University of Warsaw

Faculty of Mathematics, Informatics and Mechanics

Michał Miśkiewicz

Singularities of minimizing harmonic maps into closed manifolds

PhD dissertation

Supervisor

dr hab. Anna Zatorska-Goldstein
Institute of Applied Mathematics and Mechanics
University of Warsaw

March 2019

Author’s declaration:

I hereby declare that this dissertation is my own work.

March 6, 2019 ....

Michał Miśkiewicz

Supervisor’s declaration:

The dissertation is ready to be reviewed.

March 6, 2019 ....

dr hab. Anna Zatorska-Goldstein

Abstract

This thesis is concerned with singularities of *minimizing harmonic maps* into closed manifolds, with special emphasis on maps into the sphere $\mathbb{S}^2$. By definition, they are maps that minimize the Dirichlet energy $E(u) = \int |\nabla u|^2$ with respect to given boundary conditions. Since the 80's, such maps are known to be smooth outside a closed set of codimension at least 3, called the *singular set*; recently, its codimension 3 Hausdorff measure was shown to be locally finite.

First, we present a regularity theorem for the singular set. For maps into $\mathbb{S}^2$, we show that the singular set is indeed a codimension 3 topological submanifold (up to a set of measure zero), thus excluding possible arbitrary *gaps* in the singular set. This was previously known only for domains of dimension 4.

Next, we give various extensions of Naber and Valtorta's discrete Reifenberg theorem, a general tool in geometric measure theory that yields upper measure bounds for sets satisfying Reifenberg-type flatness conditions. We also illustrate the applications in the study of singularities.

Finally, building upon previously known local measure estimates, we study how the singularities of a minimizer $u\colon \Omega \to \mathbb{S}^2$ depend on its boundary map $\varphi = u|_{\partial\Omega}$. It is shown that the measure of the singular set can be estimated linearly in terms of the boundary energy $\int_{\partial\Omega} |\nabla\varphi|^{n-1}$, where $n = \dim \Omega$. Moreover, the singular set is stable (in Wasserstein distance) with respect to $W^{1,n-1}$-perturbations of the boundary map.

Keywords: harmonic maps, singularities, Reifenberg parametrization

AMS Subject Classification: 58E20, 35J20, 35A20

Streszczenie

Tematem niniejszej pracy są osobliwości *minimalizujących przekształceń harmonicznych* o wartościach w zamkniętej rozmaitości, ze szczególnym uwzględnieniem przekształceń w sferę $\mathbb{S}^2$. Z definicji są to przekształcenia z zadanym warunkiem brzegowym, minimalizujące energię Dirichleta $E(u) = \int |\nabla u|^2$. Od ponad 30 lat wiadomo, że przekształcenia takie są gładkie poza pewnym zbiorem domkniętym kowymiaru co najmniej 3, zwanym *zbiorem osobliwym*. Jednak dopiero w ostatnich latach wykazano, że miara Hausdorffa (kowymiaru 3) tego zbioru jest lokalnie skończona.

Zaczniemy od zbadania regularności zbioru osobliwego. Dla przekształceń w $\mathbb{S}^2$ dowiedziemy mianowicie, że zbiór osobliwy jest topologiczną podrozmaitością kowymiaru 3 (z dokładnością do zbioru miary zero), wykluczając w ten sposób możliwe *dziury* w zbiorze osobliwym. Taki wynik był dotychczas znany jedynie dla dziedzin wymiaru 4.

Następnie zaprezentujemy możliwe uogólnienia pochodzącego od Nabera i Valtorty dyskretnego twierdzenia Reifenberga. Są to ogólne narzędzia z zakresu geometrycznej teorii miary, pozwalające na uzyskanie górnych ograniczeń na miarę zbiorów spełniających odpowiednie założenia płaskości typu Reifenberga. Omówimy też zastosowanie takich twierdzeń do badania osobliwości.

Na koniec wykorzystamy i wzmocnimy dostępne lokalne oszacowania, by zbadać zależność osobliwości przekształcenia minimalizującego $u\colon \Omega \to \mathbb{S}^2$ od jego przekształcenia brzegowego $\varphi = u|_{\partial\Omega}$. Wykażemy, że miarę zbioru osobliwego można oszacować w sposób liniowy przez energię brzegową $\int_{\partial\Omega} |\nabla\varphi|^{n-1}$, gdzie $n = \dim\Omega$. Co więcej, pokażemy stabilność osobliwości (w sensie odległości Wassersteina) przy zaburzeniach przekształcenia brzegowego w normie $W^{1,n-1}$.

Acknowledgments

First of all, I would like to express my gratitude to my PhD advisors Anna Zatorska-Goldstein and Paweł Strzelecki. During these years, they taught me most of what I know in PDEs, gave me their unconditional support and guided me towards being an independent mathematician. I remember Paweł praising my choice of specialization years ago, saying that *any fool can achieve something in mathematical analysis*. I hope to prove him right someday.

I am grateful to my coauthors Katarzyna Mazowiecka and Armin Schikorra, as well as other mathematicians who helped me along the way, including Aaron Naber, Piotr Rybka and Michał Łasica. I would also like to thank Tristan Rivière for his hospitality during my stay in Zurich.

I would not follow this path if it were not for my high school teachers, Wiktor Bartol and Jerzy Bednarczuk. Thank you for showing me what mathematics is!

Special thanks are due to my friends, with whom I engaged in the Tuesday Burger Group, the Faculty Choir and all sorts of sports. Grześ Bokota, Marcin Fatyga, Maks Grab, Tomek Gródek, Kasia Kowal, Zosia Aldonka Michalik, Łukasz Rajkowski, Mateusz Rapicki – to name at least some of you – I greatly appreciate the time we spent together!

I could not be more grateful to my family – my parents, brothers, sisters, nephews and nieces. Their constant encouragement helped me overcome all the obstacles on the way.

Finally, I would like to thank the National Science Center in Poland for supporting my research via grants no. 2012/05/E/ST1/03232, 2016/21/B/ST1/03138 and 2018/28/T/ST1/00117. I also acknowledge the financial support of Warsaw Center of Mathematics and Computer Sciences.

Contents

Chapter 1

Introduction

1.1 Classical theory of harmonic maps

In the greatest generality, our object of study are maps $u\colon \mathcal{M} \to \mathcal{N}$ between two Riemannian manifolds that minimize (or are critical points of) the Dirichlet energy

$$E(u) := \int_{\mathcal{M}} |\nabla u|^2,$$

called *harmonic maps*. More precise definitions will be given in a moment; for now let us only explain that the size of the differential $\nabla u(x)\colon T_x\mathcal{M} \to T_{u(x)}\mathcal{N}$ is measured with the Hilbert-Schmidt norm and the integration takes place with respect to the volume measure on $\mathcal{M}$.

In this dissertation we investigate the regularity properties of such maps, with special emphasis on the case $\mathcal{N} = \mathbb{S}^2$ and $\dim \mathcal{M} \geqslant 3$. As we shall see, harmonic maps are in general not regular, but their singularities are by now well understood. The study of their *singular sets* is the main objective of this work.

For now, let us mention several interesting examples of harmonic maps:

- if $\mathcal{M}$ is 1-dimensional, harmonic maps are geodesics on $\mathcal{N}$;
- if $\mathcal{N} = \mathbb{R}$, harmonic maps are simply harmonic functions on $\mathcal{M}$;
- if $\mathcal{M}$ is 2-dimensional, *conformal* harmonic maps are parametrizations of minimal surfaces (i.e., critical points of the area functional).

A detailed discussion of the classical theory of (smooth) harmonic maps can be found in two survey articles by Eells and Lemaire [11, 12].

1.2 Analytic difficulties

The analysis of harmonic maps becomes substantially harder when the domain has dimension $n \geqslant 3$. To illustrate the difficulties, let us assume that the target manifold $\mathcal{N} \subseteq \mathbb{R}^N$ is a submanifold of some (possibly high-dimensional) Euclidean space; by Nash's isometric embedding theorem, this does not affect the generality of our considerations.

To start with, assume that $u \colon \mathcal{M} \to \mathcal{N}$ is a critical point of E; this means that for each perturbation $\varphi \in C_c^\infty(\mathcal{M}, \mathbb{R}^N)$ we have

$$\frac{d}{dt}\Big|_{t=0} E(\pi_{\mathcal{N}}(u + t\varphi)) = 0.$$

Note that $u + t\varphi$ is not a valid competitor as it takes values outside of $\mathcal{N}$, and thus it needs to be projected back onto $\mathcal{N}$ by the nearest-point projection $\pi_{\mathcal{N}}$. This Euler-Lagrange equation can be rewritten as

$$-\Delta_{\mathcal{M}} u = A_u^{\mathcal{N}}(\nabla u, \nabla u), \tag{1.2.1}$$

where $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator on $\mathcal{M}$ and $A_u^{\mathcal{N}}$ denotes the second fundamental form of the submanifold $\mathcal{N} \subseteq \mathbb{R}^N$ evaluated at u. To be precise, the right-hand side is to be understood as a sum $\sum_\alpha A_u^{\mathcal{N}}(\partial_\alpha u, \partial_\alpha u)$ over some orthonormal basis ∂_α of $T\mathcal{M}$.

This dissertation is focused on the special case when $\mathcal{M} \subseteq \mathbb{R}^n$ is a bounded flat domain and $\mathcal{N}$ is the standard sphere $\mathbb{S}^2$. Then, the equation (1.2.1) takes the simple form

$$-\Delta u = |\nabla u|^2 u.$$

In all possible cases of non-flat target manifolds, the quadratic non-linearity is troublesome. If one assumes that u belongs to the Sobolev space $W^{1,2}$ – which is natural in the context of minimizing E – then the right-hand side belongs merely to the space L^1, and most standard techniques of regularity theory cannot be directly applied. Indeed, in higher dimensions the solutions may be singular:

- if $u \in W^{1,2}$ is a weak solution of (1.2.1) and $\dim \mathcal{M} \leqslant 2$, then u is a smooth classical solution (Hélein [21]);
- however, if $\dim \mathcal{M} \geqslant 3$, then u may be discontinuous everywhere (Rivière [41]).

An example of a singular harmonic map is

$$\mathbb{R}^n \ni x \longmapsto \frac{x}{|x|} \in \mathbb{S}^{n-1} \qquad (n \geqslant 3).$$

Let us stress that this map is not only a critical point, but also a minimizer in the sense of Definition 1.3.2 [25]. It is worth mentioning that singularities appear also in the absence of topological obstructions – i.e., a homotopically trivial prescribed boundary map $\varphi\colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ may give rise to a singular minimizer $u\colon \mathbf{B}^n \to \mathbb{S}^{n-1}$ [16, 31].

Since full regularity is not available in the case $n \geqslant 3$, one can hope for at least a partial regularity result, i.e., smoothness outside a small *singular set*. To this end, one has to carefully distinguish between different classes of harmonic maps. Many of the results cited in this introduction also hold in case of so-called stationary and stable-stationary harmonic maps, but for simplicity we focus only on minimizing harmonic maps introduced in the next section.

The dichotomy mentioned above – discontinuity on the *singular set* and smoothness elsewhere – in some sense reduces the usual regularity problems to the study of the singular set. For this reason, from now on we will focus on estimating its dimension (which in typical situations is $n-3$) and size (i.e., Hausdorff measure), studying its manifold structure and its dependence on the boundary data.

1.3 Minimizing harmonic maps

From now on, $\mathcal{N}$ will always be a smooth closed (i.e., compact, without boundary) manifold, isometrically embedded into some Euclidean space $\mathbb{R}^N$. To simplify the considerations, we will also assume that the maps $u\colon \Omega \to \mathcal{N}$ under consideration are defined on a bounded flat domain $\Omega \subseteq \mathbb{R}^n$. Since singular behavior of u is a local phenomenon, most results in this dissertation can be generalized to a general domain $\mathcal{M}$. The main idea is that by restricting u to a sufficiently small ball $\mathbf{B}_r(p) \subseteq \mathcal{M}$ and rescaling in normal coordinates we obtain the map

$$\overline{u}\colon \mathbf{B}_1 \to \mathcal{N}, \qquad \overline{u}(x) := u(\exp_p(rx))$$

defined on the unit ball with Riemannian metric arbitrarily close to the Euclidean metric. The differences in analysis are of technical nature – e.g., the monotonicity formula (2.1.1) becomes an almost-monotonicity formula. The interested reader can find a detailed explanation in [37] and [47, Sec. 8].

To discuss partial regularity results by Schoen and Uhlenbeck [43, 44, 45], we first need to precisely define the class of harmonic maps we consider.

Definition 1.3.1. If $\Omega \subseteq \mathbb{R}^n$ is a bounded domain and $\mathcal{N} \subseteq \mathbb{R}^N$ is a closed smooth submanifold, then we define the class of Sobolev maps $W^{1,2}(\Omega, \mathcal{N})$ by

$$W^{1,2}(\Omega, \mathcal{N}) = \left\{ u \in W^{1,2}(\Omega, \mathbb{R}^N) : u(x) \in \mathcal{N} \text{ for a.e. } x \in \Omega \right\}.$$

Note that this is a weakly closed subset, but not a linear subspace of $W^{1,2}(\Omega, \mathbb{R}^N)$.

Definition 1.3.2. Let $u \in W^{1,2}(\Omega, \mathcal{N})$.

- if $E(u) \leqslant E(v)$ for each $v \in W^{1,2}(\Omega, \mathcal{N})$ with the same trace on $\partial\Omega$ as u, then u is called an *energy minimizer* or a *minimizing harmonic map* in Ω;
- if Ω can be covered by a family of balls in which u is minimizing, then u is called a *local energy minimizer* or a *locally minimizing harmonic map*.

Finally, having in mind partial regularity, we adopt the following definition.

Definition 1.3.3. Let $u\colon \Omega \to \mathcal{N}$ be a (locally) minimizing harmonic map. A point $x \in \Omega$ is called *regular* if u has a representative continuous at x, otherwise x is *singular*. We denote the set of all singular points by $\operatorname{sing} u$.

Remark 1.3.4. Equivalently, a point $x \in \Omega$ is regular if u is smooth on some neighborhood of x (see the theorem below).

The following partial regularity theorem summarizes the results obtained by Schoen and Uhlenbeck in the 1980's for the interior case [43, 45] and the boundary case [44].

Theorem 1.3.5 ([43, 44, 45])**.** *Let $u\colon \Omega \to \mathcal{N}$ be a minimizing harmonic map in a bounded domain $\Omega \subseteq \mathbb{R}^n$. Then $\operatorname{sing} u \subseteq \Omega$ is a closed subset of Hausdorff dimension at most $n-3$, and u is smooth in $\Omega \setminus \operatorname{sing} u$. Moreover,*

- *if both the boundary $\partial\Omega$ and the boundary map $u|_{\partial\Omega}$ are sufficiently smooth ($C^{1,\alpha}$ is sufficient), then u is smooth on some neighborhood of $\partial\Omega$;*
- *in case $n = 3$, $\operatorname{sing} u$ is discrete.*

In particular, if both $\partial\Omega$ and $u|_{\partial\Omega}$ are sufficiently smooth and $n = 3$, the singular set $\operatorname{sing} u$ consists of finitely many points. The higher-dimensional counterpart of this statement – that $\mathcal{H}^{n-3}(\operatorname{sing} u) < \infty$ – had been open for over 30 years and has been recently proved by Naber and Valtorta [37]. This recent breakthrough and the new methods behind it were one of the reasons I have chosen this topic for my doctoral dissertation.

One of the main results of Naber and Valtorta's article from 2017 [37] is the following. Further discussion of this work can be found in Chapter 4.

Theorem 1.3.6. *If $u\colon \mathbf{B}_{2r}(p) \to \mathcal{N}$ is a locally minimizing harmonic map, then $\operatorname{sing} u$ is a rectifiable $(n-3)$-dimensional set. Moreover, its measure in a smaller ball $\mathcal{H}^{n-3}(\operatorname{sing} u \cap \mathbf{B}_r(p))$ is bounded by r^{n-3} times a constant dependent on n, $\mathcal{N}$ and the energy $r^{2-n}\int_{\mathbf{B}_{2r}(p)} |\nabla u|^2$.*

In general the $(n-3)$-dimensional bounds on the singular set cannot be improved. A typical singularity of a minimizing map into $\mathbb{S}^2$ looks like

$$\mathbb{R}^3 \times \mathbb{R}^{n-3} \ni (x,y) \mapsto x/|x| \in \mathbb{S}^2,$$

so it has an $(n-3)$-dimensional plane as its singular set (see Corollary 2.4.2).

These are the main results available for a general smooth closed target manifold $\mathcal{N}$. A number of other interesting properties were shown for special classes of target manifolds, especially for the standard sphere $\mathbb{S}^2$.

1.4 Results for special target manifolds

There are many results for special classes of target manifolds – real analytic manifolds [47], simply-connected manifolds [17], symmetric spaces [20] – or simply for the target manifold $\mathcal{N} = \mathbb{S}^2$.

In addition to its intrinsic mathematical interest, this special case also appears in some physical models. The molecules of liquid crystals are small but relatively long, and their configuration minimizes an energy that penalizes changes of direction. Taking the averaged direction of molecules at each point, we obtain a map $u\colon \Omega \to \mathbb{R}P^2$ that minimizes a functional closely resembling the Dirichlet energy $E(u)$. Singularities of harmonic maps are related to *defects* of liquid crystals, i.e., points where the direction of molecules changes in a discontinuous way. Replacing $\mathbb{R}P^2$ (the space of directions) by $\mathbb{S}^2$ and simplifying the functional to E, we can still capture the main phenomena. An interested reader can be referred to [1] and [14].

Here, I focus on the case $\mathcal{N} = \mathbb{S}^2$ and give three results of this type. A large part of the dissertation is dedicated to these three theorems and their generalizations to higher dimensional domains. Possible generalizations to a larger class of target manifolds will also be discussed.

In the special case of maps $u\colon \mathbf{B}^4 \to \mathbb{S}^2$, Hardt and Lin [19] obtained the following remarkable structure result.

Theorem 1.4.1. *The singular set of an energy minimizer $u\colon \mathbf{B}^4 \to \mathbb{S}^2$ is locally a union of a finite set and a finite family of Hölder continuous closed curves with a finite number of crossings.*

The same claim was obtained also for maps $u\colon \mathbf{B}^5 \to \mathbb{S}^3$ (Lin-Wang [26]). To the author's knowledge, these are the only two cases where $\operatorname{sing} u$ was shown to be essentially a manifold.

Let us remark here that the classification of tangent maps from [26] makes it possible to generalize the main results of this dissertation to the case of maps into $\mathbb{S}^3$. For clarity, we focus on $\mathbb{S}^2$ and refer the interested reader to [26] and [37, Sec. 1.3], where the necessary modifications are described.

In dimension $n = 3$, when minimizing harmonic maps have only isolated singularities, further refinements of Schoen and Uhlenbeck's results were obtained.

Hardt and Lin [18] also showed that the singularities are stable under Lipschitz perturbations of the boundary map.

Theorem 1.4.2. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded smooth domain and $u\colon \Omega \to \mathbb{S}^2$ be a minimizing harmonic map with Lipschitz continuous boundary data $\varphi := u|_{\partial\Omega}$. If u_k is a sequence of minimizers with corresponding boundary maps φ_k and*

$$\varphi_k \to \varphi \text{ in } \operatorname{Lip}(\partial\Omega, \mathbb{S}^2), \qquad u_k \to u \text{ in } W^{1,2}(\Omega, \mathbb{S}^2),$$

then for large k, u_k has the same number of singularities as u, and $\operatorname{sing} u_k$ converges to $\operatorname{sing} u$ (say, with respect to Hausdorff distance).

