\{\omega_j\}_{j=1}^n$ being a $\mathfrak{M}$-d.d. of ω $(\omega_1, \ldots, \omega_n, \omega' \setminus \omega)$ is a $\mathfrak{M}$-d.d. of ω', so

$$\sum_{j=1}^n |\mu(\omega_j)| \leq \left(\sum_{j=1}^n |\mu(\omega_j)|\right) + |\mu(\omega' \setminus \omega)| \leq (\mathrm{var}\,\mu)(\omega'),$$

which gives $(\mathrm{var}\,\mu)(\omega) \leq (\mathrm{var}\,\mu)(\omega')$.

2. For any $n \in \mathbb{N}$ we have

$$\sum_{k=1}^n |\mu(\omega_k)| \leq (\mathrm{var}\,\mu)\Big(\dot{\bigcup_{k=1}^n} \omega_k\Big) \leq (\mathrm{var}\,\mu)(\omega),$$

by 1., but $\sum_{h=1}^{+\infty} |\mu(\omega_h)| = \lim_{n \to +\infty} \sum_{k=1}^n |\mu(\omega_k)|$, so we get 2.

3. $\omega = \bigcup_{k \geq 1} \omega_k$ for $\omega_k = \begin{cases} \omega & k = 1 \\ \emptyset & k > 1 \end{cases}$, so by 2.

$$(\mathrm{var}\,\mu)(\omega) \geq \sum_{h=1}^{\infty} |\mu(\omega_h)| = |\mu(\omega)| + |\mu(\emptyset)| \cdot \sum_{h=1}^{+\infty} 1 = +\infty$$

4. Suppose, that for some $A, B \in \mathfrak{M}$, $A \cap B = \emptyset$

$(\mathrm{var}\,\mu)(A \cup B) < (\mathrm{var}\,\mu)(A) + (\mathrm{var}\,\mu)(B)$. So all the $(\mathrm{var}\,\mu)(A)$, $(\mathrm{var}\,\mu)(B)$, $\mathrm{var}(A \cup B)$ must be finite, and let $\varepsilon := RHS - LHS$ in the above inequality - i.e. $\varepsilon > 0$. Choose

$(\omega_1, \dots \omega_n)$ and $(\omega_1', \dots, \omega_m')$ being such $\mathfrak{M}$-d.d. of A and B, resp., that $\sum_{j=1}^{n} |\mu(\omega_j)| > (\mathrm{var}\,\mu)(A) - \varepsilon/2$ and $\sum_{j=1}^{m} |\mu(\omega_j')| > (\mathrm{var}\,\mu)(B) - \varepsilon/2$. So $(\omega_1, \dots, \omega_n, \omega_1', \dots, \omega_m')$ is such $\mathfrak{M}$-d.d. of $A \cup B$, that $\sum_{j=1}^{n} |\mu(\omega_j)| + \sum_{j=1}^{m} |\mu(\omega_j')| >$ $(\mathrm{var}\,\mu)(A) + (\mathrm{var}\,\mu)(B) - \varepsilon$. This contradicts our assumption.

5. is obvious by 1.

6. By 4. we need only find the proof of sub-additivity. Let $\omega_1, \dots, \omega_n$ be a $\mathfrak{M}$-d.d. of $A \cup B$. Defining $\omega_{jA} := \omega_j \cap A$ and $\omega_{jB} := \omega_j \cap B$, $j = 1, \dots n$ we get $(\omega_{1A}, \dots, \omega_{nA})$, $(\omega_{1B}, \dots, \omega_{nB})$ being $\mathfrak{M}$-d.d. of A and B, resp.

But
$$\sum_{j=1}^{n} |\mu(\omega_j)| = \sum_{j=1}^{n} |\mu(\omega_{jA}) + \mu(\omega_{jB})| \leq \sum_{j=1}^{n} |\mu(\omega_{jA})| + \sum_{j=1}^{n} |\mu(\omega_{jB})| \leq (\mathrm{var}\,\mu)(A) + (\mathrm{var}\,\mu)(B),$$
which gives
$$(\mathrm{var}\,\mu)(A \cup B) \leq \text{---}''\text{---} + \text{---}''\text{---}.$$

7. The proof of sub-σ-additivity *) is almost the same, as in 6. $\longrightarrow$. If $\{A_n\}_{n \geq 1}$ is a $\mathfrak{M}$-d.d. of ω then for any $n \geq 1$ we have $(\mathrm{var}\,\mu)(\omega) \geq (\mathrm{var}\,\mu)(\bigcup_{k=1}^{n} A_k)$ by 1., but by 4. $(\mathrm{var}\,\mu)(\bigcup_{k=1}^{n} A_k) \geq \sum_{k=1}^{n} (\mathrm{var}\,\mu)(A_k)$, which gives super-$\sigma$-additivity *).