Even more, there exist bi-Lipschitz transformations η_k *of* Ω *mapping* $\operatorname{sing} u$ *to* $\operatorname{sing} u_k$ *and such that* $\|\eta_k - \operatorname{id}\|_{\mathrm{Lip}} \to 0$ *and* $\|u - u_k \circ \eta_i\|_{C^\beta} \to 0$ *for some small* $\beta > 0$.

Almgren and Lieb estimated the number of singularities in terms of the boundary map [1].

Theorem 1.4.3. *Let* $\Omega \subseteq \mathbb{R}^3$ *be a bounded smooth domain and* $u\colon \Omega \to \mathbb{S}^2$ *be a minimizing harmonic map with boundary data* $\varphi \in W^{1,2}(\partial\Omega, \mathcal{N})$. *Then*

$$\# \operatorname{sing} u \leqslant C(\Omega) \int_{\partial\Omega} |\nabla\varphi(x)|^2 \,\mathrm{d}\mathcal{H}^2(x).$$

Note that a simple non-linear estimate $\# \operatorname{sing} u \leqslant C(\Omega, \|\varphi\|_{\mathrm{Lip}})$ follows already from Hardt and Lin's Theorem 1.4.2 (see [18, Sec. 4]). However, this linear estimate only uses the $W^{1,2}$-norm of the boundary map, which does not control the distance of singularities from the boundary.

1.5 Discussion of results

General goals

Considering the whole theory of harmonic maps, a number of fundamental questions has already been answered. For low dimensions ($n = 1, 2$), this theory is classical and the solutions are smooth, while for higher dimensions ($n \geqslant 3$) one needs to study minimizing maps (instead of merely critical points) and singularities appear. Again, the case $n = 3$ (when singularities are isolated) is very well understood, while for $n > 3$ (when singularities are $(n-3)$-dimensional) a major breakthrough took place in the last 5 years. For special cases of target manifolds, especially for $\mathbb{S}^2$, some further results are available. They rely heavily on the classification of *tangent maps* into $\mathbb{S}^2$ (Theorem 2.4.1), carried out by Brezis, Coron and Lieb [4].

This discussion motivates the following general questions, which I aim to address in this disertation.

- Do Theorems 1.4.1, 1.4.2, 1.4.3 generalize to domains of an arbitrary dimension $n \geqslant 3$?
- Can similar results be shown for other target manifolds? What special properties of minimizing harmonic maps into $\mathbb{S}^2$ are crucial?
- In case the singular set is proved to be a topological manifold (as in Theorem 1.4.1), how regular is it?

Structure of the dissertation

Chapters 1, 2, 4 have an introductory character, and only Sections 4.2, 4.3 there include new results or proofs. The author's results from [33] are presented in Chapter 3, while Chapter 5 is based on the article published in Annales Academiæ Scientiarum Fennicæ Mathematica [32]. Chapters 6 and 7 discuss the recent results obtained in collaboration with Katarzyna Mazowiecka and Armin Schikorra [30] (see also [29]).

Let us now discuss the contents of this dissertation in more detail.

In the course of the proofs, we shall use – among others – many tools developed in the 1980's (most of them already in [43]), such as the ε-regularity theorem, monotonicity formula and strong $W^{1,2}$-compactness of the class of minimizing maps. These are introduced in Chapter 2. The notion of tangent maps – which describe the infinitesimal behavior of minimizing harmonic maps, especially around a singular point – is discussed with special care.

Chapter 3 generalizes Theorem 1.4.1 to higher-dimensional domains; it is based on the author's work [33]. The main result (Corollary 3.1.5) is the following. For any minimizing map $u\colon \Omega \to \mathbb{S}^2$ defined on $\Omega \subseteq \mathbb{R}^n$, one can distinguish the top-dimensional part of the singular set $\operatorname{sing}_* u \subseteq \operatorname{sing} u$ (see (2.3.1)), which is a subset of full $\mathcal{H}^{n-3}$-measure. Then, $\operatorname{sing}_* u$ is proved to be an open subset and a topological $(n-3)$-dimensional manifold of Hölder class $C^{0,\gamma}$ for every $\gamma \in (0, 1)$.

In order to extract the topological obstruction responsible for preventing gaps in the singular set of maps into $\mathbb{S}^2$, we study properties of possible singularities of maps into an arbitrary closed Riemannian manifold $\mathcal{N}$. We distinguish particular homotopy classes of tangent maps $\mathbb{R}^3 \to \mathcal{N}$ (called here *indecomposable classes*, see Definition 3.2.4), with which we are able to prove an analogous result (see Theorem 3.1.3 for a precise statement). Consider a minimizing map $u\colon \Omega \to \mathcal{N}$ and an indecomposable homotopy class $\alpha \in \pi_2(\mathcal{N})$. Then the set of points where u has a singularity of type α and *energy density close to optimal* forms an open subset of $\operatorname{sing} u$ and a topological $(n-3)$-dimensional manifold of Hölder class $C^{0,\gamma}$ for some $\gamma > 0$.

This formulation might seem complicated, but it sheds some light on the analytic and topological properties of singularities responsible for regularity of the singular set. In the particular case of $\mathcal{N} = \mathbb{S}^2$, Theorem 2.4.1 implies that almost all singularities have the same indecomposable homotopy type and optimal energy density, and Corollary 3.1.5 follows.

The important contributions of Naber and Valtorta [37] are crucial to all further developments described here, thus the whole Chapter 4 is devoted to them. Let us remark that the three ingredients needed to prove the measure bound in Theorem 1.3.6 are the L^2-approximation theorem (Theorem 4.1.5), the discrete Reifenberg theorem (Theorem 4.1.3) and an appropriate covering theorem (see [37, Lemma 8.1]).

In addition to the estimates on the whole singular set (as in Theorem 1.3.6), bounds on its $(n-4)$-dimensional part are also available (see Corollary 4.2.2). I would like to thank Aaron Naber for pointing out that the results of Chapter

3 can be combined with [37] in this way.

To give a flavor of Naber and Valtorta's methods, we investigate the special case considered in Chapter 3 of a minimizing map $u\colon \mathbf{B}^n \to \mathbb{S}^2$ close to its tangent map (i.e., δ-flat in the sense of Definition 3.3.3). In Theorem 4.3.1 (a very weak version of their main result), we show that in this case the original proof simplifies significantly, and the measure bound on $\operatorname{sing} u$ can be obtained without a sophisticated covering argument. This also emphasizes the importance of the discrete Reifenberg theorem.

Chapter 5 discusses various extensions of the discrete Reifenberg theorem from [37]; these were published in the author's paper [32]. Theorems of this kind have wide applicability in the study of singular sets in various geometric problems [36, 22, 5], and in particular to singularities of minimizing harmonic maps [37]. They are also interesting in themselves as general results in geometric measure theory.

The main result of this chapter is phrased in terms of so called Jones' height excess numbers. Fixing a Radon measure μ on $\mathbb{R}^n$ and some dimension $0 < k < n$, the quantity

$$\beta_{\mu,2}^2(x,r) := \inf\left\{ r^{-(k+2)} \int_{\mathbf{B}_r(x)} d^2(y,V)\,\mathrm{d}\mu(y) : V \text{ is a } k\text{-dim affine plane} \right\}$$

measures how far $\mu \llcorner \mathbf{B}_r(x)$ is from being supported on some k-dimensional plane ($d(\cdot,V)$ denotes the distance to V). The main result of this chapter (Theorem 5.1.1) states that (under some technical assumptions on μ) the condition

$$r^{-k}\int_{\mathbf{B}_r(x)}\int_0^r \beta_{\mu,2}^2(y,s)\frac{\mathrm{d}s}{s}\,\mathrm{d}\mu(y) \leqslant J \quad \text{on each ball } \mathbf{B}_r(x)$$

implies the bound $\mu(\mathbf{B}_r(x)) \leqslant C(n)(1+J^{1/2})r^k$ on every ball.

These technical assumptions are automatically satisfied if μ is the Hausdorff measure on some k-dimensional set (i.e., $\mu = \mathcal{H}^k \llcorner S$) or if it is a discrete measure $\mu = \sum_j \omega_k r_j^k \delta_{x_j}$ associated to some family $\{\mathbf{B}_{r_j}(x_j)\}$ of disjoint balls. The name *discrete Reifenberg theorem* comes from the fact that the proof follows by a careful application of the classical Reifenberg construction [40], first in the case when μ is a discrete measure. Indeed, a version for more general measures (Theorem 5.5.1, Remark 5.5.3) follows easily from the discrete case. Simple modifications allow also for the use of $\beta_{\mu,q}$-numbers with $q \geqslant 2$ (which involve the distance $d(\cdot, V)$ to the power q), and for weakened assumptions (Theorem 5.5.4). I remark here that Edelen, Naber and Valtorta [9, 10] later published even more general versions of Theorem 5.1.1 (see also the lecture notes [35]).

The last two chapters describe the results of my joint work with Katarzyna Mazowiecka and Armin Schikorra [30]. The main result of Chapter 6 is Theorem 6.1.1, a higher-dimensional counterpart of Almgren and Lieb's Theorem 1.4.3. If $\Omega \subseteq \mathbb{R}^n$ is a bounded smooth domain and $u\colon \Omega \to \mathbb{S}^2$ is a minimizing map with boundary data $\varphi \in W^{1,n-1}(\partial\Omega, \mathbb{S}^2)$, then

$$\mathcal{H}^{n-3}(\operatorname{sing} u) \leqslant C(\Omega) \int_{\partial\Omega} |\nabla\varphi(x)|^{n-1} \, \mathrm{d}\mathcal{H}^{n-1}(x).$$

As in the case $n = 3$, a non-linear estimate $\mathcal{H}^{n-3}(\operatorname{sing} u) \leqslant C(\Omega, \|\varphi\|_{\mathrm{Lip}})$ is much easier to obtain (see Theorem 6.1.3). Thus, the power of our result lies in the linear dependence on the energy, and in the use of $W^{1,n-1}$-norm of the boundary map, which again does not control the distance of singularities from the boundary.

The strategy of the proof is close to the original, based on refined boundary regularity results of the following type: if a minimizer $u\colon \mathbf{B}_1^+ \to \mathbb{S}^2$ has a boundary map $\varphi\colon \mathbf{B}_1^{n-1} \to \mathbb{S}^2$ with small energy, then some region of $\mathbf{B}_1^+$ is free of singularities. However, the crucial ingredient here is the *hot spot lemma* due to Almgren and Lieb [1, Thm. 2.4], generalized to higher dimensions (Theorem 6.2.2). It yields the same regularity conclusion with a weakened assumption – we only assume that the energy of φ on $\mathbf{B}_1^{n-1} \setminus \mathbf{B}_\varepsilon$ is small, while its behavior on the small ball $\mathbf{B}_\varepsilon$ (called the hot spot) can be arbitrarily wild.

The original paper of Almgren and Lieb [1] relies on the classification of singularities of maps into $\mathbb{S}^2$ (Theorem 2.4.1) to show a lower bound on the distance between two singularities. Replacing this bound by Naber and Valtorta's Theorem 1.3.6, we are able to obtain a similar result in an arbitrary dimension. A similar strategy was used in the context of minimal surfaces by Edelen in [8], where he combined interior measure bounds due to Naber and Valtorta [36] with boundary regularity results to obtain global bounds on the singular set.

Moreover, the only special property of $\mathbb{S}^2$ needed in course of the proof is the extension property (Theorem 2.5.1), which can be shown for a wider class of target manifolds. Thus, the final result holds for maps into any closed simply connected Riemannian manifold $\mathcal{N}$.

Finally, a higher-dimensional counterpart of Hardt and Lin's stability theorem is proved in Chapter 7 (Theorem 7.1.1). With the same assumptions on Ω, u and φ as above, if u_k is a sequence of minimizers with boundary data φ_k and

$$u_k \to u \text{ in } W^{1,2}, \quad \varphi_k \to \varphi \text{ in } W^{1,n-1},$$

then

$$\mathcal{H}^{n-3} \llcorner \operatorname{sing} u_k \xrightarrow{d_W} \mathcal{H}^{n-3} \llcorner \operatorname{sing} u,$$

where d_W denotes the 1-Wasserstein distance (7.1.1) between Hausdorff measures on singular sets of u_k and u. In particular, the total measure $\mathcal{H}^{n-3}(\operatorname{sing} u_k)$ tends to $\mathcal{H}^{n-3}(\operatorname{sing} u_k)$.

Note that this recovers most of Hardt and Lin's Theorem 1.4.2 in the case $n = 3$ (except for the diffeomorphism statement). Indeed, $\mathcal{H}^0$ is simply the counting measure, so Wasserstein convergence implies that $\# \operatorname{sing} u_k = \# \operatorname{sing} u$ for large k and that $\operatorname{sing} u_k$ converges to $\operatorname{sing} u$ with respect to Hausdorff distance. However, generalizing the diffeomorphism statement to higher dimensions is very hard – even the bi-Lipschitz regularity of $\operatorname{sing}_* u$ is an open problem for $n > 3$.

As in the original paper [18], the heart of the argument lies in the *local* case. If we restrict u to a small enough ball around a singularity, it is close to its tangent map, and after rescaling the problem reduces to the following (which is the content of Lemma 7.4.1). If $u\colon \mathbf{B}_{80} \to \mathbb{S}^2$ is close enough to its tangent map (again, δ-flat in the sense of Definition 3.3.3), then

$$(1-\varepsilon)\omega_{n-3} \leqslant \mathcal{H}^{n-3}(\operatorname{sing} u \cap \mathbf{B}_1) \leqslant (1+\varepsilon)\omega_{n-3}.$$

This means that the measure of $\operatorname{sing} u \cap \mathbf{B}_1$ is close to the measure of the singular set of its tangent map in $\mathbf{B}_1$, which is an $(n-3)$-dimensional disc. The proof is very similar to that of Theorem 4.3.1 and follows the lines of Naber and Valtorta's work, only that Theorem 4.1.4 (rectifiable Reifenberg) is used instead of Theorem 4.1.3 (discrete Reifenberg). This time however, the results of Chapter 3 are essential to the proof of the sharp measure estimate above. Therefore, a generalization to a larger class of target manifolds seems challenging.

However, it should be possible to further refine the norm of the boundary map in both Theorem 6.1.1 and Theorem 1.4.2. Our work in progress involves replacing the $W^{1,n-1}(\partial\Omega)$ norm in these two theorems by $W^{1,p}(\partial\Omega)$ with any $p > 2$, using more sophisticated geometric tools. As examples in [31] show, both results fail in the case $p < 2$.

Chapter 2

Basic properties of minimizing harmonic maps

This chapter introduces most notions and results needed in the sequel. These mostly come from the seminal work of Schoen and Uhlenbeck [43], but the presentation here mostly follows Simon's lecture notes [48]. In what follows, $u\colon \mathbf{B}^n \to \mathcal{N}$ is an energy minimizing map into a closed Riemannian manifold $\mathcal{N}$.

2.1 Regularity of energy minimizers

A central object in the study of singularities is the rescaled energy

$$\theta_u(x,r) := r^{2-n} \int_{\mathbf{B}_r(x)} |\nabla u|^2 \quad \text{for } \mathbf{B}_r(x) \subseteq \Omega,$$

which can be shown to be monotone in r:

$$\tfrac{\partial}{\partial r}\theta_u(x,r) = 2\int_{\partial \mathbf{B}_r(x)} \frac{|\nabla u \cdot (y-x)|^2}{|y-x|^n} \geqslant 0. \tag{2.1.1}$$

For the sake of exposition, we show only a weaker version of this formula. Choosing a competitor $v(x) = u(r \cdot \frac{x}{|x|})$, we see that $v = u$ on $\partial \mathbf{B}_r$ and

$\int_{\mathbf{B}_r} |\nabla v|^2 = \frac{r}{n-2} \int_{\partial \mathbf{B}_r} |\nabla_T u|^2$, where $\nabla_T u$ denotes the differential restricted to directions tangent to $\partial \mathbf{B}_r$. It follows from minimality of u that

$$\int_{\mathbf{B}_r} |\nabla u|^2 \leqslant \frac{r}{n-2} \int_{\partial \mathbf{B}_r} |\nabla_T u|^2,$$

$$\text{thus} \quad \frac{\partial}{\partial r}\theta_u(x,r) = r^{2-n} \int_{\partial \mathbf{B}_r} |\nabla u|^2 - (n-2) r^{1-n} \int_{\mathbf{B}_r} |\nabla u|^2$$
$$\geqslant r^{2-n} \int_{\partial \mathbf{B}_r} |\nabla u|^2 - r^{2-n} \int_{\partial \mathbf{B}_r} |\nabla_T u|^2$$
$$= \int_{\partial \mathbf{B}_r} \frac{|\nabla u \cdot (y-x)|^2}{|y-x|^n}.$$

A refined reasoning based on the so-called stationary equation shows that the derivative of $\theta_u(x,r)$ is exactly twice the right-hand side; the proof can be found in [48, Sec. 2.4]. Either way, it is evident that $\theta_u(x,r)$ is constant in r if and only if u is a homogeneous (i.e., radially constant) map.

The monotonicity formula (2.1.1) enables us to define the energy density at x:

$$\theta_u(x,0) := \lim_{r \to 0} \theta_u(x,r),$$

which is by definition an upper semicontinuous function (in both $x \in \mathbf{B}^n$ and $u \in W^{1,2}$) [48, 2.11]. Obviously, $\theta_u(x,0) = 0$ at regular points.

The main regularity statement of [43] is the following ε-regularity theorem:

$$\text{there is } \varepsilon(n,\mathcal{N}) > 0 \text{ s.t. } \theta_u(x,2r) < \varepsilon \Rightarrow u \text{ is smooth on } \mathbf{B}_r(x), \tag{2.1.2}$$
$$\text{in particular } \theta_u(x,0) < \varepsilon \Rightarrow x \notin \operatorname{sing} u.$$

We also note two compactness theorems for a sequence u_k of energy minimizers in Ω:

- if $u_k \rightharpoonup u$ in $W^{1,2}(\Omega)$, then u is an energy minimizer in any subdomain $\Omega' \Subset \Omega$ and the convergence is actually locally strong in $W^{1,2}(\Omega')$ [27] (see [48, Sec. 2.9]),

- if $u_k \rightharpoonup u$ in $W^{1,2}(\Omega)$, then the convergence is uniform on compact sets disjoint from $\operatorname{sing} u$ [43, Proposition 4.6].

As a historical note, let us mention that the first statement was proved by Luckhaus a few years after Schoen and Uhlenbeck's work. Its use significantly simplifies the analysis, even if it could be avoided.

It follows from upper semicontinuity and ε-regularity that the singular set of a minimizer $u\colon \Omega \to \mathcal{N}$ is relatively closed in Ω. The $W^{1,2}$-compactness theorem above yields even more for a sequence of minimizers u_k:

$$y_k \in \operatorname{sing} u_k,\ u_k \to u \text{ in } W^{1,2},\ y_k \to y \quad \Longrightarrow \quad y \in \operatorname{sing} u. \tag{2.1.3}$$

2.2 Tangent maps

To study the infinitesimal behavior of u at a singular point x, we introduce the notion of *tangent maps*. It is a close analogue of *tangent cones* used to describe singularities of minimal surfaces and other possibly non-smooth geometric objects.

Given an energy minimizer $u\colon \mathbf{B}^n \to \mathcal{N}$ and a point $x \in \mathbf{B}^n$, consider the family of rescaled maps $u_r(y) = u(x + ry)$. By the results from the previous section (monotonicity formula and compactness of minimizers), each sequence $r_j \to 0$ has a subsequence for which u_{r_j} converges in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ to some local energy minimizer φ, called a tangent map of u at x (possibly dependent on the choice of the subsequence, and thus non-unique). By monotonicity formula, this limit map is homogeneous, i.e., $\varphi(\lambda x) = \varphi(x)$ for all $\lambda > 0$, $x \in \mathbb{R}^n$. Moreover, the energy of φ is consistent with the energy density of u in the sense that $\theta_u(x, 0) = \theta_\varphi(0, r)$ for any $r \geqslant 0$.

Example 2.2.1. (a) If x is a regular point, then evidently every tangent map φ is constant, mapping $\mathbb{R}^n$ to the point $u(x) \in \mathcal{N}$. By ε-regularity theorem

(2.1.2), the reverse implication is also true – if u has a constant tangent map at x, then $\theta_u(x,0)=0$ and u is smooth around x.

(b) The map $u\colon \mathbf{B}^3 \to \mathbb{S}^2$ given by $u(x)=x/|x|$ is energy minimizing. Since it is homogeneous, u is its own (and unique) tangent map at 0.

For a homogeneous energy minimizer $\varphi\colon \mathbb{R}^n \to \mathcal{N}$, the energy density $\theta_\varphi(y,0)$ is maximal at $y=0$; moreover, equality $\theta_\varphi(y)=\theta_\varphi(0)$ at some other point y leads to higher symmetry: $\varphi(x+ty)=\varphi(x)$ for all $t\in\mathbb{R}$, $x\in\mathbb{R}^n$. Let $S(\varphi)$ be defined by

$$S(\varphi)=\{y\in\mathbb{R}^n : \theta_\varphi(y)=\theta_\varphi(0)\}\,.$$

Then $S(\varphi)$ is a linear subspace of $\mathbb{R}^n$ describing the symmetries of φ:

$$\varphi(x+y)=\varphi(x) \quad \text{for all } x\in\mathbb{R}^n,\ y\in S(\varphi).$$

For non-constant φ, we have $S(\varphi)\subseteq \operatorname{sing}\varphi$. If $\dim S(\varphi)=n-3$, we note that this inclusion is necessarily an equality.

Since the symmetries described above will play an important role later, we adopt the following definition (see [37, Def. 1.1]).

Definition 2.2.2. A map $\varphi\colon\mathbb{R}^n\to\mathcal{N}$ is *symmetric* with respect to a k-dimensional linear plane $V\subseteq\mathbb{R}^n$ if it is homogeneous ($\varphi(\lambda x)=\varphi(x)$ for $\lambda>0$ and $x\in\mathbb{R}^n$) and $\varphi(x+y)=\varphi(x)$ for all $y\in V$ and $x\in\mathbb{R}^n$. It is called *k-symmetric* if any such V exists; the space of all such functions will be denoted by $\mathrm{sym}_{n,k}$.

2.3 Top-dimensional part of the singular set

If u is an energy minimizer, for each $j=0,1,2,\dots,n-1$ we define

$$\begin{aligned} S_j &= \{y\in\operatorname{sing}u : \dim S(\varphi)\leqslant j \text{ for all tangent maps } \varphi \text{ of } u \text{ at } y\}\\ &= \{y\in\operatorname{sing}u :\ \text{no tangent map } \varphi \text{ of } u \text{ at } y \text{ is } (j+1)\text{-symmetric}\}\,, \end{aligned}$$

which leads to the classical stratification of the singular set

$$S_0 \subseteq S_1 \subseteq \ldots \subseteq S_{n-3} = S_{n-2} = S_{n-1} = \operatorname{sing} u.$$

It is known [43] that each S_j has Hausdorff dimension at most j, in particular

$$\dim_H \operatorname{sing} u \leqslant n - 3.$$

Because of this, we are mostly interested in the top-dimensional part of the singular set:

$$\begin{aligned} \operatorname{sing}_* u &= S_{n-3} \setminus S_{n-4} \\ &= \{y \in \operatorname{sing} u : \dim S(\varphi) = n-3 \text{ for some tangent map } \varphi \text{ of } u \text{ at } y\}. \end{aligned} \tag{2.3.1}$$

Note that

$$\dim_H(\operatorname{sing} u \setminus \operatorname{sing}_* u) \leqslant n - 4.$$

Definition 2.3.1. Following [48], we shall call any homogeneous energy minimizing $\varphi\colon \mathbb{R}^n \to \mathcal{N}$ with $\dim S(\varphi) = n-3$ (i.e., $(n-3)$-symmetric) a *homogeneous cylindrical map* (abbreviated HCM).

2.4 Classification of tangent maps into $\mathbb{S}^2$

For maps into $\mathbb{S}^2$, all possible homogeneous minimizers $\varphi\colon \mathbb{R}^3 \to \mathbb{S}^2$ were classified by Brezis, Coron and Lieb [4].

Theorem 2.4.1 ([4, Thm. 7.1, 7.3, 7.4])**.** *All homogeneous locally minimizing harmonic maps $\varphi\colon \mathbb{R}^3 \to \mathbb{S}^2$ take the form*

$$\varphi(x) = \frac{qx}{|qx|}$$

for some linear isometry q of $\mathbb{R}^3$.

Note that $\int_{\mathbf{B}_1} |\nabla\varphi|^2 = 8\pi$ does not depend on the choice of q, in particular the energy density $\theta_u(x,0)$ can take only two values: 0 at regular points and 8π at singular points.

In higher dimensional domains, a full classification of tangent maps is not available, but one can at least describe all HCMs ($(n-3)$-symmetric minimizers); they are responsible for $\mathcal{H}^{n-3}$-almost all singularities. This is done by combining Theorem 2.4.1 with a simple but important observation due to Hardt and Lin [19, Lemma 2.1].

Corollary 2.4.2. *The map*

$$\mathbb{R}^3 \times \mathbb{R}^{n-3} \ni (x,y) \stackrel{\Psi}{\longmapsto} \frac{x}{|x|} \in \mathbb{S}^2 \tag{2.4.1}$$

is the only locally minimizing $(n-3)$-symmetric harmonic map from $\mathbb{R}^n$ to $\mathbb{S}^2$, up to linear isometries of $\mathbb{R}^n$. That is, any such map takes the form $\Psi \circ q$ for some linear isometry q of $\mathbb{R}^n$.

Its energy density will be denoted by

$$\Theta := \int_{\mathbf{B}_1} |\nabla\Psi|^2 \,\mathrm{d}x. \tag{2.4.2}$$

As before, we may observe that if $u\colon \Omega \to \mathbb{S}^2$ is a minimizing harmonic map in $\Omega \subseteq \mathbb{R}^n$ and $x \in \operatorname{sing}_* u$, then $\theta_u(x,0) = \Theta$.

2.5 Uniform boundedness of minimizers

The last two sections gather results on boundary behavior of minimizers. For convenience, we denote the upper half-space by $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$, and the upper half-ball by $\mathbf{B}^+_r = \mathbf{B}_r \cap \mathbb{R}^n_+$. For any $r > 0$ we write $T_r = \mathbf{B}_r \cap \partial\mathbb{R}^n_+$ for the flat part and $S^+_r = \partial\mathbf{B}_r \cap \mathbb{R}^n_+$ for the curved part of the boundary of the half ball $\mathbf{B}^+_\rho$.

The following extension property of maps into $\mathbb{S}^2$ is crucial in establishing Theorem 6.1.1. A proof can be found in [15].

Theorem 2.5.1 (Extension Property). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and let $v \in W^{1,2}(\Omega, \mathbb{R}^3)$ with $v(x) \in \mathbb{S}^2$ for a.e. $x \in \partial\Omega$. Then there exists a map $u \in W^{1,2}(\Omega, \mathbb{S}^2)$,*

$$u\Big|_{\partial\Omega} = v\Big|_{\partial\Omega}$$

such that

$$\|\nabla u\|_{L^2(\Omega)} \leqslant C\, \|\nabla v\|_{L^2(\Omega)}$$

with a dimensional constant $C(n) > 0$.

Remark 2.5.2. As shown in [17], an analogous statement holds for any simply connected manifold $\mathcal{N}$ in place of $\mathbb{S}^2$.

We obtain the following as a corollary of Theorem 2.5.1 (see [29, Sec. 3]). We remark here that a similar argument works for domains close to $\mathbf{B}_1^+$ (i.e., bounded by a C^2 graph with small constant). For clarity, we focus on the flat boundary case.

Corollary 2.5.3. *If $u \colon \mathbf{B}_r \to \mathbb{S}^2$ is a minimizing harmonic map, then the following estimate holds*

$$\|\nabla u\|_{L^2(\mathbf{B}_r)} \lesssim \sqrt{r^{\frac{n-1}{2}} \|\nabla_T u\|_{L^2(\partial \mathbf{B}_r)}}. \tag{2.5.1}$$

If $u \colon \mathbf{B}_r^+ \to \mathbb{S}^2$ is a minimizing harmonic map with $u = \varphi$ on the flat part of the boundary T_r, then the following estimate holds

$$\|\nabla u\|_{L^2(\mathbf{B}_r^+)} \lesssim \sqrt{r^{\frac{n-1}{2}} \|\nabla_T u\|_{L^2(S_r^+)} + r^{\frac{n-1}{2}} \|\nabla \varphi\|_{L^2(T_r)}}. \tag{2.5.2}$$

Sketch of proof. Consider the first statement with $r = 1$. The square root comes from interpolation – since $\|u\|_{L^\infty} = 1$, we have $[u]_{W^{1/2,2}(\partial \mathbf{B}_1)} \lesssim \sqrt{\|\nabla u\|_{L^2(\partial \mathbf{B}_1)}}$. By the trace theorem, there exists an extension $v \in W^{1,2}(\mathbf{B}_1, \mathbb{R}^3)$ such that $\|\nabla v\|_{L^2(\mathbf{B}_1)} \lesssim [u]_{W^{1/2,2}(S_1)}$. By the extension property, we can actually take v in the space $W^{1,2}(\mathbf{B}_1, \mathbb{S}^2)$, and the claim follows from energy comparison.

The same reasoning works on the half-ball $\mathbf{B}_1^+$, and the general case follows by rescaling. □

One of the main consequences is the following – slightly surprising – result.

Theorem 2.5.4 (Uniform Boundedness of Minimizers)**.** *Let $u \in W^{1,2}(\mathbf{B}_R(0), \mathbb{S}^2)$ be a minimizing harmonic map. Then for any $r < R$,*

$$r^{2-n} \int_{\mathbf{B}_r(0)} |\nabla u|^2 \,\mathrm{d}x \leqslant C(n) \frac{R}{R-r},$$

where C is an absolute constant.

Also, let $u \in W^{1,2}(\mathbf{B}_{2r}^+(0), \mathbb{S}^2)$ be a minimizing harmonic map with $u = \varphi$ on the flat part of the boundary T_{2r}. Then

$$r^{2-n} \int_{\mathbf{B}_r^+(0)} |\nabla u|^2 \,\mathrm{d}x \leqslant C(n) \left(1 + r^{\frac{3-n}{2}} \|\nabla \varphi\|_{L^2(T_{2r})}\right).$$

Proof. We focus on the boundary estimate, which is more delicate. In the absence of the boundary term, the calculations are more straightforward and one easily obtains the more precise asymptotics.

Denote $D(\rho) := \|\nabla u\|^2_{L^2(\mathbf{B}_\rho^+)}$ and $A := r^{\frac{n-1}{2}} \|\nabla \varphi\|_{L^2(T_{2r})}$. Observing that $D'(\rho) = \|\nabla u\|^2_{L^2(S_\rho^+)}$, we can restate Corollary 2.5.3 as the inequality

$$D(\rho) \leqslant C\left(\rho^{\frac{n-1}{2}} \sqrt{D'(\rho)} + A\right) \qquad \text{for } 0 < \rho \leqslant 2r.$$

Since our aim is an estimate $D(r) \lesssim r^{n-2} + A$, we may assume that $D(r) \geqslant 2CA$ with C as above. Then

$$D(\rho) \leqslant 2C\rho^{\frac{n-1}{2}} \sqrt{D'(\rho)} \qquad \text{for } r \leqslant \rho \leqslant 2r.$$

Rewriting this as the differential inequality $(-D(\rho)^{-1})' \geqslant 4C^{-2}\rho^{1-n}$ and integrating, we obtain

$$D(r)^{-1} - D(2r)^{-1} \geqslant 4C^{-2} \int_r^{2r} \rho^{1-n} \,\mathrm{d}\rho.$$

The final claim now follows from $D(2r)^{-1} \geqslant 0$. □

2.6 Boundary regularity

All of the boundary regularity statements were already present (sometimes implicitly) in Almgren and Lieb's paper [1]. Very similar results (usually with stronger assumptions on the boundary map) can be found in the literature, and the necessary modifications in our case are minor. A discussion of these modification can be found in [29, 30]; here we gather all the necessary results and briefly sketch the main ideas.

Recall that a weakly convergent sequence of minimizers is actually strongly convergent in compactly contained subdomain. However, global estimates require convergence on a domain that reaches the boundary; to this end, one additionally needs to assume convergence of the boundary map (see [34]).

Theorem 2.6.1 (strong convergence of minimizers at the boundary)**.** *Consider a sequence of minimizing harmonic maps $u_i \in W^{1,2}(\mathbf{B}^+, \mathbb{S}^2)$ and denote their traces $\varphi_i := u_i|_{T_1}$. Assume additionally that φ_i converges to φ in $W^{1,2}(T_1)$. Then, up to taking a subsequence, we find $u\colon \mathbf{B}^+ \to \mathbb{S}^2$ such that $u_i \to u$ strongly in $W^{1,2}(\mathbf{B}_r^+, \mathbb{S}^2)$ for every $r \in (0,1)$. Moreover, u is a minimizing harmonic map in each such ball $\mathbf{B}_r^+$.*

Remark 2.6.2. A technical modification of this reasoning allows us to consider in Theorem 2.6.1 a sequence of maps u_i defined on converging Lipschitz domains with non-flat boundaries. This will be used in Theorem 6.2.4. For the sake of exposition, the author chose to downplay the role of curved boundaries.

As a first corollary of the compactness result above, we have

Theorem 2.6.3 (interior regularity for almost constant boundary data)**.** *For each bounded smooth domain $\Omega \subseteq \mathbb{R}^n$ and each $\sigma > 0$, there is $\varepsilon(\Omega, \sigma) > 0$ so that the following holds. If $u \in W^{1,2}(\Omega, \mathbb{S}^2)$ is a minimizing harmonic map with trace $\varphi := u|_{\partial\Omega}$ and*

$$\int_{\partial\Omega} |\nabla\varphi|^{n-1} \, \mathrm{d}\mathcal{H}^{n-1} \leqslant \varepsilon,$$

then u is smooth in the interior region $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geqslant \sigma\}$.

Proof. Assume this is not true – there is a sequence of minimizers u_k satisfying $\int_{\partial\Omega} |\nabla\varphi_k|^{n-1} \to 0$, but with a singularity $y_k \in \operatorname{sing} u_k$ in the interior region. By interior compactness, we may assume that $u_k \to u$ locally in $W^{1,2}(\Omega)$, and y_k tends to some $y \in \operatorname{sing} u$ in the interior region.

To reach a contradiction, we also need convergence at the boundary. According to Remark 2.6.2, we can apply Theorem 2.6.1 on Ω, as without loss of generality φ_k tends to a constant. But then u is a minimizer with constant boundary conditions, thus u is constant and in particular smooth in the interior region. □

The first step towards to boundary regularity theory is the following uniform boundary regularity theorem for constant boundary data (see [1, Theorem 1.10]). The other results can be derived from it by a contradiction argument.

Theorem 2.6.4 (Boundary regularity)**.** *There is $\lambda(n) > 0$ such that the following holds. If $u \in W^{1,2}(\mathbf{B}_1^+, \mathbb{S}^2)$ is a minimizer and its trace φ on T_1 is constant, then u is smooth in a small neighborhood $T_{1/2} \times (0, \lambda)$ of the boundary.*

Corollary 2.6.5. *There is another constant $\varepsilon(n) > 0$ such that the smallness condition $\int_{T_1} |\nabla\varphi^2| \leqslant \varepsilon$ implies smoothness of u on an even smaller neighborhood $T_{1/2} \times (\lambda/2, \lambda)$.*

Again, a similar statement holds for a general domain with flat enough boundary.

Proof. This follows from Theorem 2.6.4 by a contradiction argument based on Theorem 2.6.1 (see the proof of Theorem 2.6.3). The necessity of restricting to $T_{1/2} \times (\lambda/2, \lambda)$ comes from the fact that otherwise the sequence of singularities may converge to a boundary point. □

Theorem 2.6.6 (Uniform boundary regularity for singular boundary data)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded smooth domain. Then there are constants σ (depending on Ω), ε, λ (as in Corollary 2.6.5) such that the following holds. If $u\colon \Omega \to \mathbb{S}^2$ is* *a minimizing harmonic map with the trace φ on $\partial\Omega$, which satisfies the smallness condition*

$$\int_{T_\rho(p)} |\nabla_T \varphi|^{n-1} \, \mathrm{d}\mathcal{H}^{n-1} \leqslant \varepsilon$$

for some $p \in \partial\Omega$ and $\rho \leqslant \sigma$, then u is smooth in $\mathbf{B}_{\lambda\rho}(p) \cap \Omega$.

Sketch of proof. We choose $\sigma(\Omega) > 0$ so that $\partial\Omega$ is flat enough in balls of size $\rho \leqslant \sigma$ (after rescaling to unit size). Applying Corollary 2.6.5 on $\mathbf{B}_\rho(p)$, we obtain regularity in a small strip. Thanks to scaling-invariance of the $W^{1,n-1}$ norm on the boundary, the smallness condition also holds on all smaller balls. The final claim now follows by a covering argument. □

Chapter 3

Hölder regularity of the singular set

3.1 Introduction

Singularities of energy minimizing harmonic maps

As already mentioned, (locally) minimizing harmonic maps between manifolds may have singularities if the domain dimension is 3 or higher. The most well-known example is the map

$$\mathbb{R}^3 \times \mathbb{R}^{n-3} \ni (x, y) \xrightarrow{\ \Psi\ } x/|x| \in \mathbb{S}^2.$$

Since all the considerations in this chapter are local in nature, we shall drop the word *locally*.

In general, any energy minimizer u is smooth outside the closed singular set $\operatorname{sing} u$ of Hausdorff dimension $n-3$ or less, n being the dimension of the domain (Schoen, Uhlenbeck [43, 45]). The phenomenon of singularities is now well-understood in dimension 3, when singularities form a discrete set. In recent years, there has been a substantial progress concerning the case $n \geqslant 4$. Naber and Valtorta [37] have proved that the singular set has locally finite $(n-3)$-dimensional Hausdorff measure and is $(n-3)$-rectifiable, i.e., can be essentially covered by countably many Lipschitz images of $\mathbb{R}^{n-3}$; the latter was already known (due to Simon [47]) in the case when the target manifold is real-analytic.

The results cited above are mostly concerned with the size of the singular set, but do not imply *lower bounds* on the singular set. In particular, the possibility that the singular set is an arbitrary subset of an $(n-3)$-dimensional manifold (with many small gaps) is not excluded by [37, 43, 45, 47].

Lower bounds on the size are indeed possible in the presence of a topological obstruction; the following example is simple but instructive.