8. Suppose first, that $\mathbb{K} = \mathbb{R}$. Let $(\omega_1, \dots, \omega_n)$ be a $\mathfrak{M}$-d.d. of ω. Define $I := \{i \in \{1, \dots, n\} : \mu(\omega_i) \geq 0\}$, and $J := \{1, \dots, n\} \setminus I$.

---

*) which is defined as follows ... ($\longrightarrow$)

Then $\sum_{k=1}^{n} |\mu(\omega_k)| = \sum_{i\in I} \mu(\omega_i) + \sum_{j\in J} (-\mu(\omega_j)) =$

$$= \mu\left(\bigcup_{i\in I} \omega_i\right) - \mu\left(\bigcup_{j\in J} \omega_j\right) = \left|\mu\left(\bigcup_{i\in I}\omega_i\right)\right| + \left|\mu\left(\bigcup_{j\in J}\omega_j\right)\right| \leq$$

$\leq 2 \sup\{|\mu(\tilde\omega)| : \tilde\omega \in \mathfrak{M}, \tilde\omega \subset \omega\}$. This gives: 8. for $\mathbb{K}=\mathbb{R}$.

When $\mathbb{K} = \mathbb{C}$, let us study $\mathrm{Re}\,\mu$ and $\mathrm{Im}\,\mu$. Obviously: $\mathrm{Re}\,\mu, \mathrm{Im}\,\mu \in \ell_{add}(\mathfrak{M})$ and both are real functions. We can easily check ($\longrightarrow$ Ⓐ), that

$(\mathrm{var}\,\mu)(\omega) \leq (\mathrm{var}\,\mathrm{Re}\,\mu)(\omega) + (\mathrm{var}\,\mathrm{Im}\,\mu)(\omega)$,

which finishes the proof, by the part of 8. just proved for $\mathbb{R}$.

9. „$\Leftarrow$" is obvious by 1. To get „$\Rightarrow$" we should use 8. with $\omega := \Omega$.

$\boxed{\mu}$

Let us collect now the most important results for the case $\mu \in \ell^\infty_{add}(\mathfrak{M})$ (in particular - for $\mu \in \ell_{\sigma\text{-}add}(\mathfrak{M})$).

Theorem („On variation of (bounded) $\mathbb{K}$-additive measure")

If $\mu \in \ell^\infty_{add}(\mathfrak{M})$, then

(1) $\mathrm{var}\,\mu \in \ell^\infty_{add}(\mathfrak{M})$, $\mathrm{var}\,\mu \geq 0$;

(2) $\forall_{\substack{\omega, \tilde\omega \in \mathfrak{M} \\ \tilde\omega \subset \omega}} \quad \frac{1}{c_{\mathbb{K}}}(\mathrm{var}\,\mu)(\omega) \leq |\mu(\tilde\omega)| \leq (\mathrm{var}\,\mu)(\omega)$;

(3) $\frac{1}{c_{\mathbb{K}}}(\mathrm{var}\,\mu)(\Omega) \leq \|\mu\|_\infty \leq (\mathrm{var}\,\mu)(\Omega)$;

(4) if, moreover, $\mu \in l_{\sigma\text{-add}}(\mathfrak{M})$ with $\mathfrak{M}$ - a σ-algebra, then $\operatorname{var}\mu \in l_{\sigma\text{-add}}(\mathfrak{M})$,

where $C_{\mathbb{K}} = \begin{cases} 2 & \text{for } \mathbb{K} = \mathbb{R} \\ 4 & \text{for } \mathbb{K} = \mathbb{C} \end{cases}$.

Proof

We get:
- (1) from 6. and 8. of Fact,
- (2) from 8. and 5. of Fact,
- (3) from the above (2) with $\omega = \Omega$
- (4) from 7. of Fact and from the above (1) + Fact 1 page APP-4.

$\boxed{\Box}$

A1.2 The norm space $(l^{\infty}_{add}(\mathfrak{M}), \|\ \|_{var})$

As we shall see, the previous norm $\|\ \|_{\infty}$ in $l^{\infty}_{add}(\mathfrak{M})$ can be replaced by a „more complicated", but also more convenient one. Define for $\mu \in l^{\infty}_{add}(\mathfrak{M})$

$$\|\mu\|_{var} := (\operatorname{var}\mu)(\Omega).$$

By Theorem „On variation..." p. (3) $\|\mu\|_{var} \in [0;+\infty)$ so $\|\cdot\|_{var} : l^{\infty}_{add}(\mathfrak{M}) \longrightarrow [0;+\infty)$.

Fact

$\|\ \|_{var}$ is a norm in $l^{\infty}_{add}(\mathfrak{M})$ and $\|\ \|_{var} \equiv \|\ \|_{\infty}$.