Example 3.1.1. Consider the smooth boundary map $\varphi\colon \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2$ given by $\varphi(x,y) = x$ and (some) $u\colon \mathbf{B}^3 \times \mathbb{S}^1 \to \mathbb{S}^2$ minimizing the energy in the class of maps equal to φ on the boundary. Restricting u to a slice $\mathbf{B}^3 \times \{y\}$ and applying Brouwer's theorem, we see that each such slice contains a singular point. This shows that $\mathcal{H}^1(\operatorname{sing} u) \geqslant \mathcal{H}^1(\mathbb{S}^1) = 2\pi$. In this particular case one can actually prove that $u(x,y) = x/|x|$, but the presented reasoning applies also to every $\varphi'\colon \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2$ homotopic to φ.

In the special case of maps $u\colon \mathbf{B}^4 \to \mathbb{S}^2$, Hardt and Lin [19] obtained the following remarkable result.

Theorem 3.1.2. *The singular set of an energy minimizer $u\colon \mathbf{B}^4 \to \mathbb{S}^2$ is locally a union of a finite set and a finite family of Hölder continuous closed curves with a finite number of crossings.*

The same claim was obtained also for maps $u\colon \mathbf{B}^5 \to \mathbb{S}^3$ (Lin-Wang [26]). To the author's knowledge, these are the only two cases where $\operatorname{sing} u$ was shown to be essentially a manifold.

The above theorem relies on the classification of tangent maps from $\mathbb{R}^3$ into $\mathbb{S}^2$ carried out by Brezis, Coron and Lieb [4] (see Theorem 2.4.1); for $\mathbb{S}^3$, a similar classification was obtained by Nakajima [39]. These maps describe the infinitesimal behavior of u at a typical point of $\operatorname{sing} u$.

Main results

In this chapter, we aim to extract the topological obstruction responsible for preventing gaps in the singular set of maps into $\mathbb{S}^2$. To this end, we distinguish particular homotopy classes of tangent maps $\mathbb{R}^3 \to \mathcal{N}$ (called here *indecomposable classes*) for any closed Riemannian manifold $\mathcal{N}$.

To each homotopy class $\alpha \in \pi_2(\mathcal{N})$ we assign its lowest energy level $\Theta(\alpha)$ and call α indecomposable if $\Theta(\alpha) < \infty$ and α cannot be represented as a sum of homotopy classes $\alpha_j \in \pi_2(\mathcal{N})$ with strictly smaller energy levels $\Theta(\alpha_j)$. We then restrict our attention to singularites with fixed topological type α – we define $\operatorname{sing}_\alpha u$ to be the set of points at which some tangent map of u has type α. Rigorous definitions are given in Section 3.2.

Another goal is to generalize the result of Hardt and Lin [19] to higher dimensional domains. The difficulty lies in the fact that the singular set is stratified – it decomposes into parts of different dimensions. For $u\colon \mathbf{B}^4 \to \mathbb{S}^2$, there are only two strata: one is formed by Hölder continuous curves and the other by their crossing points and a finite number of additional isolated points. In the theorem below, we were only able to study the top-dimensional part $\operatorname{sing}_* u$ of the singular set. Again, the necessary notions are introduced in Section 2.2.

For simplicity, we only consider the standard Euclidean ball $\mathbf{B}^n$ as the domain, but the results hold true for any manifold. This is due to the fact that we only consider the infinitesimal behavior of maps. A detailed explanation can be found in [37] and [47, Sec. 8].

Theorem 3.1.3. *Let $u\colon \mathbf{B}^n \to \mathcal{N}$ be an energy minimizing map into a closed Riemannian manifold $\mathcal{N}$, $\alpha \in \pi_2(\mathcal{N})$ be an indecomposable homotopy class, and $\Theta(\alpha)$ be its lowest energy level. Then for each exponent $0 < \gamma < 1$ there is $\delta(\gamma, n, \alpha, \mathcal{N}) > 0$ such that the set*

$$\left\{ x \in \operatorname{sing}_\alpha u : \lim_{r \to 0} r^{2-n} \int_{\mathbf{B}_r(x)} |\nabla u|^2 < \Theta(\alpha) + \delta \right\}$$

forms an open subset of $\operatorname{sing} u$ *and it is a topological* $(n-3)$*-dimensional manifold of Hölder class* $C^{0,\gamma}$*.*

In the case when $\mathcal{N}$ is a real-analytic manifold, Simon [47, Lemma 4.3] showed that the set of possible energy densities $\lim_{r\to 0} r^{2-n} \int_{\mathbf{B}_r(x)} |\nabla u|^2$ is discrete. This allows us to slightly strenghten the statement above. The same conclusion holds also if $\mathcal{N}$ satisfies the integrability assumption introduced in [48, Ch. 3.13].

Corollary 3.1.4. *If* $u\colon \mathbf{B}^n \to \mathcal{N}$ *is an energy minimizing map into a real-analytic manifold* $\mathcal{N}$ *and* $\alpha \in \pi_2(\mathcal{N})$ *is an indecomposable homotopy class, then*

$$\left\{ x \in \operatorname{sing}_\alpha u : \lim_{r\to 0} r^{2-n} \int_{\mathbf{B}_r(x)} |\nabla u|^2 = \Theta(\alpha) \right\}$$

forms an open subset of $\operatorname{sing} u$ *and it is a topological* $(n-3)$*-dimensional manifold of Hölder class* $C^{0,\gamma}$ *with any* $0 < \gamma < 1$*.*

Specializing to the case $\mathcal{N} = \mathbb{S}^2$ and recalling the classification of tangent maps [4], we obtain a partial generalization of Theorem 3.1.2 [19] to arbitrary dimensions:

Corollary 3.1.5. *If* $u\colon \mathbf{B}^n \to \mathbb{S}^2$ *is an energy minimizing map, then the top-dimensional part* $\operatorname{sing}_* u$ *forms an open subset of* $\operatorname{sing} u$ *and it is a topological* $(n-3)$*-dimensional manifold of Hölder class* $C^{0,\gamma}$ *with any* $0 < \gamma < 1$*.*

An outline

Section 3.2 starts with the definition of indecomposable homotopy classes of maps from $\mathbb{S}^2$ into $\mathcal{N}$. Since Theorem 3.1.3 only concerns the singularities of indecomposable types, it is worthwhile to investigate the existence of such classes, which we do in Proposition 3.2.7. Indeed, we show that for any $\mathcal{N}$ the second homotopy group $\pi_2(\mathcal{N}, p)$ is generated (up to the action of $\pi_1(\mathcal{N}, p)$ on $\pi_2(\mathcal{N}, p)$) by indecomposable homotopy classes. This is very close to the classical (slightly weaker) result due to Sacks and Uhlenbeck [42] (see also [49]):

smooth harmonic maps from $\mathbb{S}^2$ into $\mathcal{N}$ generate the whole group $\pi_2(\mathcal{N}, p)$ (up to the action of $\pi_1(\mathcal{N}, p)$).

To obtain bi-Hölder-equivalence with a Euclidean ball, we employ Reifenberg's topological disc theorem [40] (see also [46]). We recall its statement and the so-called Reifenberg flatness condition in Section 3.3. We also introduce a flatness condition for an energy minimizer u which includes Reifenberg flatness for $\operatorname{sing} u$, but also forces u to be close to a tangent map.

The main results are proved in Section 3.4. The difficulty in applying Reifenberg's theorem to $\operatorname{sing} u$ lies in showing that this set has no gaps. This is done in Lemma 3.4.1; this is also the point where our topological assumptions play a role. Then we are able to show that if u satisfies the flatness condition on the ball $\mathbf{B}_2(0)$, it also satisfies the same condition on each smaller ball $\mathbf{B}_r(0)$ (Corollary 3.4.5) and on each ball $\mathbf{B}_1(z)$ centered at a point $z \in \mathbf{B}_1$ with enough energy density (Proposition 3.4.7). Combining these results, we check the hypotheses of Reifenberg's theorem and establish Theorem 3.1.3.

In addition, Section 3.5 we show that the δ-flatness property is stable with respect to small $W^{1,2}$-perturbations of the map. This simple result (first proved in [30]) will play a crucial role in the proof of Theorem 4.2.2 (measure bound on the S_{n-4} stratum of the singular set) and Theorem 7.1.1 (stability of the singular set).

Some other interesting observations not needed for the proof of Theorem 3.1.3 are gathered in Section 3.6.

3.2 Indecomposable homotopy classes

Decompositions of tangent maps

Consider now a HCM $\varphi_0 \colon \mathbb{R}^n \to \mathcal{N}$ with $S(\varphi_0) = \mathbb{R}^{n-3} \times \mathbf{0}$. This map actually depends only on 3 variables, i.e. $\varphi_0(x, y) = \varphi_1(y)$ for some homogeneous $\varphi_1 \colon \mathbb{R}^3 \to \mathcal{N}$. By [19, Lemma 2.1], the map φ_1 defined in this way is energy minimizing if and only if φ_0 is. Since φ_1 is homogeneous, it is uniquely determined by its restriction to the unit sphere $\varphi_2 \colon \mathbb{S}^2 \to \mathcal{N}$, which is a smooth harmonic map.

From now on, we shall abuse the notation and use the same symbol for all three maps $\varphi_0, \varphi_1, \varphi_2$; the precise meaning should be clear from the context. Note that their energies differ by a multiplicative constant:

$$\int_{\mathbb{S}^2} |\nabla\varphi_2|^2 = \int_{\mathbf{B}_1^3} |\nabla\varphi_1|^2 = C(n) \int_{\mathbf{B}_1^n} |\nabla\varphi_0|^2,$$

so energy comparison does not lead to confusion.

Homotopy type of a HCM always refers to the map $\varphi_2 \colon \mathbb{S}^2 \to \mathcal{N}$ (as φ_0, φ_1 are discontinuous and defined on contractible domains). For a general HCM φ_0 we may choose a rotation q that maps $S(\varphi_0)$ to $\mathbb{R}^{n-3} \times \mathbf{0}$ and thus reduce to the previous case. We then say that φ_0 has homotopy type α if $\varphi_0 \circ q^{-1}$ restricted to $\mathbf{0} \times \mathbb{S}^2$ has type α.

Remark 3.2.1. There is a subtle ambiguity here. Depending on the choice of q, we may obtain two homotopy types that differ by a composition with the antipodal map, i.e. both $[\varphi_2(x)]$ and $[\varphi_2(-x)]$.

Using this terminology, singular points in $\operatorname{sing}_* u$ can be classified according to their energy density and the homotopy type of a tangent map. Since we only consider basepoint-free homotopies, we denote by $\pi_2(\mathcal{N})$ the set of homotopy classes of continuous maps $\mathbb{S}^2 \to \mathcal{N}$.

Remark 3.2.2. To avoid confusion, we distinguish the set of basepoint-free homotopy classes $\pi_2(\mathcal{N})$ from the second fundamental group $\pi_2(\mathcal{N}, p)$. Note that in general $\pi_2(\mathcal{N})$ does not carry a group structure, as it is the quotient of the action of $\pi_1(\mathcal{N}, p)$ on $\pi_2(\mathcal{N}, p)$.

Definition 3.2.3. For any homotopy type $\alpha \in \pi_2(\mathcal{N})$ we let

$$\operatorname{sing}_\alpha u = \{y \in \operatorname{sing} u : \text{ some tangent map of } u \text{ at } y \text{ is a HCM of type } \alpha\}.$$

We also denote its lowest energy level by

$$\Theta(\alpha) := \inf \left\{ \int_{\mathbf{B}_1^n} |\nabla \varphi|^2 : \varphi \text{ is a HCM of type } \alpha \right\}.$$

A simple compactness argument shows that this infimum is either infinite (if no HCM has type α) or achieved by some minimal HCM. We also let

$$\operatorname{sing}_{\geqslant \Theta} u = \{y \in \operatorname{sing} u : \theta_u(y, 0) \geqslant \Theta\}.$$

which is a closed set by upper semicontinuity of $\theta_u(\cdot, 0)$.

At this point we cannot exclude the case when there are many homotopically different tangent maps at one point. However, this cannot happen under an additional assumption described below (see Remark 3.4.10). Again, the decomposition in Definition 3.2.4 is to be understood up to the action of $\pi_1(\mathcal{N})$, as described in Section 3.2 (see also the formulation of [42, Thm. 5.9]).

Definition 3.2.4. Consider $\alpha \in \pi_2(\mathcal{N})$ with $\Theta(\alpha) < \infty$. This homotopy class is called decomposable if there is a decomposition

$$\alpha = \alpha_1 + \ldots + \alpha_k \quad \text{in } \pi_2(\mathcal{N}),$$

where $\Theta(\alpha_j) < \Theta(\alpha)$ for each $j = 1, \ldots, k$. Otherwise α is called indecomposable.

Note that the above criterion does not depend on the dimension n, but only on the manifold $\mathcal{N}$.

As a special case, α is indecomposable if $\Theta(\alpha)$ is the smallest among all non-trivial homotopy types. In this case the proof is much easier (see the remark below Lemma 3.4.1).

Remark 3.2.5. Similar decompositions of this type appear naturally as a result of the *bubbling phenomenon* when one tries to minimize the energy in a given homotopy class. More precisely, recall that by [42] (see also [44, 49]) any smooth map $\varphi\colon \mathbb{S}^2 \to \mathcal{N}$ can be decomposed as $[\varphi] = [\varphi_1] + \ldots + [\varphi_k]$, where each φ_j is a harmonic map and

$$\sum_{j=1}^{k} \int_{\mathbb{S}^2} |\nabla \varphi_j^2| \leqslant \int_{\mathbb{S}^2} |\nabla \varphi^2|.$$

Motivated by these decompositions, one could replace the condition $\Theta(\alpha) > \max_j \Theta(\alpha_j)$ in Definition 3.2.4 by $\Theta(\alpha) \geqslant \sum_j \Theta(\alpha_j)$, thus enlarging the set of indecomposable classes. A natural conjecture here would be that Theorem 3.1.3 continues to hold in this case, but the author was not able to verify it.

Example 3.2.6. By Theorem 2.4.1 (classification of tangent maps from [4]), the only HCMs into the sphere $\mathbb{S}^2$ are isometries $\varphi\colon \mathbb{S}^2 \to \mathbb{S}^2$. Thus, for $\alpha \in \pi_2(\mathbb{S}^2)$ we have

$$\Theta(\alpha) = \begin{cases} 0 & \text{for } \alpha = 0, \\ 4\pi & \text{for } \alpha = [\pm\,\mathrm{id}], \\ \infty & \text{otherwise.} \end{cases}$$

By Definition 3.2.4, the indecomposable classes here are 0, $[\mathrm{id}]$, $[-\,\mathrm{id}]$. Note that these classes generate the whole group $\pi_2(\mathbb{S}^2)$ (see Proposition 3.2.7 for the general case).

An existence theorem

We show that the set of all indecomposable homotopy classes generates $\pi_2(\mathcal{N})$. Similarly to [42, Thm. 5.9], we only consider basepoint-free homotopies, so this statement should be understood as generating $\pi_2(\mathcal{N}, p)$ up to the action of $\pi_1(\mathcal{N}, p)$. In other words, for any $\alpha \in \pi_2(\mathcal{N})$ there are indecomposable homotopy classes $\alpha_1, \ldots, \alpha_k \in \pi_2(\mathcal{N})$ and a continuous map

$$u\colon \mathbf{B}^3 \setminus \bigcup_{j=1}^{k} \mathbf{B}_j \to \mathcal{N}$$

such that $u|_{\partial \mathbf{B}} \in \alpha$ and $u|_{\partial \mathbf{B}_j} \in \alpha_j$, where $\mathbf{B}_j \Subset \mathbf{B}$ are smaller disjoint balls.

This can be divided into two steps as follows. The first one is due to Schoen and Uhlenbeck [44, Prop. 3.3].

Proposition 3.2.7. *Let $\mathcal{N}$ be a closed Riemannian manifold. Then*

(a) the set of all HCMs $\varphi\colon \mathbb{S}^2 \to \mathcal{N}$ generates $\pi_2(\mathcal{N})$,
(b) each HCM $\varphi\colon \mathbb{S}^2 \to \mathcal{N}$ as an element of $\pi_2(\mathcal{N})$ can be decomposed into indecomposable homotopy classes.

Proof. To show part (a), fix $\alpha \in \pi_2(\mathcal{N})$ and choose a smooth map $\varphi\colon \mathbb{S}^2 \to \mathcal{N}$ of this type. Then there exists (possibly non-unique) $u \in W^{1,2}(\mathbf{B}_1^3, \mathcal{N})$ such that

$$\int_{\mathbf{B}_1} |\nabla u|^2 = \min \left\{ \int_{\mathbf{B}_1} |\nabla v|^2 : v \in W^{1,2}(\mathbf{B}_1, \mathcal{N}),\ v = \varphi \text{ on } \mathbb{S}^2 \right\};$$

note that the set of admissible maps is indeed non-empty, as it contains the map $x \mapsto \varphi(x/|x|)$. Such a minimizer has at most a finite number of interior singularities $p_1, \ldots, p_k \in \mathbf{B}_1$. At each p_j there is a (possibly non-unique) tangent map φ_j, which is necessarily a HCM; by uniform convergence away from the singularity, u restricted to $\partial \mathbf{B}_r(p_j)$ is homotopic to φ_j for some arbitrary small r (in consequence, also for all sufficiently small r). This yields the decomposition

$$[\varphi] = [\varphi_1] + \ldots + [\varphi_k] \quad \text{in } \pi_2(\mathcal{N}).$$

Part (b) follows from the definition by a compactness argument, which allows us to exclude infinite decompositions. Consider any homotopy type $\alpha \in \mathcal{N}$ represented by a HCM, i.e. with $\Theta(\alpha) < \infty$. First let us show that there are only finitely many homotopy types $\beta \in \pi_2(\mathcal{N})$ with $\Theta(\beta) \leqslant \Theta(\alpha)$. Indeed, otherwise we would have an infinite sequence of HCMs $\varphi_k \colon \mathbf{B}_1^3 \to \mathcal{N}$ with distinct homotopy types and uniformly bounded energy. Without loss of generality, φ_k converges to some HCM φ in $W^{1,2}(\mathbf{B}_1)$, but also in $C^0(\mathbb{S}^2)$. This shows that almost all φ_k have the same homotopy type as φ, which is a contradiction.

If α is decomposable, we have $\alpha = \alpha_1 + \ldots + \alpha_k$, where $\Theta(\alpha_j) < \Theta(\alpha)$ for each j. Decomposing further each α_j whenever possible, and iterating this procedure until all obtained homotopy types are indecomposable, we arrive at claim (b). One only needs to note that this procedure stops after at most N steps, where N is the number of homotopy types from the last paragraph. Indeed, any branch of the decomposition tree is a sequence $\beta_0, \beta_1, \beta_2, \ldots$ with $\beta_0 = \alpha$ and $\Theta(\beta_{j+1}) < \Theta(\beta_j)$ for each j, so it contains at most N elements. □

We remark that a similar decomposition was first obtained by Sacks and Uhlenbeck [42] (see also [49]): smooth harmonic maps from $\mathbb{S}^2$ into $\mathcal{N}$ generate the whole group $\pi_2(\mathcal{N}, p)$ up to the action of $\pi_1(\mathcal{N}, p)$. It may be that some homotopy classes in $\pi_2(\mathcal{N})$ do not contain any harmonic map. Since here we only consider harmonic maps $\varphi \colon \mathbb{S}^2 \to \mathcal{N}$ for which the homogeneous extension $\varphi \colon \mathbf{B}^3 \to \mathcal{N}$ is energy minimizing, the result discussed above is slightly more general.