Proof

The "non-degeneracy" condition for $\| \; \|_{var}$ holds (e.g.) from (3) of Theorem "On variation..." p. APP-9.
The condition of "|homogenity|" ($\|\lambda\mu\|_{var} = |\lambda| \, \|\mu\|_{var}$) is clear by the definiton of $(\operatorname{var}\mu)(\Omega)$. Let us prove the triangle inequality:

$$\|\mu+\nu\|_{var} = (\operatorname{var}(\mu+\nu))(\Omega) = \sup\Big\{\sum_{j=1}^{n} |\mu(\omega_j)+\nu(\omega_j)| : \{\omega_j\}_{j=1}^{n} \text{ is a } \mathfrak{M}\text{-d.d. of } \Omega, \; n\in\mathbb{N}\Big\},$$

but $\sum_{j=1}^{n} |\mu(\omega_j)+\nu(\omega_j)| \leq \sum_{j=1}^{n} |\mu(\omega_j)| + \sum_{j=1}^{n} |\nu(\omega_j)| \leqslant \|\mu\|_{var} + \|\nu\|_{var}$ for any $\{\omega_j\}_{j=1}^{n}$ — an $\mathfrak{M}$-d.d. of Ω, which gives $\|\mu+\nu\|_{var} \leq \|\mu\|_{var} + \|\nu\|_{var}$.
Now the equivalence of $\| \; \|_{var}$ and $\| \; \|_{\infty}$ is just (3) of Theorem "On variation...". $\square$

Corollary

$(\ell^{\infty}_{add}(\mathfrak{M}), \| \; \|_{var})$ is a Banach space. If moreover $\mathfrak{M}$ is a σ-algebra, then also $(\ell_{\sigma\text{-add}}(\mathfrak{M}), \| \; \|_{var})$ is a Banach space.

Proof It follows from the equivalence of $\| \; \|_{var}$ and $\| \; \|_{\infty}$ and Fact 2 page APP-4. $\square$

The "more convenience" of $\| \ \|_{var}$, which we announced at the begining of this subsection, will be seen when we try to define the integration with respect to a $\mathbb{K}$-additive measure from $\ell^\infty_{add}(\mathfrak{M})$.

A1.3 The integration with respect to $\mathbb{K}$-additive measure

Suppose again that $\mathfrak{M}$ is an algebra of subsets of $\Omega \neq \emptyset$ and that $\mu \in \ell^\infty_{add}(\mathfrak{M})$. We shall define "$\int_\Omega f \, d\mu$" for some (scalar) functions f on Ω, starting from simple functions f – similarly as in the definition of the integral with respect to a measure. Let $S(\Omega, \mathfrak{M})$ be a set of all simple ($\mathfrak{M}$-measurable simple) functions, i.e. is a linear subspace of $\mathcal{M}_b(\Omega, \mathfrak{M})$ space*) given by

$$S(\Omega, \mathfrak{M}) := \operatorname{lin}\{\chi_\omega : \omega \in \mathfrak{M}\}$$

(where χ_ω is a characteristic (= indicator) function of ω: $\chi_\omega(t) = \begin{cases} 1 & t\in\omega \\ 0 & t\notin\omega \end{cases}$). Each $f \in S(\Omega, \mathfrak{M})$ posesses a unique representation of the form

$$f = \sum_{k=1}^{m} \lambda_k \chi_{\omega_k}$$

where: $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\sum_{k=1}^{0} \ldots := 0$, (and for $m \geq 1$) the numbers $\lambda_1, \ldots \lambda_m \in \mathbb{K} \setminus \{0\}$ are mutually different and the sets $\omega_1, \ldots, \omega_k \in \mathfrak{M}$ mutually disjoint.

---

*) see *) on APP-15 !

(to get this, we just take the $f(\Omega)\setminus\{0\}$ which is a finite set and $\lambda_1, \dots, \lambda_m$ is any numeration of this set ($m = 0$ if it is $\emptyset$, generally $m := \#(f(\Omega)\setminus\{0\})$ and $\omega_k := f^{-1}(\{\lambda_k\})$ for $k = 1, \dots, m$).

For such f we define:

$$\int_\Omega f\, d\mu := \sum_{k=1}^{m} \lambda_k\, \mu(\omega_k). \qquad (1)$$

So we have defined an operation/function:

$$\int_\Omega \cdot\, d\mu \;:\; S(\Omega, \mathfrak{M}) \longrightarrow \mathbb{K}.$$

$\boxed{\text{Fact}}$

$\int_\Omega \cdot\, d\mu \in \big(S(\Omega, \mathfrak{M})\big)^{\#}$ and

$$\forall_{f \in S(\Omega, \mathfrak{M})} \quad \Big|\int_\Omega f\, d\mu\Big| \leq \|f\|_\infty \cdot \|\mu\|_{var}. \qquad (2)$$

This means that $\int_\Omega \cdot\, d\mu \in \big(S(\Omega, \mathfrak{M})\big)^*$ and

$\|\int_\Omega \cdot\, d\mu\| \leq \|\mu\|_{var}$ *), where we treat $S(\Omega, \mathfrak{M})$ as the norm subspace of the norm space $\mathcal{M}_b(\Omega, \mathfrak{M})$ (with the norm $\|\ \|_\infty$ **)).