3.3 Notions of flatness and Reifenberg's topological disc theorem

Hölder regularity of the singular set will be obtained by an application of Reifenberg's topological disc theorem [40] (see also [46]). To state it, we first need the following notion of flatness (for our purposes restricted to codimension 3).

Definition 3.3.1. A set $A \subseteq \mathbb{R}^n$ is said to be ε-Reifenberg flat in the ball $\mathbf{B}_r(x)$

(with respect to L) if

$$A \cap \mathbf{B}_r(x) \subseteq \mathbf{B}_{r\varepsilon} L \quad \text{and} \quad L \cap \mathbf{B}_r(x) \subseteq \mathbf{B}_{r\varepsilon} A$$

for some $(n-3)$-dimensional affine plane L through x.

The above condition means exactly that the normalized Hausdorff distance on $\mathbf{B}_r(x)$ from A to some $(n-3)$-dimensional affine plane through x is not larger than ε.

Theorem 3.3.2 (Reifenberg's topological disc theorem)**.** *For each Hölder exponent $0 < \gamma < 1$ there is $\varepsilon(n, \gamma) > 0$ such that the following holds. If a closed set $A \subseteq \mathbb{R}^n$ containing the origin is ε-Reifenberg flat in each ball $\mathbf{B}_r(x)$ with $x \in A \cap \mathbf{B}_1$ and $r < 1$, then the set $A \cap \mathbf{B}_1$ is bi-Hölder equivalent to the closed unit ball $\mathbf{B}^{n-3} \subseteq \mathbb{R}^{n-3}$ with exponent γ.*

We shall also make repeated use of the following condition for energy minimizing maps.

Definition 3.3.3. Fix an indecomposable homotopy class $\alpha \in \pi_2(\mathcal{N})$ and let $\Theta = \Theta(\alpha)$ be its lowest energy level (as in Definition 3.2.3). We say that an energy minimizer u is δ-flat in the ball $\mathbf{B}_r(x)$ (of type α) if

1. x is a singular point of u and $\Theta \leqslant \theta_u(x, 0) \leqslant \theta_u(x, r) \leqslant \Theta + \delta$,
2. $\operatorname{sing} u$ is $\frac{1}{10}$-Reifenberg flat in $\mathbf{B}_r(x)$ with respect to some L, and u restricted to $(x + L^{\perp}) \cap \partial \mathbf{B}_{r/2}(x)$ has homotopy type α.

Note that this definition is scale-invariant in the following sense: u is δ-flat in $\mathbf{B}_r(x)$ if and only if the rescaled map $\overline{u}(y) = u(x + ry)$ is δ-flat in $\mathbf{B}_1$. Also note that u is smooth outside the tube around L by and thus the homotopy type is well-defined.

The main feature of Definition 3.3.3 is that δ-flatness in a ball trivially ensures that Condition 1 is satisfied in all smaller concentric balls, and one only needs to check Condition 2 (see Corollary 3.4.5). This is why the constant $\frac{1}{10}$ in Condition 2 was chosen independently of δ.

In fact, one could relax the $\frac{1}{10}$-Reifenberg condition in 2 to the one-sided condition $\operatorname{sing} u \cap \mathbf{B}_r(x) \subseteq \mathbf{B}_{r/10}L$. This gives effectively the same notion of δ-flatness, as used in [30].

From now on, we consider a non-trivial indecomposable class α and its lowest energy level $\Theta = \Theta(\alpha)$ to be fixed.

3.4 Regularity of the singular set

Persistence of indecomposable singularities

As a first step, we show that if an energy minimizer u restricted to some sphere has homotopy type α, then $\operatorname{sing} u$ satisfies the flatness condition of Definition 3.3.3 and the energy density of u cannot drop in a smaller ball. Note that the claim of Lemma 3.4.1 is essentially stronger than the corresponding condition $L \cap \mathbf{B}_1 \subseteq \mathbf{B}_\varepsilon(\operatorname{sing}_{\geqslant\Theta} u)$ appearing in Definition 3.3.1.

Observe that some tubular neighborhood $\mathbf{B}_\eta\mathcal{N} \subseteq \mathbb{R}^M$ admits a continuous retraction $\pi_\mathcal{N}$ onto $\mathcal{N}$. As a consequence, if two continuous functions f, g into $\mathcal{N} \subseteq \mathbb{R}^M$ are close enough in supremum norm, then

$$(t, x) \mapsto \pi_\mathcal{N}(tf(x) + (1-t)g(x))$$

yields a homotopy between them.

Lemma 3.4.1. *Assume that* $\operatorname{sing} u \cap \mathbf{B}_1 \subseteq \mathbf{B}_\varepsilon L$ *for some* $0 < \varepsilon < \frac{1}{5}$ *and some* $(n-3)$*-dimensional plane* L *through* 0*. Assume further that* u *restricted to the sphere* $L^\perp \cap \partial\mathbf{B}_{1/2}$ *has homotopy type* α*. Then*

$$L \cap \mathbf{B}_{1-\varepsilon} \subseteq \pi_L(\operatorname{sing}_{\geqslant\Theta} u \cap \mathbf{B}_1),$$

where π_L denotes the orthogonal projection onto L. In particular, $\operatorname{sing}_{\geqslant\Theta} u$ *is* ε*-Reifenberg flat in* $\mathbf{B}_1$.

Before giving the full proof, let us consider the special case when $\Theta(\alpha)$ is the lowest among all non-trivial homotopy types; in particular, this property holds if we consider maps into the standard sphere $\mathbb{S}^2$. In this case, the proof is simpler and does not depend on the deep results of Naber and Valtorta [37].

For each $y \in L \cap \mathbf{B}_{1-\varepsilon}$, the restriction of u to the sphere $(y + L^{\perp}) \cap \partial\mathbf{B}_{\varepsilon}(y)$ has homotopy type α, therefore it cannot be continuously extended to the ball $(y + L^{\perp}) \cap \mathbf{B}_{\varepsilon}(y)$. This shows the weaker inclusion $L \cap \mathbf{B}_{1-\varepsilon} \subseteq \pi_L(\operatorname{sing} u \cap \mathbf{B}_1)$. Recall that $\mathcal{H}^{n-3}$-a.e. point $z \in \operatorname{sing} u$ belongs to $\operatorname{sing}_* u$ and hence $\theta_u(z, 0) \geqslant \Theta$ due to our additional assumption. Since $\operatorname{sing}_{\geqslant\Theta} u$ is a closed set, we obtain the stronger inclusion.

Proof of Lemma 3.4.1. For simplicity, let us assume $L = \mathbb{R}^{n-3} \times \mathbf{0}$ (by composing u with a rotation, if necessary). Assume for the contrary that $L \cap \mathbf{B}_{1-\varepsilon}$ is not covered by the projection of $\operatorname{sing}_{\geqslant\Theta} u \cap \overline{\mathbf{B}}_{1-\varepsilon/2}$. Since the latter is a compact set, it has to be disjoint with some small cylinder $\mathbf{B}_{\delta}^{n-3}(z) \times \mathbb{R}^3$ with $|z| \leqslant 1 - \varepsilon$.

Recall that by the recent important work of Naber and Valtorta [37] discussed in the next chapter (see Theorem 4.1.1), the set $\operatorname{sing} u$ has finite $\mathcal{H}^{n-3}$ measure (locally, away from the boundary). Moreover, the set $\operatorname{sing} u$ is $(n-3)$-rectifiable and for $\mathcal{H}^{n-3}$-a.e. $y \in \operatorname{sing} u$ there exists an $(n-3)$-dimensional tangent plane $\operatorname{Tan}(\operatorname{sing} u, y)$ coinciding with $S(\varphi)$ for every tangent map φ of u at y. Let us temporarily assume that these tangent planes are transversal to $\mathbf{0} \times \mathbb{R}^3$, i.e.

$$\operatorname{Tan}(\operatorname{sing} u, y) \pitchfork \mathbf{0} \times \mathbb{R}^3 \quad \text{for } \mathcal{H}^{n-3}\text{-a.e. } y \in \operatorname{sing} u \cap \mathbf{B}_{1-\varepsilon/2}. \tag{3.4.1}$$

We shall need Eilenberg's inequality, a Fubini-type inequality valid for any $\mathcal{H}^{n-3}$-measurable set A with finite measure (see [28, 7.7-7.8] or [13, 2.10.25-

27]):

$$\int_{\mathbf{B}^{n-3}_{\delta}(z)} \mathcal{H}^0(A \cap \pi_L^{-1}(y))\,\mathrm{d}y \leqslant \omega_{n-3}\mathcal{H}^{n-3}(A)$$

Applying the above inequality twice – once with A as the singular set and once with A as its exceptional part of measure zero – we learn that for almost every $y \in \mathbf{B}^{n-3}_{\delta}(z)$ the slice $\operatorname{sing} u \cap \mathbf{B}_1 \cap \pi_L^{-1}(y)$ consists of finitely many points, at each of them the tangent plane exists and is transverse to $\mathbf{0} \times \mathbb{R}^3$ (i.e. the direction of slicing).

Let us choose one such y and denote these singular points by $p_1, \ldots, p_k$. Let also

$$L_j := \operatorname{Tan}(\operatorname{sing} u, p_j), \quad r_0 := \tfrac{1}{2} \min_{i \neq j} \left(|p_i - p_j|, \varepsilon - |p_i - y| \right).$$

For each $j = 1, \ldots, k$, there is a HCM $\varphi_j \colon \mathbb{R}^n \to \mathcal{N}$ with $S(\varphi_j) = L_j$ such that the sequence of rescaled maps $u_{r_i}(x) = u(p_j + r_i x)$ converges to φ_j in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ for some sequence $r_i \to 0$. Note that by our assumption, φ_j has energy density strictly less than Θ. Since this convergence is uniform away from L_j, maps u_{r_i} and φ_j are homotopic on $L_j^{\perp} \cap \partial\mathbf{B}_1$ for large enough i. Tilting $L_j^{\perp}$ to $\mathbf{0} \times \mathbb{R}^3$ and rescaling, we get that for some small $r_j < r_0$ the restriction of u to $\pi_L^{-1}(y) \cap \partial\mathbf{B}_{r_j}(p_j)$ has the homotopy type of φ_j. Recalling that u restricted to $\mathbf{0} \times \mathbb{R}^3 \cap \partial\mathbf{B}_{1/2}$ (and hence also to $\pi_L^{-1}(y) \cap \partial\mathbf{B}_{\varepsilon}(y)$) has homotopy type α, we conclude

$$\alpha = [\varphi_1] + \ldots + [\varphi_k] \quad \text{in } \pi_2(\mathcal{N}),$$

where each φ_j has energy density smaller than Θ, which is a contradiction with the assumption that α is indecomposable.

To finish the proof, we need to get rid of the additional assumption (3.4.1). This is done by using the following simple transversality lemma.

Lemma 3.4.2. *Let $n = a + b$, consider the Grassmannian $G(n,a)$ with the standard Haar measure λ and $G(n,b)$ with a finite positive Borel measure μ. Then the set*

$$\{E \in G(n,a) : \mu(\{F \in G(n,b) : E \not\pitchfork F\}) > 0\})$$

has zero λ measure.

Postponing its proof for the moment, we complete the reasoning as follows. Choose $a = 3$, $b = n - 3$, and let μ be the measure $\mathcal{H}^{n-3} \llcorner \operatorname{sing} u \cap \mathbf{B}_{1-\varepsilon/2}$ pushed-forward by the map $\operatorname{Tan}(\operatorname{sing} u, \cdot)$, i.e.

$$\mu(U) = \mathcal{H}^{n-3}(\{y \in \operatorname{sing} u \cap \mathbf{B}_{1-\varepsilon/2} : \operatorname{Tan}(\operatorname{sing} u, y) \in U\}) \quad \text{for } U \subseteq G(n, n-3).$$

Then the set in Lemma 3.4.2 has measure zero and in particular its complement is dense. In result, we can choose $E \in G(n, 3)$ so that $E \pitchfork F$ for μ-a.e. $F \in G(n, n-3)$, with E arbitrarily close to $\mathbf{0} \times \mathbb{R}^3$. This amounts to satisfying (3.4.1) with a slightly tilted direction of slicing. Recall that $\operatorname{sing}_{\geqslant \Theta} u \cap \mathbf{B}_{1-\varepsilon/2}$ is disjoint with the cylinder $\mathbf{B}^{n-3}_{\delta}(z) \times \mathbb{R}^3$, in consequence it is also disjoint with some smaller cylinder in direction $E^{\perp}$. It is easy to see that the rest of the proof remains unchanged. □

Proof of Lemma 3.4.2. First note that here $E \pitchfork F$ means these two linear subspaces intersect only at the origin. Thus, for each $F \in G(n, b)$ the set of all $E \in G(n, a)$ non-transversal to F is a finite union of smooth submanifolds of $G(n, a)$ of positive codimension

$$\{E \in G(n, a) : E \not\pitchfork F\} = \bigcup_{c=1}^{\min(a,b)} \{E \in G(n, a) : \dim E \cap F = c\},$$

hence it has zero λ measure. Applying Fubini's theorem, we get

$$\begin{aligned} \int_{G(n,a)} & \mu(\{F \in G(n, b) : E \not\pitchfork F\}) \, \mathrm{d}\lambda(E) \\ &= \int_{G(n,a)} \int_{G(n,b)} \mathbf{1}_{E \not\pitchfork F} \, \mathrm{d}\mu(F) \, \mathrm{d}\lambda(E) \\ &= \int_{G(n,l)} \lambda(\{E \in G(n, a) : E \not\pitchfork F\}) \, \mathrm{d}\mu(F) \\ &= 0, \end{aligned}$$

so the integrand has to be zero for λ-a.e. $E \in G(n, a)$. □

Propagation of δ-flatness to finer scales

In this section we investigate some important consequences of Definition 3.3.3. Assuming that an energy minimizing map u is δ-flat in $\mathbf{B}_1$ (with small $\delta > 0$), we shall see that $\operatorname{sing} u$ is actually more flat than a priori assumed (Lemma 3.4.3), u is also δ-flat in all smaller concentric balls (Corollary 3.4.5), and that $0 \in \operatorname{sing}_\alpha u$ (Corollary 3.4.6).

Lemma 3.4.3. *For every $\varepsilon > 0$ there is $\delta_1(\varepsilon) > 0$ such that if u is δ_1-flat in $\mathbf{B}_2$, then $\|u - \varphi\|_{W^{1,2}(\mathbf{B}_1)} \leqslant \varepsilon$ for some HCM φ of homotopy type α with energy density Θ. Moreover, $\operatorname{sing} u$ is ε-Reifenberg flat in $\mathbf{B}_1$ with respect to the $(n-3)$-dimensional plane $S(\varphi)$.*

Remark 3.4.4. For clarity, the conclusion above is stated on the twice smaller ball, but one can obtain a stronger conclusion ($\|u - \varphi\|_{W^{1,2}(\mathbf{B}_{2-\varepsilon})} \leqslant \varepsilon$ and ε-Reifenberg condition in $\mathbf{B}_{1-\varepsilon}$) by the same argument. However, restricting to a smaller ball is necessary due to the local nature of the $W^{1,2}$-compactness theorem, as well as possible singularities on $\partial \mathbf{B}_2$.

Proof. We employ the usual contradiction argument. Let u_k be a sequence of minimizing harmonic maps such that u_k is $1/k$-flat in $\mathbf{B}_2$; we may assume $\operatorname{sing} u$ is $\frac{1}{10}$-Reifenberg flat with respect to a fixed plane L. Choosing a subsequence, we have $u_k \to \varphi$ in $W^{1,2}_{\mathrm{loc}}(\mathbf{B}_2)$ for some energy minimizing φ. By condition 1 in Definition 3.3.3, φ is homogeneous with energy density Θ. By Lemma 3.4.1, for each k the set $\operatorname{sing}_{\geqslant \Theta} u_k$ is $\frac{1}{10}$-Reifenberg flat in $\mathbf{B}_1$ with respect to L. Taking the limit and exploiting the upper semicontinuity of $\theta_\cdot(\cdot, 0)$ with respect to both the map and the point, we conclude that the set

$$S(\varphi) \equiv \operatorname{sing}_{\geqslant \Theta} \varphi$$

is not contained in any $(n-4)$-dimensional plane. On the other hand, it is itself a linear subspace of dimension at most $n-3$, so we learn that φ is a HCM of homotopy type α; the homotopy property follows from uniform convergence away from L. For large enough k, u_k is ε-close to φ in $W^{1,2}(\mathbf{B}_1)$ and its singular set is contained in $\mathbf{B}_\varepsilon S(\varphi)$ (this is a consequence of upper semicontinuity of

$\theta_\cdot(\cdot, 0)$ and ε-regularity (2.1.2)), which finishes the proof by another application of Lemma 3.4.1. □

Corollary 3.4.5. *If $\delta \leqslant \delta_1(\frac{1}{20})$ and u is δ-flat in $\mathbf{B}_2$, then u is also δ-flat in any smaller ball $\mathbf{B}_r$ centered at 0 with $0 < r \leqslant 1$.*

Proof. Condition 1 of Definition 3.3.3 is trivially satisfied. As for condition 2, it follows from Lemma 3.4.3 that $\operatorname{sing} u$ is $\frac{1}{20}$-Reifenberg flat in $\mathbf{B}_1$, hence $\frac{1}{10}$-flat in any ball $\mathbf{B}_r$ with $\frac{1}{2} \leqslant r \leqslant 1$. In consequence, u is δ-flat in each of these balls. Then the claim follows by iteration of Lemma 3.4.3 rescaled to smaller and smaller balls. □

Corollary 3.4.6. *If $\delta \leqslant \delta_1(\frac{1}{20})$ and u is δ-flat in $\mathbf{B}_1$, then every tangent map to u at 0 is a HCM of type α. In particular, $0 \in \operatorname{sing}_\alpha u$.*

Proof. Let φ be any tangent map to u at 0, i.e. a $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$-limit of rescaled functions $u_k(x) = u(r_k x)$ for some sequence $r_k \to 0$; any thus obtained φ is homogeneous. By Corollary 3.4.5, each u_k is δ-flat in $\mathbf{B}_1$ (with the set $\operatorname{sing} u_k$ $\frac{1}{10}$-Reifenberg flat with respect to some L_k), so the claim follows from Lemma 3.4.1 as in the proof of Lemma 3.4.3. The only difference is that the planes L_k may change, but without loss of generality $L_k \to L$ in $G(n, n-3)$, which is enough to conclude that $\operatorname{sing}_{\geqslant \Theta} \varphi$ spans an $(n-3)$-dimensional plane. □

Moving the ball center

Proposition 3.4.7. *For every $\varepsilon > 0$ there is $\delta_2(\varepsilon) > 0$ such that if u is δ_2-flat in $\mathbf{B}_2$, then u is $\delta_1(\varepsilon)$-flat in each of the balls $\mathbf{B}_r(z)$ with $z \in \operatorname{sing}_{\geqslant \Theta} u \cap \mathbf{B}_1$ and $0 < r \leqslant 1/2$. Moreover, the sets $\operatorname{sing}_\alpha u$ and $\operatorname{sing}_{\geqslant \Theta} u$ restricted to the ball $\mathbf{B}_1$ coincide.*

Proof. Choose $\delta_2 := \min(\delta_1(\varepsilon), \delta_1(\eta/2))$ according to Lemma 3.4.3, where $\eta > 0$ is to be fixed in a moment. Applying Lemma 3.4.3 rescaled to the ball $\mathbf{B}_2$, denote by φ the approximating HCM and let $L = S(\varphi)$; according to Remark 3.4.4, we may assume the conclusion actually holds on the larger ball $\mathbf{B}_{3/2}$. To obtain the first claim, we first show that $\theta_u(z, 1/2) \leqslant \Theta + \delta_1(\varepsilon)$ for each $z \in \mathbf{B}_1 \cap \mathbf{B}_\eta L$. First,

$$\int_{\mathbf{B}_{1/2}(z)} |\nabla u|^2 \leqslant \int_{\mathbf{B}_{1/2}(z)} |\nabla \varphi|^2 + C\eta^2,$$

by Lemma 3.4.3. If $z' = \pi_L(z)$, then $|z - z'| < \eta$ and

$$\int_{\mathbf{B}_{1/2}(z)} |\nabla \varphi|^2 \leqslant \int_{\mathbf{B}_{1/2+\eta}(z')} |\nabla \varphi|^2 = (1 + 2\eta)^{n-2}\Theta$$

by L-invariance of φ in z'-direction. If η is chosen small enough (depending on $\delta_1(\varepsilon)$), we obtain $\theta_u(z, 1/2) \leqslant \Theta + \delta_1(\varepsilon)$.