---

*) In fact it $= \|\mu\|_{var}$ $\longrightarrow$ $\triangle\!\!\!\!\mathrm{M}$

**) but, (please,) distinguish it from $\|\ \|_\infty$ in $\ell^\infty(\mathfrak{M})$!!!

Proof

The linearity can be proved in the exactly the same way, as the linearity of the „usual" integral with resp. to a „usual" measure on the simple functions space (it is a simple, but not very short proof, but try to invent a more elegant argumentation ... → ⚠).

To get (2) we can simply estimate using (1):

$$\left|\int_\Omega f\,d\mu\right| \leq \sum_{k=1}^{m} |\lambda_k|\,|\mu(\omega_k)| \leq \left(\max_{k=1,\dots m} |\lambda_k|\right)\cdot\sum_{k=1}^{m}|\mu(\omega_k)|$$

for f of the form from page APP-12, $f \neq 0$ (for $f = 0$ (2) is true...), but $\max_{k=1,\dots,m} |\lambda_k| = \|f\|_\infty$ and $\omega_1, \dots \omega_k$ are mutually disjoint, so $\sum_{k=1}^{m}|\mu(\omega_k)| \leq \left(\sum_{k=1}^{m}|\mu(\omega_k)|\right) + \left|\mu\left(\Omega \setminus \bigcup_{k=1}^{m}\omega_k\right)\right| \leq (\mathrm{Var}\,\mu)(\Omega) = \|\mu\|_{var}$. $\boxed{\square}$

Now we can make use of Theorem „On extension of bounded operators" (p. OF-40) – it gives a unique continuous extension of the functional $\int_\Omega \cdot \, d\mu$ from the space $S(\Omega, \mathfrak{M})$ onto its closure $\overline{S(\Omega, \mathfrak{M})}$ in $\mathcal{M}_b(\Omega, \mathfrak{M})$!

Moreover, the norm of this extension will be again $\leq \|\mu\|_{var}$.

The following fact can be easily proved $\longrightarrow$ △

$\boxed{\text{Fact}}$

$$\overline{S(\Omega, \mathfrak{M})} = \mathcal{M}_b(\Omega, \mathfrak{M}).$$

— The proof is almost the same as for „classical" Measure Theory case with $\mathfrak{M}$ — a σ-algebra (see Fact (C) „On approximation by simple functions" formulated — but not proved... — on p. PB-27).

With the above result we get the main result of our construction, but first we define / denote:

for any $f \in \mathcal{M}_b(\Omega, \mathfrak{M})$ its integral with respect to μ is the value (at f) of the unique continuous extension of the functional $\int_\Omega \cdot \, d\mu$ defined by (1) on $S(\Omega, \mathfrak{M})$. It is denoted by the same symbol $\int_\Omega f \, d\mu$!

And also the extension of $\int_\Omega \cdot \, d\mu$ onto $\mathcal{M}_b(\Omega, \mathfrak{M})$ is denoted here by $\int_\Omega \cdot \, d\mu$.

$*$) — from p. APP-12: in fact the space $\mathcal{M}_b(\Omega, \mathfrak{M})$ was not defined for any algebra $\mathfrak{M}$, but only for σ-algebras — — see p. PB-31 and PB-46 (for Banach space information...)

VERTE $\longrightarrow$

$\boxed{\text{Theorem}}$ ("On integral")

The $\int_\Omega \cdot \, d\mu$ on $\mathcal{M}_b(\Omega, \mathfrak{M})$ is a continuous linear functional, being the extension of the functional $S(\Omega, \mathfrak{M})$ given by (1) and

$$\left\| \int_\Omega f d\mu \right\| \leqslant \|f\|_\infty \|\mu\|_{var}$$

for any $f \in \mathcal{M}_b(\Omega, \mathfrak{M})$.

$\boxed{\text{Proof}}$ — is placed before the formulation ... $\boxed{L}$

---

from p. APP-15 : the definition for $\mathfrak{M}$ - an algebra is analogic (continuation) but masurability with respect to $\mathfrak{M}$ should be defind: f is $\mathfrak{M}$- measurable when $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ is, and for real f it means that $f^{-1}((-\infty; c])$ is in $\mathfrak{M}$ for any $c \in \mathbb{R}$.