Since each point $z \in \operatorname{sing}_{\geqslant\Theta} u \cap \mathbf{B}_1$ lies in $\mathbf{B}_\eta L$ by Lemma 3.4.3, the above reasoning shows that condition 1 of Definition 3.3.3 holds for the ball $\mathbf{B}_{1/2}(z)$. Condition 2 is satisfied by our assumptions, so this ball is $\delta_1(\varepsilon)$-flat. Then Corollary 3.4.5 implies $\delta_1(\varepsilon)$-flatness of u also in all smaller balls $\mathbf{B}_r(z)$ (to be precise, δ_1-flatness for radii in the interval $(\frac{1}{4}, \frac{1}{2})$ comes from the assumption of Reifenberg flatness).

By Corollary 3.4.6 we now have $z \in \operatorname{sing}_\alpha u$ for each $z \in \operatorname{sing}_{\geqslant\Theta} \cap \mathbf{B}_1$. The inverse inclusion $\operatorname{sing}_\alpha u \subseteq \operatorname{sing}_{\geqslant\Theta} u$ is evident from the definition of $\Theta(\alpha)$. □

Corollary 3.4.8. *Under the assumptions of Proposition 3.4.7, the whole singular set* $\operatorname{sing} u$ *restricted to the ball* $\mathbf{B}_{1/2}$ *coincides with* $\operatorname{sing}_{\geqslant\Theta} u$ *(and hence with* $\operatorname{sing}_\alpha u$*).*

Proof. Assume that the ball $\mathbf{B}_{1/2}$ contains a point $p \in \operatorname{sing} u \setminus \operatorname{sing}_{\geqslant\Theta} u$. We may choose a point $z \in \operatorname{sing}_{\geqslant\Theta} u$ closest to p (as it is a closed set) and set $r = 2|p - z|$. Clearly $z \in \mathbf{B}_1$ and $0 < r \leqslant 1$, so u is $\delta_1(\varepsilon)$-flat in $\mathbf{B}_r(z)$. Choose $L = L(z, r)$ according to Definition 3.3.1. Then by Lemma 3.4.1 there is a point $z' \in \operatorname{sing}_{\geqslant\Theta} u \cap \mathbf{B}_r(z)$ such that $\pi_L(z') = \pi_L(p)$. Since both $|\pi_L(p) - p|$ and $|\pi_L(z') - z|$ are less than $\frac{r}{10}$, the triangle inequality yields a contradiction with minimality of z. □

In order to apply the above results, one needs to know that u is δ-flat in at least one ball.

Lemma 3.4.9. *Let $\delta > 0$. If $0 \in \operatorname{sing}_\alpha u$ and $\theta_u(0,0) < \Theta + \delta$, then there is $r > 0$ such that u is δ-flat in $\mathbf{B}_r$.*

Proof. Note that condition 1 of Definition 3.3.3 is trivially satisfied for small enough r.

By definition of $\operatorname{sing}_\alpha u$, some sequence of rescaled functions $u_k(x) = u(r_k x)$ converges in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ to a HCM φ of homotopy type α for some sequence $r_k \to 0$. For large enough k, we have $\operatorname{sing} u_k \cap \mathbf{B}_1 \subseteq \mathbf{B}_{1/10} S(\varphi)$. Since the convergence is uniform away from $S(\varphi)$, u_k restricted to $S(\varphi)^\perp \cap \partial\mathbf{B}_{1/2}$ has homotopy type α, so condition 2 follows from Lemma 3.4.1. Rescaling, we see that u is δ-flat in $\mathbf{B}_{r_k}$ for large enough k. □

Remark 3.4.10. Combining Lemma 3.4.9 with Corollary 3.4.6, we see that *some* can be changed to *any* in the definition of $\operatorname{sing}_\alpha$, if only we restrict ourselves to points with energy density close to optimal. That is, if $y \in \operatorname{sing}_\alpha u$ and $\theta_u(y,0) < \Theta + \delta_1(\frac{1}{20})$, then every tangent map of u at y is a HCM of type α.

We are now ready to prove the main theorem.

Proof of Theorem 3.1.3. Fix the Hölder exponent $0 < \gamma < 1$ and choose the value $\varepsilon(\gamma, n) > 0$ according to Reifenberg's topological disc theorem (Theorem 3.3.2), then fix δ to be $\delta_2(\varepsilon)$ from Proposition 3.4.7.

Choose a point $p \in \operatorname{sing}_\alpha u$ such that $\theta_u(p,0) < \Theta + \delta$. According to Lemma 3.4.9, u is $\delta_2(\varepsilon)$-flat in some small enough ball $\mathbf{B}_{2r}(p)$. By Proposition 3.4.7 we now know that the set $\operatorname{sing}_\alpha u \cap \mathbf{B}_r(p)$ is closed and ε-flat in each ball $\mathbf{B}_s(z)$ centered at $z \in \operatorname{sing}_\alpha u \cap \mathbf{B}_r(p)$ with radius $0 < s < r/2$. Applying Theorem 3.3.2, we conclude that $\operatorname{sing}_\alpha u \cap \mathbf{B}_r(p)$ is bi-Hölder equivalent (with exponent γ) to an $(n-3)$-dimensional ball.

By upper semicontinuity of $\theta_u(\cdot, 0)$ we can ensure $\theta_u(y, 0) < \Theta + \delta$ for all $y \in \mathbf{B}_r(p)$ (just by taking r small enough), which together with Corollary 3.4.8 shows that the set in question forms an open subset of $\operatorname{sing} u$. □

3.5 Stability of δ-flatness

The following proposition states that the δ-flatness property is stable with respect to small $W^{1,2}$-perturbations of the map. This simple result will play a crucial role in the proof of Theorem 4.2.2 (measure bound on the S_{n-4} stratum of the singular set) and Theorem 7.1.1 (stability of the singular set).

As before, consider an indecomposable homotopy type $\alpha \in \pi_2(\mathcal{N})$ and its lowest energy level $\Theta = \Theta(\alpha)$.

Proposition 3.5.1 (Stability of δ-flatness)**.** *For each $\varepsilon > 0$ there is $\delta > 0$ such that the following holds. If u is δ-flat in the ball $\mathbf{B}_1$ and $u_k \to u$ in $W^{1,2}(\mathbf{B}_1)$, then for k large enough there is $x_k \in \operatorname{sing} u_k \cap \mathbf{B}_\varepsilon$ such that u_k is ε-flat in the ball $\mathbf{B}_{1-\varepsilon}(x_k)$.*

Proof. Choose $\varepsilon'(n, \varepsilon) > 0$ small enough, more precisely such that

$$\varepsilon' < \varepsilon/2, \quad (1 - 2\varepsilon')^{2-n}(\Theta + \varepsilon/2) \leqslant \Theta + \varepsilon.$$

By taking δ small enough, we may assume by Lemma 3.4.3 (and Remark 3.4.4) that

$$\operatorname{sing} u \cap \mathbf{B}_{1-\varepsilon'/2} \subseteq \mathbf{B}_{\varepsilon'/2}(L)$$

for some $(n-3)$-dimensional linear plane L. Since singular points converge again to singular points (see (2.1.3)), we have for all large k,

$$\operatorname{sing} u_k \cap \mathbf{B}_{1-\varepsilon'} \subseteq \mathbf{B}_{\varepsilon'/2}(L). \tag{3.5.1}$$

Recall that $u_k \to u$ locally uniformly outside the singular set, and thus

$$u_k \rightrightarrows u \quad \text{in } \mathbf{B}_{1-\varepsilon'} \setminus \mathbf{B}_{\varepsilon'/2}(L).$$

In particular, u_k and u restricted to $L^\perp \cap \partial \mathbf{B}_{1/2}$ have the same homotopy type for large k.

By Lemma 3.4.1

$$L \cap \mathbf{B}_{1-2\varepsilon'} \subseteq \pi_L(\operatorname{sing}_{\geqslant \Theta} u_k \cap \mathbf{B}_{1-\varepsilon'}).$$

Combined with (3.5.1), this means that we may find $x_k \in \operatorname{sing}_\Theta u_k$ such that $|x_k| \leqslant \frac{1}{2}\varepsilon'$ and $\theta_{u_k}(x_k, 0) \geqslant \Theta$.

The last condition to show is $\theta_{u_k}(x_k, 1-\varepsilon) \leqslant \Theta + \varepsilon$. By strong convergence, for large enough k,

$$\int_{\mathbf{B}_{1-\varepsilon'}} |\nabla u_k|^2 \leqslant \varepsilon/4 + \int_{\mathbf{B}_1} |\nabla u|^2.$$

Thus

$$\begin{aligned}(1-2\varepsilon')^{2-n} \int_{\mathbf{B}_{1-2\varepsilon'}(x_k)} |\nabla u_k|^2 &\leqslant (1-2\varepsilon')^{2-n}\left(\varepsilon/4 + \int_{\mathbf{B}_1} |\nabla u|^2\right) \\ &\leqslant (1-2\varepsilon')^{2-n}(\Theta + \delta + \varepsilon/4),\end{aligned}$$

which does not exceed $\Theta + \varepsilon$ if only $\delta \leqslant \varepsilon/4$. By the monotonicity formula, we conclude that $\theta_{u_k}(x_k, 1-\varepsilon) \leqslant \theta_{u_k}(x_k, 1-2\varepsilon') \leqslant \Theta + \varepsilon$ and hence that u_k is ε-flat in the ball $\mathbf{B}_{1-\varepsilon}(x_k)$. □

3.6 Additional results

In this subsection we discuss two elementary observations that give a better description of δ-flatness, but were not needed earlier in the proof of Theorem 3.1.3. Again, we fix an indecomposable homotopy type $\alpha \in \pi_2(\mathcal{N})$ and its lowest energy level $\Theta = \Theta(\alpha)$.

The following lemma shows that Condition 2 in Definition 3.3.3 can be dropped if one assumes a priori that $x \in \operatorname{sing}_\alpha u$. This gives us an equivalent condition for δ-flatness.

Lemma 3.6.1. *Assume that* $0 \in \operatorname{sing}_\alpha u$. *If* $\delta \leqslant \delta_1(\frac{1}{20})$ *and* $\theta_u(0,2) \leqslant \Theta + \delta$, *then* u *is* δ*-flat in* $\mathbf{B}_1$.

Proof. Inspecting the proof of Lemma 3.4.3, we see that condition 2 of Definition 3.3.3 was only needed to ensure required symmetry of approximating homogeneous minimizer φ. Hence it would be enough to assume condition 2 of Definition 3.3.3 in a smaller ball $\mathbf{B}_{1/2}$, and δ-flatness in $\mathbf{B}_1$ follows as in Lemma 3.4.3.

By Lemma 3.4.9, there is $r > 0$ (possibly very small) such that u is δ-flat in $\mathbf{B}_r$. Applying the reasoning above, we see it is also δ-flat in every ball $\mathbf{B}_s$ with $r \leqslant s \leqslant \min(1, 2r)$. An iteration of this argument (as in Corollary 3.4.5, but in the opposite direction) leads to the claim. □

The last lemma gives a uniform bound (independent of u) for the rate of convergence $\theta_u(x,r) \to \theta_u(x,0)$ when $r \to 0$, assuming $\theta_u(x,r)$ is already close to $\theta_u(x,0)$. This assumption cannot be dropped, if only there exist tangent maps $\varphi\colon \mathbb{R}^n \to \mathcal{N}$ with $\dim_H \operatorname{sing} \varphi = n-3$ which are not HCMs.

An additional assumption is needed to ensure that the energy density is not greater than Θ. This assumption is automatically satisfied if $\mathcal{N}$ is real-analytic or integrable in the sense of [48, Ch. 3.13]; see the remark preceding Corollary 3.1.4.

Lemma 3.6.2. *Let us assume that* $0 \in \operatorname{sing}_\alpha u$ *and* $\theta_u(0,1) \leqslant \Theta + \delta_3$ *with* $\delta_3(n,\alpha,\mathcal{N}) > 0$ *sufficiently small, and assume additionally that* Θ *is an isolated energy level for HCMs of type* α. *Then for every* $\delta > 0$ *there is* $r(\delta, n, \mathcal{N}) > 0$ *such that* $\theta_u(0,r) \leqslant \Theta + \delta$ *(in consequence,* u *is* δ*-flat in* $\mathbf{B}_r$*).*

Proof. We choose $\delta_3 > 0$ smaller than $\delta_1(\frac{1}{20})$ from Lemma 3.4.3 and such that

$$\int_{\mathbf{B}_1^n} |\nabla\varphi|^2 \notin (\Theta, \Theta + \delta_3]$$

for each HCM φ of type α.

For the sake of contradiction, assume there is a sequence of such energy minimizing maps u_k with

$$\Theta + \delta \leqslant \theta_{u_k}(0, 1/k) \leqslant \theta_{u_k}(0, 1) \leqslant \Theta + \delta_3.$$

Taking a subsequence, we obtain a limit map u such that

$$\Theta + \delta \leqslant \theta_u(0, 0) \leqslant \theta_u(0, 1) \leqslant \Theta + \delta_3.$$

It follows from Lemma 3.6.1 that each u_k is $\delta_1(\frac{1}{20})$-flat in $\mathbf{B}_{1/2}$, hence so is u and by Corollary 3.4.6 we infer $0 \in \operatorname{sing}_\alpha u$. In particular the energy density $\theta_u(0, 0)$ is either Θ or greater than $\Theta + \delta_3$, a contradiction. □

Chapter 4

Regularity results of Naber and Valtorta

4.1 Important tools and results

Here we discuss the results of Naber and Valtorta [37] needed in the sequel. A simplified presentation of these is available in their later article [38].

The main result of [37] is the following more precise restatement of Theorem 1.3.6 discussed in the introduction. As in the previous chapter, all the cited results also hold for local energy minimizers, and so we drop the distinction between local and global minimizers.

Theorem 4.1.1 ([37, Thm. 1.5, 1.6]). *Let* $u\colon \mathbf{B}_{2r} \to \mathcal{N}$ *be energy minimizing and* $r^{2-n}\int_{\mathbf{B}_{2r}} |\nabla u|^2 \leqslant \Lambda$*. Then there exists a constant* $C(n, \mathcal{N}, \Lambda) > 0$ *such that* $\mathcal{H}^{n-3}(\operatorname{sing} u \cap \mathbf{B}_r) \leqslant Cr^{n-3}$*.*

Moreover, $\operatorname{sing} u$ *is a rectifiable* $(n-3)$*-dimensional set and for* $\mathcal{H}^{n-3}$*-a.e. singular point* $p \in \operatorname{sing} u$ *there exists a unique* $(n-3)$*-dimensional plane* L *such that every tangent map of* u *at* p *is symmetric with respect to* L*.*

In the special case of $\mathcal{N} = \mathbb{S}^2$, uniform boundedness of minimizers (Theorem

2.5.4) implies that the energy assumption is redundant. The simple corollary below is a key ingredient in the proof of Theorem 6.1.1.

Corollary 4.1.2. *If $u\colon \mathbf{B}_{2r} \to \mathbb{S}^2$ is an energy minimizer, then the uniform measure bound $\mathcal{H}^{n-3}(\operatorname{sing} u \cap \mathbf{B}_r) \leqslant Cr^{n-3}$ holds with some constant $C(n) > 0$.*

In order to prove the stability theorem (Theorem 7.1.1), one needs more refined measure estimates. Note that for the tangent map ψ, the singular set is an $(n-3)$-plane and so $\mathcal{H}^{n-3}(\operatorname{sing} \psi \cap \mathbf{B}_r) = \omega_{n-3} r^{n-3}$. If u is close to ψ, one could expect its singular set to have similar measure (see Lemma 7.4.1). To this end, we will need two more results, which are essential ingredients of [37].

To state them, we first recall the definition of Jones' height excess β-numbers. Choosing a Borel measure μ in $\mathbb{R}^n$, a dimension $0 < k < n$ and an exponent $p \geqslant 1$, we can define for each ball $\mathbf{B}_r(x)$ the quantity

$$\beta_{\mu,k,p}(x,r) := \inf_L \left(r^{-k-p} \int_{\mathbf{B}_r(x)} \operatorname{dist}(y,L)^p \,\mathrm{d}\mu(y) \right)^{1/p},$$

where the infimum is taken over all k-dimensional affine planes $L \subseteq \mathbb{R}^n$. This measures how far the support of μ is from a k-dimensional plane (on the ball $\mathbf{B}_r(x)$).

The role of β in obtaining upper bounds is discussed in detail in Chapter 5. In the applications however, we shall not work directly with this definition, but rather rely on the two theorems below, since they encompass all the geometric information we need.

The next two theorems are general geometric results. The first one plays a central role in proving measure bounds on the singular set. The reasoning sketched in the next section illustrates this application, and the whole Chapter 5 is devoted to various extensions of this theorem.

Theorem 4.1.3 (discrete Reifenberg [37, Thm. 3.4])**.** *There are dimensional constants $C(n), \delta(n) > 0$ such that the following holds. Let $\{\mathbf{B}_{r_j}(x_j)\}$ be a disjoint* *collection of balls in $\mathbf{B}_2$, and let $\mu = \sum_j \omega_k r_j^k \delta_{x_j}$ be its associated discrete measure. If for each ball $\mathbf{B}_r(x) \subseteq \mathbf{B}_2$ we have*

$$\int_{\mathbf{B}_r(x)} \int_0^r \beta^2_{\mu,k,2}(y,s) \frac{\mathrm{d}s}{s} \,\mathrm{d}\mu(y) \leqslant \delta r^k,$$

then $\mu(\mathbf{B}_1) \leqslant C$.

The second is concerned with the special case when μ is the Hausdorff measure on some k-dimensional set S. In this case, the same assumptions yield rectifiability and sharp measure estimates. Interestingly, these measure estimates were not used at all in [37], but are essential for our stability theorem (Theorem 7.1.1).

Theorem 4.1.4 (rectifiable Reifenberg [37, Thm 3.3])**.** *For every $\varepsilon > 0$ there is $\delta(n,\varepsilon) > 0$ such that the following holds. Let $S \subseteq \mathbb{R}^n$ be a $\mathcal{H}^k$-measurable subset and assume that for each ball $\mathbf{B}_r(x) \subseteq \mathbf{B}_2$*

$$\int_{\mathbf{B}_r(x)} \int_0^r \beta^2_{\mu,k,2}(y,s) \frac{\mathrm{d}s}{s} \,\mathrm{d}\mu(y) \leqslant \delta r^k,$$

where μ denotes the measure $\mathcal{H}^k \llcorner S$. Then $\mu(\mathbf{B}_1) \leqslant (1+\varepsilon)\omega_k$ and S is a k-rectifiable set.

As a side remark, let us note that in our application (Chapters 6 and 7) the set S will satisfy the so-called Reifenberg condition and so one could work with the $W^{1,p}$-Reifenberg theorem [37, Thm 3.2] instead.

For the last main tool developed in [37], recall the notion of k-symmetric maps from Definition 2.2.2. Recall also that the tangent map Ψ from (2.4.1), which is $(n-3)$-symmetric but not $(n-2)$-symmetric, hence it belongs to $\mathrm{sym}_{n,k}$ for all $k = 0, 1, \ldots, n-3$ but not for $k = n-2$. In future applications (Theorems 4.3.1, 7.4.1), we will fix

$$\varepsilon := 2 \operatorname{dist}_{L^2(\mathbf{B}_{10})}(\Psi,\ \mathrm{sym}_{n,n-2})$$

and choose $\delta > 0$ accordingly.

Theorem 4.1.5 (L^2-best approximation [37, Thm 7.1]). *For every $\varepsilon > 0$ there are $\delta(n,\varepsilon) > 0$ and $C(n,\varepsilon) > 0$ such that the following holds. If $u\colon \mathbf{B}_{10} \to \mathbb{S}^2$ is energy minimizing,*

$$\operatorname{dist}_{L^2(\mathbf{B}_{10})}(u,\ \operatorname{sym}_{n,0}) \leqslant \delta,$$
$$\operatorname{dist}_{L^2(\mathbf{B}_{10})}(u,\ \operatorname{sym}_{n,k+1}) \geqslant \varepsilon,$$

then for any finite measure μ on $\mathbf{B}_1$ we have

$$\beta^2_{\mu,k,2}(0,1) \leqslant C \int_{\mathbf{B}_1} (\theta_u(y,8) - \theta_u(y,1))\,\mathrm{d}\mu(y).$$

Again, the formulation in [37] involves an energy bound. However, Theorem 2.5.4 shows a uniform bound on $\int_{\mathbf{B}_9} |\nabla u|^2$ and thus we obtain the stronger formulation above.

4.2 A measure bound on lower strata

The measure bound $\mathcal{H}^{n-3}(\operatorname{sing} u \cap \mathbf{B}_r) \leqslant Cr^{n-3}$ in Theorem 4.1.1 concerns the whole singular set, but estimates for lower strata S_k (introduced in Section 2.3) are also available. This section is devoted to discussion of these results and their corollaries.

First, we need to refine the notion of k-symmetry (Definition 2.2.2) to k-almost-symmetry, and the stratification S_k (from Section 2.3) to quantitative stratification. Although it would make some statements more concise, I chose to avoid these notions outside of this section.

A map f is called (k,ε)-symmetric on a ball $\mathbf{B}_r(p)$ in its domain if

$$\int_{\mathbf{B}_1} |f(p+rx) - \overline{f}(x)|^2 \,\mathrm{d}x \leqslant \varepsilon^2$$

for some k-symmetric map $\overline{f}\colon \mathbf{B}_1 \to \mathcal{N}$. In this language, the assumptions of Theorem 4.1.5 could be stated as follows: *u is $(0,\delta)$-symmetric but not $(k+1,\varepsilon)$-symmetric on $\mathbf{B}_{10}$.*

Cheeger and Naber [6] introduced the following *quantitative stratification*:

$$\begin{aligned} S^k_\varepsilon &= \Big\{ y \in \operatorname{sing} u : u \text{ is not } (k+1,\varepsilon)\text{-symmetric on any } \mathbf{B}_s(y) \text{ with } s \in (0,1] \Big\} \\ &= \Big\{ y \in \operatorname{sing} u : \int_{\mathbf{B}_1} |f(y+sx) - \overline{f}(x)|^2 \,\mathrm{d}x > \varepsilon^2 \\ &\qquad\qquad \text{for each } k\text{-symmetric map } \overline{f} \text{ and } s \in (0,1] \Big\}. \end{aligned}$$

In its full generality, the measure bound derived in [37] concerns a larger set $S^k_{\varepsilon,r}$ (defined by restricting the radius above to $r \leqslant s \leqslant 1$). One can check [37, Sec. 9.3] that $S^k_{\varepsilon,r} \searrow S^k_\varepsilon$ and $S^k_\varepsilon \nearrow S_k$, so the classical stratification is recovered.

Theorem 4.2.1 ([37, Thm 1.4])**.** *Let $u\colon \mathbf{B}_{2r} \to \mathcal{N}$ be energy minimizing and $r^{2-n}\int_{\mathbf{B}_{2r}} |\nabla u|^2 \leqslant \Lambda$. Then S^k_ε is a k-rectifiable set and $\mathcal{H}^{n-3}(S^k_\varepsilon \cap \mathbf{B}_r) \leqslant Cr^k$ for some constant $C(\varepsilon, n, \mathcal{N}, \Lambda) > 0$.*

For general $\mathcal{N}$, Theorem 4.2.1 does not imply that the k-th stratum S_k has finite k-dimensional measure. This is only possible for the top-dimensional stratum, and the reason is that the ε-regularity theorem (2.1.2) can be rephrased as the inclusion $\operatorname{sing} u \subseteq S^{n-3}_\varepsilon$ for some small ε; see [6, Thm. 2.4] and [37, Sec. 10.1].

The results of Chapter 3 can be employed in a similar fashion, implying the inclusion $S_{n-4} \subseteq S^{n-4}_\varepsilon$ and hence the measure bounds.

Corollary 4.2.2. *There is $\varepsilon(n) > 0$ such that the following holds. If $u\colon \mathbf{B}_{2r} \to \mathbb{S}^2$ is a minimizing harmonic map, then $S_{n-4} \subseteq S^{n-4}_\varepsilon$. In consequence, the uniform measure bound $\mathcal{H}^{n-4}(S_{n-4} \cap \mathbf{B}_r) \leqslant Cr^{n-4}$ holds for some constant $C(n) > 0$.*

Proof. For the sake of contradiction, assume that such an inclusion is in general not true. By rescaling, this means that there is a sequence of minimizers

$u_k \colon \mathbf{B}_2 \to \mathbb{S}^2$ which are $(n-3, \frac{1}{k})$-symmetric on $\mathbf{B}_2$, but $0 \in S_{n-4}$. By composing with a rotation, we may also assume that $\|u_k - f_k\|_{L^2(\mathbf{B}_2)} \leqslant 2^{n/2}\varepsilon$ for some f_k symmetric with respect to a fixed $(n-3)$-dimensional plane $\mathbf{0} \times \mathbb{R}^{n-3}$.

By taking a subsequence, u_k converges in $W^{1,2}_{\text{loc}}(\mathbf{B}_2)$ to a minimizer u. It follows from the assumptions that u is also symmetric with respect to $\mathbf{0} \times \mathbb{R}^{n-3}$. By Theorem 2.4.2, there are two possibilities:

Case 1. The map u is constant. Then by $W^{1,2}$-convergence, the energy $\int_{\mathbf{B}_1} |\nabla u_k|^2$ tends to zero and by ε-regularity (2.1.2) u_k is smooth around 0. This is a contradiction, since 0 is assumed to be a singular point.

Case 2. The map u has the form $(x, y) \mapsto \pm\frac{x}{|x|}$. In this case, let us fix small $\delta > 0$ according to Proposition 3.4.7. By $W^{1,2}$-convergence and Proposition 3.5.1 (stability of δ-flatness) we infer that for large enough k the map u_k is δ-flat on a ball $\mathbf{B}_1(p_k)$ centered at some point $p_k \in \mathbf{B}_{1/4}$. Now Corollary 3.4.8 implies that any singular point in $\mathbf{B}_{1/4} \subseteq \mathbf{B}_{1/2}(p_k)$ lies in the top-dimensional part $\operatorname{sing}_* u_k$. This is a contradiction, since 0 is a singularity in the lower stratum S_{n-4}.

□

4.3 A measure bound on the singular set for $\mathcal{N} = \mathbb{S}^2$

The following is a toy case of Theorem 4.1.1. We restrict our attention to the target manifold $\mathcal{N} = \mathbb{S}^2$, and to the case when u is close to the tangent map Ψ (2.4.1) in the sense of Definition 3.3.3. Its proof is meant to illustrate the methods of [37] and in particular emphasize the importance of the discrete Reifenberg theorem (Theorem 4.1.3) in the study of singularities. Thanks to results of Chapter 3, the main tools described in the previous section can be applied directly, without additional covering arguments.

A closely analogous argument will be used to prove the local stability theorem (Lemma 7.4.1). For this reason, I chose to make the following proof a bit sketchy at parts where it coincides with the proof of Lemma 7.4.1.

Theorem 4.3.1. *There are constants $C_0(n) > 0$, $\delta(n) > 0$ such that the following is true. If $u\colon \mathbf{B}_{80} \to \mathbb{S}^2$ is energy minimizing and δ-flat in $\mathbf{B}_{80}$ (see Definition 3.3.3), then the (upper) Minkowski content estimate $\mathcal{M}^{n-3}(\operatorname{sing} u \cap \mathbf{B}_1) \leqslant C_0$ holds.*

The (upper) Minkowski content mentioned above can be defined as

$$\mathcal{M}^s(A) = \limsup_{\varepsilon \to 0} P(A, \varepsilon)\varepsilon^s, \text{ where}$$
$$P(A, \varepsilon) = \max\left\{k : \text{there exist } k \text{ disjoint balls } \mathbf{B}_\varepsilon(x_i) \text{ centered at } x_i \in A\right\}$$

is the packing number. Estimates for Minkowski content $\mathcal{M}^s$ are harder to obtain than for Hausdorff measure $\mathcal{H}^s$, since $\mathcal{H}^s(A) \lesssim \mathcal{M}^s(A)$ in general.

As an example, consider the set $A = \{0\} \cup \{1/n : n \in \mathbb{N}\}$, which has Hausdorff dimension 0 and Minkowski dimension $1/2$; indeed, one can check that $P(A, \varepsilon) \geqslant (2\varepsilon)^{-1/2}$. See [28, Ch. 5] for a detailed discussion and comparison.

The use of $\mathcal{M}^s$ instead of $\mathcal{H}^s$ here is mostly for convenience, but it also helps illustrate the power of these methods. Indeed, with just a little more effort one can get here packing content estimates, which are even stronger than Minkowski content estimates (again, see [28, Ch. 5]) and are crucial in some applications [35].

Proof. We follow the general outline of Naber and Valtorta's work [37, Sec. 1.4].

In order to derive a Minkowski content estimate, fix a finite disjoint family of balls $\mathbf{B}_{r_0}(x_j)$ with centers $x_j \in \operatorname{sing} u \cap \mathbf{B}_1$ and radii $r_0 \leqslant 1/2$. Denote the associated discrete measure by $\mu := \sum_j r_0^{n-3}\delta_{x_j}$. We choose C_0 to be the constant from the discrete Reifenberg theorem (Theorem 4.1.3) and prove inductively that

$$\mu(\mathbf{B}_r(x)) \leqslant C_0 r^{n-3} \quad \text{for all } x \in \mathbf{B}_2 \text{ with } r_0 \leqslant r \leqslant r_1. \tag{4.3.1}$$

Once this is shown for $r_1 = 1$ (with arbitrarily small r_0 and an arbitrary family of balls), the proof of the Minkowski content estimate is complete.

The induction takes place with respect to r_1. For $r_1 = r_0$, (4.3.1) obviously holds, even with constant 1 instead of C_0. Without loss of generality, we focus on the last inductive step – i.e., we assume that the estimate (4.3.1) is already known for $r_1 = 1/2$.

However, since any ball can be covered by at most $C(n)$ balls of 8 times smaller radius, we see (4.3.1) holds for $8r_1$ with a worse constant $C(n) \cdot C_0$. This *weak upper bound* does not establish the inductive claim, but is enough to justify the estimates that follow.

With $\delta_1 > 0$ to be fixed later, by Proposition 3.4.7 we can choose δ small enough so that all singular points in $\mathbf{B}_{40}$ lie in the top-dimensional part $\operatorname{sing}_* u$, moreover u is also δ_1-flat in each ball $\mathbf{B}_r(z)$ with $z \in \operatorname{sing} u \cap \mathbf{B}_{40}$ and $0 < r \leqslant 20$.

It follows from δ_1-flatness that we can apply the L^2-best approximation Theorem 4.1.5 on each of these balls and obtain

$$\beta^2(z,s) \leqslant C(n) s^{-(n-3)} \int_{\mathbf{B}_s(z)} \left(\theta_u(y,8s) - \theta_u(y,s)\right) \,\mathrm{d}\mu(y)$$

for each ball $\mathbf{B}_s(z) \subseteq \mathbf{B}_2$ with $z \in \operatorname{sing} u$, where β denotes $\beta_{\mu,n-3,2}$.

Integrating this estimate over $\mathbf{B}_r(x)$, exchanging the order of summation and exploting the weak upper bound, we obtain

$$\int_{\mathbf{B}_r(x)} \beta^2(z,s) \,\mathrm{d}\mu(z) \lesssim \int_{\mathbf{B}_{2r}(x)} \left(\theta_u(y,8s) - \theta_u(y,s)\right) \mathrm{d}\mu(y).$$

When the above is integrated with respect to s, we obtain a telescopic sum. Indeed, the substitution $s \mapsto 8s$ together with monotone convergence $\theta_u(y,s) \searrow \theta_u(y,0)$ give us

$$\begin{aligned}\int_0^r (\theta_u(y,8s) - \theta_u(y,s))\frac{\mathrm{d}s}{s} &= \int_r^{8r} (\theta_u(y,s) - \theta_u(y,0))\frac{\mathrm{d}s}{s} \\ &\leqslant \ln(8)\delta_1,\end{aligned}$$

as $\theta_u(y,8r) - \theta_u(y,0) \leqslant \delta_1$ for all considered y and r.

Now we are ready to combine the above estimates:

$$\begin{aligned}\int_{\mathbf{B}_r(x)} \int_0^r \beta^2(z,s)\frac{\mathrm{d}s}{s}\,\mathrm{d}\mu(z) &\lesssim \int_0^r \int_{\mathbf{B}_{2r}(x)} (\theta_u(y,8s) - \theta_u(y,s))\,\mathrm{d}\mu(y)\frac{\mathrm{d}s}{s} \\ &\leqslant \int_{\mathbf{B}_{2r}(x)} \ln(8)\delta_1\,\mathrm{d}\mu(y) \\ &\lesssim \delta_1 r^{n-3},\end{aligned}$$

where we used the weak upper bound again in the last line. Assuming that $\delta_1 \leqslant \delta_2(n)/C(n)$, where $\delta_2(n)$ is chosen as in the discrete Reifenberg theorem (Theorem 4.1.3), we see that the assumptions of this theorem are satisfied and we infer the upper estimate $\mu(\mathbf{B}_1) \leqslant C_0$.

Since the estimate does not depend on the choice of balls, we infer the packing number bound $P(\operatorname{sing} u \cap \mathbf{B}_1, r_0)r_0^{n-3} \leqslant C_0$. The estimate is also independent of r_0, thus the Minkowski content bound $\mathcal{M}^{n-3}(\operatorname{sing} u \cap \mathbf{B}_1) \leqslant C_0$ follows. □

Chapter 5

Discrete Reifenberg-type theorem

5.1 Introduction

Reifenberg-type theorems

Classical Reifenberg's theorem states that if a closed set $S \subseteq \mathbb{R}^n$ is well approximated by affine k-planes (in the sense of Hausdorff distance) at all balls centered in S, then S is bi-Hölder equivalent with a plane. It was proved by Reifenberg in 1960 [40] in his work on the Plateau problem (see also [46]).

Here we consider approximation in the sense of Hausdorff semi-distance, i.e. sets with holes are allowed.

The quality of this approximation is measured by Jones' height excess numbers β. Fix natural numbers $1 \leqslant k < n$ and let μ be a Radon measure on $\mathbb{R}^n$; the basic example is $\mu = \mathcal{H}^k \llcorner S$, where S is a k-dimensional set and $\mathcal{H}^k$ is the k-dimensional Hausdorff measure. We define

$$\beta_{\mu,q}(x,r) = \inf_{V^k} \left(r^{-(k+q)} \int_{\mathbf{B}_r(x)} d^q(y, V^k)\, \mathrm{d}\mu(y) \right)^{1/q}. \tag{5.1.1}$$

This is the L^q norm of $d(y, V^k)/r$ on $\mathbf{B}_r(x)$ with respect to the measure $r^{-k}\mu$, where V^k is the best affine k-plane.

In order to obtain an upper bound on the measure μ, a uniform bound on $\beta_q(x, r)$ is not sufficient (see Example 5.2.1). The upper bound can follow from a bound on Jones' square function

$$J_{\mu,q}(x, r) = \int_0^r \beta^2_{\mu,q}(x, s)\frac{\mathrm{d}s}{s}. \tag{5.1.2}$$

In dimension 1, Jones' Traveling Salesman Theorem [23] shows the connection between a version of this function and 1-dimensional Hausdorff measure bounds. The geometric importance of $J_{\mu,q}$ is also illustrated by Example 5.2.2. The subscript μ shall be omitted when it is clear from the context.

There are many results concerning the consequences of a bound on Jones' square function. David and Toro [7] showed that if S satisfies the assumptions of Reifenberg's theorem and $J_{\mathcal{H}^k \llcorner S,1}(x, 1)$ is uniformly bounded, then the parametrization of S obtained in Reifenberg's theorem is Lipschitz continuous. Azzam and Tolsa [50], [3] characterized rectifiable measures by the condition $J_{\mu,2}(x, 1) < \infty$ μ-a.e., assuming that the upper-density is positive and finite μ-a.e.

Our aim here is to obtain upper bounds on the measure μ. In this direction, Naber and Valtorta [37] proved that there is $\delta(n) > 0$ such that if

$$r^{-k}\int_{\mathbf{B}_r(x)} J_{\mu,2}(y, r)\,\mathrm{d}\mu(y) \leqslant \delta^2$$

holds for any ball $\mathbf{B}_r(x) \subseteq \mathbf{B}_2$, then $\mu(\mathbf{B}_1) \leqslant C(n)$. This was proved in two cases: when μ is a discrete measure and when $\mu = \mathcal{H}^k \llcorner S$. In the latter case, the authors also obtained rectifiability of S; see Chapter 4 for a more detailed discussion of their results.

However, it was the discrete version (Theorem 4.1.3 [37, Th. 3.4]) that was used to obtain an upper bound on the singular set $\mathcal{H}^k(\operatorname{sing} u)$ of a harmonic map

u in terms of its Dirichlet energy. Application to singular sets of solutions of nonlinear PDEs is one of the main motivations of this chapter.

Recently, Azzam and Schul [2] have generalized Jones' work to sets of higher dimensions. One of their results bounds the k-dimensional Hausdorff measure $\mu = \mathcal{H}^k \llcorner S$ of a closed set $S \subseteq \mathbf{B}_1 \subseteq \mathbb{R}^n$ in terms of $J_{\mu,q}(0,2)$. The set S is assumed to be lower content regular; this property implies that for some $c, r_0 > 0$

$$\mu(\mathbf{B}_r(x)) \geqslant cr^k \qquad \text{for all } x \in S,\ 0 < r < r_0.$$

The precise definitions and statements are slightly more involved, as they employ the outer measures $\mathcal{H}^k_\delta$ instead of $\mathcal{H}^k$; we refer the reader to [2] for details. Thanks to this modification the authors avoid assuming a priori that μ is finite.

Similar results were also obtained by Edelen, Naber and Valtorta in their paper [9], which improves their previous work [37]. They prove a variant of Theorem 5.1.1 under somewhat different assumptions and also show rectifiability of the measure in case the lower-density is suitably controlled.

Basic notation

The measure of k-dimensional unit ball is ω_k and $\lambda \mathbf{B}_r(x) = \mathbf{B}_{\lambda r}(x)$ is used to denote the scaled ball.

If $\mathcal{S} = \{\mathbf{B}_j\}$ is a collection of balls, then $\mathsf{Cent}\,\mathsf{S}$ stands for the set of centers of these balls and $\lambda\mathcal{S} = \{\lambda \mathbf{B}_j\}$ is the collection of scaled balls with the same centers. We denote the union by

$$\bigcup \mathcal{S} = \bigcup_j \mathbf{B}_j.$$

As in [7], we use the normalized local Hausdorff distance

$$d_{x,r}(E,F) = \frac{1}{r}\,\mathrm{dist}_H(E \cap \mathbf{B}_r(x), F \cap \mathbf{B}_r(x)),$$

where dist_H is the standard Hausdorff distance.

Statement of the main results

The following is a slightly improved version of Naber and Valtorta's Theorem 4.1.3 [37, Th. 3.4]. The main difference is that the upper bound J is not assumed to be small. Moreover, the theorem holds for any $2 \leqslant q < \infty$.

Theorem 5.1.1 (discrete Reifenberg). *Let $\mathsf{S} = \{\mathbf{B}_{r_j}(x_j)\}$ be a collection of disjoint balls in $\mathbf{B}_2$, $\mu = \sum_j \omega_k r_j^k \delta_{x_j}$ be its associated measure and let $\beta_q(x,r)$, $J_q(x,r)$ be defined as in (5.1.1), (5.1.2), where $2 \leqslant q < \infty$. Assume that for each ball $\mathbf{B}_r(x) \subseteq \mathbf{B}_2$ we have*

$$r^{-k} \int_{\mathbf{B}_r(x)} J_q(y,r)\,\mathrm{d}\mu(y) \leqslant J. \tag{5.1.3}$$

Then the following estimate holds:

$$\mu(\mathbf{B}_1) = \sum_{x_j \in B_1} \omega_k r_j^k \leqslant C(n,q) \cdot \max\left(1, J^{\frac{q}{q+2}}\right). \tag{5.1.4}$$

The choice of the normalizing constant ω_k is motivated by the comparison of μ with k-dimensional Hausdorff measure, but has no importance for the theorem.

The proof of Theorem 5.1.1 follows the lines of [37]. This generalization is made possible by relaxing the inductive claim in the construction and carefully keeping track of the constant.

This observation also leads to other possible extensions, discussed in Section 5.5. First, Theorem 5.5.1 and Remark 5.5.3 generalize the above to measures μ with controlled upper-density, in particular to the case $\mu = \mathcal{H}^k \llcorner S$. Second, Theorem 5.5.4 shows that, with minor modifications, the proof applies also with (5.1.3) replaced by a weaker assumption

$$\fint_{\mathbf{B}_r(x)} J_q(y,r)\,\mathrm{d}\mu(y) \leqslant J.$$

Outline of the proof of Theorem 5.1.1

The main tool is Reifenberg's construction of surfaces $T_0, T_1, T_2, \ldots$ approximating the support of μ. The bound on Jones' square function J_q (5.1.3) enables us to prove that this approximation is efficient. There are three key properties that we need:

- The total area $|T_i|$ of the approximating surface is estimated from above via β_q numbers (see (5.4.1)).
- The measures μ and $\mathcal{H}^k \llcorner T_i$ are comparable on (at least some) balls $\mathbf{B}_{r_i}(x)$ centered near T_i (see (5.4.2)).
- The region outside some neighborhood of T_i has small measure μ (see (5.4.3)).

It is intuitive that these three imply some bound on the measure μ. Indeed, once they are derived, we shall see at the end of Section 5.4 that the final estimate is an easy consequence.

5.2 Examples

Reifenberg's theorem states that any ε-Reifenberg flat set is α-Hölder equivalent with a k-plane. This leads to finite Hausdorff measure in dimension k/α.

As $\varepsilon \to 0$, α tends to 1 and the dimension bound k/α gets arbitrarily close to k. The example below shows that under these assumptions this bound cannot be improved.

Example 5.2.1 (flat snowflake)**.** Fix a small angle θ and consider a modification of the Koch curve (a snowflake): each segment is divided into three segments of equal length and the middle segment is replaced by two segments, each of them at angle θ to the original segment (the original construction is obtained for $\theta = \pi/6$). We denote the curve obtained by starting with a unit segment and iterating the above procedure by K.

If θ is small, K is ε-Reifenberg flat and α-Hölder equivalent with a segment. For $\theta \approx 0$ we have $\varepsilon \approx \theta \approx 0$ and $\alpha \approx 1$. Still, the Hausdorff dimension of K is greater than 1. This example shows that Reifenberg's theorem is optimal – ε-Reifenberg flatness condition does not imply a bound on the k-dimensional Hausdorff measure.

Since ε-Reifenberg flatness condition is not enough to imply a bound on the k-dimensional Hausdorff measure, we investigate an improved example taken from [7]. It suggests that the proper hypothesis is a bound on Jones' square function (5.1.2).

Example 5.2.2 (very flat snowflake)**.** Modify the previous example by taking another angle θ_i at each stage i of the construction. After N stages we have a curve of length

$$\prod_{i=1}^{N} \frac{2 + \frac{1}{\cos\theta_i}}{3} = \prod_{i=1}^{N} \left(1 + \frac{1}{6}\theta_i^2 + o(\theta_i^2)\right).$$

The product is convergent if and only if the sum $\sum_i \theta_i^2$ converges. The measure $\lambda^1(K)$ of the limit curve can be bounded in terms of this sum.

Since the angles θ_i are comparable with β_q numbers taken on the corresponding balls, this shows that indeed the exponent 2 in the definition of Jones' square function J_q (5.1.2) is natural. It also suggests that this function can be used to bound the k-dimensional measure; indeed, a result of this type was proved in [7]. In this paper we relax this assumption by concerning a bound on the average $\fint_{\mathbf{B}_r(x)} J_q(y,r)\,\mathrm{d}\mu(y)$ or on $r^{-k}\int_{\mathbf{B}_r(x)} J_q(y,r)\,\mathrm{d}\mu(y)$ for each ball $\mathbf{B}_r(x)$.

5.3 Technical constructions

The tools discussed in this section are well known and most of them are cited from [37]. Some technical corrections were made in Lemmata 5.3.2, 5.3.3 (counterparts of [37, 4.7, 4.8]). These corrections come from the fact that the ball $\mathbf{B}_1$ cannot be covered by finitely many balls $\mathbf{B}_\rho(x_i)$ contained in $\mathbf{B}_1$. Thus one is forced to work with a weaker condition $x_i \in \mathbf{B}_1$, in consequence the balls are contained in a slightly larger ball $\mathbf{B}_{1+\rho}$.

Properties of β numbers

Recall the definitions

$$\beta_q^q(x,r) = \inf_{V^k} r^{-(k+q)} \int_{\mathbf{B}_r(x)} d^q(y,V^k)\,\mathrm{d}\mu(y), \tag{5.1.1}$$

$$J_q(x,r) = \int_0^r \beta_q^2(x,s)\frac{\mathrm{d}s}{s}. \tag{5.1.2}$$

Due to the factor $r^{-(k+q)}$ these quantities are scale invariant. Indeed, if ν is a scaled version of μ, i.e. $\nu(\cdot) = \lambda^{-k}\mu(\lambda\cdot)$, then $\beta_{\nu,q}(0,r) = \beta_{\mu,q}(0,\lambda r)$ and $J_{\nu,q}(0,r) = J_{\mu,q}(0,\lambda r)$. This scaling occurs e.g. if ν,μ are discrete measures corresponding to collections of balls S, $\lambda\mathsf{S}$, or k-dimensional Hausdorff measure restricted to sets S, λS.

First we note the basic continuity property of β_q. For any $y \in \mathbf{B}_r(x)$ we have $\mathbf{B}_r(x) \subseteq \mathbf{B}_{2r}(y)$ and it follows from the definition that

$$\beta_q^q(x,r) \leqslant 2^{k+2}\beta_q^q(y,2r) \quad \text{for } y \in \mathbf{B}_r(x). \tag{5.3.1}$$

This simple observation leads to an equivalent form of Jones' square function.

Remark 5.3.1. Fix some $\rho \in (0,1)$ and let $r_\alpha = \rho^\alpha$ for $\alpha = 0,1,2,\ldots$. Then any bound on Jones' square function is (up to a constant depending on ρ) equivalent to a bound on

$$\sum_{r_\alpha \leqslant 2r} \beta_q^2(x, r_\alpha).$$

Proof. Similarly to (5.3.1), we have

$$\beta_q^q(x,r_1) \leqslant (r_2/r_1)^{k+q}\beta_q^q(x,r_2) \quad \text{for } r_1 \leqslant r_2.$$

Take arbitrary $s \in (0,r)$ and choose α such that $\rho^{\alpha+1} \leqslant s < \rho^\alpha$. Then

$$c(\rho)\beta_q^2(x,\rho^{\alpha+1}) \leqslant \beta_q^2(x,s) \leqslant C(\rho)\beta_q^2(x,\rho^\alpha)$$
$$\text{and} \quad c(\rho) \leqslant \int_{\rho^{\alpha+1}}^{\rho^\alpha} \frac{\mathrm{d}s}{s} \leqslant C(\rho),$$

which shows the equivalence. ☐

Denote the auxiliary numbers

$$\delta_q^2(x,r) = r^{-k}\int_{\mathbf{B}_r(x)} \beta_q^2(y,r)\,\mathrm{d}\mu(y). \tag{5.3.2}$$

Note that assumption (5.1.3) together with Remark 5.3.1 yields a very rough estimate $\delta_q^2(x,r) \leqslant CJ$. Moreover,

$$\delta_q^2(x_1,r_1) \leqslant C(r_1/r_2)\delta_q^2(x_2,r_2) \qquad \text{if } \mathbf{B}_{r_1}(x_1) \subseteq \mathbf{B}_{r_2}(x_2).$$

Yet another corollary of (5.3.1) can be obtained by taking the average over all $y \in \mathbf{B}_r(x)$:

$$\beta_q^2(x,r) \leqslant C(k,q)\fint_{\mathbf{B}_r(x)} \beta_q^2(y,2r)\,\mathrm{d}\mu(y).$$

If one assumes a lower bound $\mu(\mathbf{B}_r(x)) \geqslant \tau(n)Mr^k$ (as it will be satisfied in the applications), this can be further estimated by

$$\begin{aligned}\fint_{\mathbf{B}_r(x)} \beta_q^2(y,2r)\,\mathrm{d}\mu(y) &\leqslant \frac{1}{\tau M r^k}\int_{\mathbf{B}_r(x)} \beta_q^2(y,2r)\,\mathrm{d}\mu(y)\\ &= C(n,\tau)M^{-1}\delta_q^2(x,2r).\end{aligned} \tag{5.3.3}$$

Finally, an estimate for β_q^q can be obtained by

$$\begin{aligned}\beta_q^q(x,r) = \left(\beta_q^2(x,r)\right)^{q/2} \qquad &\lesssim \left(\fint_{\mathbf{B}_r(x)} \beta_q^2(y,2r)\,\mathrm{d}\mu(y)\right)^{q/2}\\ \lesssim \left(M^{-1}\delta_q^2(x,2r)\right)^{q/2} &\lesssim M^{-\frac{q}{2}}J^{\frac{q-2}{2}}\delta_q^2(x,2r),\end{aligned} \tag{5.3.4}$$

where the symbol $\lesssim$ denotes an inequality up to a multiplicative constant, possibly dependent on n, q, τ, ρ.

Comparison of L^q-best planes via β_q

Due to compactness of the Grassmannian $G(k,n)$ and continuity of $d(y,V)$, there exists a k-plane minimizing $\int_{\mathbf{B}_r(x)} d^q(y,V)\,\mathrm{d}\mu$ (there may be more than one). We choose any of the L^q-best planes and denote it by $V(x,r)$.

We will estimate the distances between the L^q-best planes on different balls using β_q numbers. More precisely, we want to prove that the distance between $V(x_1,r_1)$ and $V(x_2,r_2)$ is estimated via β_q numbers if r_1, r_2 are comparable and controlled by $|x_1 - x_2|$.

In the case of the standard β_∞ numbers this is an elementary geometric problem. As shown by simple examples in [37], in case of β_q numbers one is forced to assume some kind of Ahlfors-David regularity of the measure μ. Here we use the condition $\tau M r^k \leqslant \mu(\mathbf{B}_r) \leqslant Mr$ because we want to study the dependence on M with $\tau(n)$ fixed.

Lemma 5.3.2. *There exists $\rho_0(n,\tau)$ such that for $\rho \leqslant \rho_0$ the following holds. If*

$$\mu(\mathbf{B}_\rho(x)) \leqslant \rho^k$$

holds for all $x \in \mathbf{B}_1$ and $\mu(\mathbf{B}_1) \geqslant \tau$, then for every affine plane $V \leqslant \mathbb{R}^n$ of dimension $\leqslant k-1$, there exists a point $x \in \mathbf{B}_1$ such that

$$d(x,V) > 10\rho, \quad \mu(\mathbf{B}_\rho(x)) \geqslant C(n,\rho) > 0.$$

Now we can prove the aforementioned tilt-excess result. We denote $\kappa = \frac{1}{1-\rho}$ so that $\kappa\mathbf{B}_\rho(x) \subseteq \kappa\mathbf{B}_1(0)$ for any $x \in \mathbf{B}_1(0)$.

Lemma 5.3.3. *Fix $\tau \in (0,1)$ and $\rho(n,\tau)$ as in Lemma 5.3.2; denote $\kappa = \frac{1}{1-\rho}$. Let μ be a positive Radon measure. Assume that $\mu(\mathbf{B}_1) \geqslant \tau M$ and that $\mu(\mathbf{B}_{\rho^2}(y)) \leqslant M\rho^{2k}$ for every $y \in \mathbf{B}_\kappa$. Additionally, let $x \in \mathbf{B}_1$ be such that $\mu(\mathbf{B}_\rho(x)) \geqslant \tau M \rho^k$.*

Then if $d(x, V(0,\kappa)) \leqslant \rho/2$ or $d(x, V(x,\kappa\rho)) \leqslant \rho/2$, then the distance between the L^q-best planes is estimated by

$$d^q_{x,\rho}(V(0,\kappa), V(x,\kappa\rho)) \leqslant C(n,q,\rho,\tau)M^{-1}\left(\beta^q_q(0,\kappa) + \beta^q_q(x,\kappa\rho)\right).$$

We present a sketch of proof, referring to [37, Lemma 4.8] for a more detailed explanation.

Sketch of proof. We assume that $d(x, V(0,\kappa)) \leqslant \rho/2$; in the other case one has to exchange the roles od $V(0,\kappa)$ and $V(x,\kappa\rho)$. Consider first the case $M = 1$.

We choose $k+1$ points $y_0, \ldots, y_k \in \mathbf{B}_\rho(x)$ with $\mu(\mathbf{B}_{\rho^2}(y_i)) \geqslant c(n,\tau)$. Denote by p_i the center of mass of $\mathbf{B}_{\rho^2}(y_i)$ and let p'_i be its projection onto $V(0,\kappa)$. We require p'_i to effectively span $V(0,\kappa) \cap \mathbf{B}_\rho(x)$, i.e.

$$d(p'_{i+1}, \operatorname{span}(p'_0, \ldots, p'_i)) > 8\rho^2.$$

This is done by inductive application of Lemma 5.3.2 and the elementary inequality $|y_i - p_i| \leqslant \rho^2$. Jensen's inequality yields

$$d^q(p_i, V(0,\kappa)) \leqslant C\beta_q^q(0,\kappa),$$
$$d^q(p_i, V(x,\kappa\rho)) \leqslant C\beta_q^q(x,\kappa\rho),$$

hence all points p_i' are close to $V(x,\kappa\rho)$. Since these points effectively span $V(0,\kappa)\cap\mathbf{B}_\rho(x)$, it can be shown that this k-plane is contained in a small neighborhood of $V(x,\kappa\rho)\cap\mathbf{B}_\rho(x)$. Since these two planes have the same dimension, the assumption $d(x, V(0,\kappa)) \leqslant \rho/2$ ensures that the inclusion works both ways (see [37, Lemma 4.2]). This completes the case $M = 1$.

Now consider a measure μ satisfying the assumptions for some $M > 0$. Then the above reasoning can be applied for the measure $\nu = M^{-1}\mu$, satisfying similar assumptions with 1 instead of M. Since μ, ν have the same L^q-best planes and $\beta^q_{\mu,q}(y,r) = M\beta^q_{\nu,q}(y,r)$ on any ball $\mathbf{B}_r(y)$, the claim follows. □

In the proof of Theorem 5.1.1, the values of τ, ρ shall be fixed depending only on the dimension n.

Bi-Lipschitz diffeomorphism construction

Here we introduce the construction later used to obtain the approximating surfaces in the proof of Theorem 5.1.1.

For some $r > 0$, let $\mathsf{J} = \{\mathbf{B}_r(x_i)\}$ be a finite collection of balls such that $\frac{1}{2}\mathsf{J}$ is disjoint. For each ball choose a k-dimensional affine plane V_i and denote the orthogonal projection onto V_i by π_i. As in [7], one can choose a locally finite smooth partition of unity $\lambda_i\colon \mathbb{R}^n \to [0,1]$ subordinate to the cover $\bigcup 4\mathsf{J}$ satisfying

1. $\sum_i \lambda_i \equiv 1$ in $\bigcup 3\mathsf{J}$,

2. $\lambda_i \equiv 0$ outside $4\mathbf{B}_r(x_i)$ for all i,

3. $||\nabla\lambda_i||_\infty \leqslant C(n)/r$,

4. the partition is completed with the smooth function $\psi = 1 - \sum_i \lambda_i$ and $||\nabla\psi||_\infty \leqslant C(n)/r$.

Definition 5.3.4. Given J, λ_i, p_i, V_i as above, define the smooth map

$$\sigma\colon \mathbb{R}^n \to \mathbb{R}^n, \qquad \sigma(x) = \psi(x)x + \sum_i \lambda_i(x)\pi_i(x).$$

The map σ interpolates between the identity and the projections onto the affine planes V_i. Note that $\sigma = \mathrm{id}$ outside of the union $\bigcup 4\mathsf{J}$, as on this region we have $\psi \equiv 1$. On the other hand, if V_i are all close to some V, then σ is close to the orthogonal projection onto V in the region $\bigcup 3\mathsf{J}$. This will be made precise in Lemma 5.3.6.

Lemma 5.3.6 is a modified version of [37, Lemma 4.12]. It is essentially a counterpart of the squash lemma used to prove classical Reifenberg's theorem. The crucial additional part of the following is the bi-Lipschitz estimate for σ that is quadratic in δ_0, δ_1; this should be compared to the measure estimate in Example 5.2.2 and the definition (5.1.2) of Jones's square function. In order to obtain this quadratic estimate, let us first consider the following geometric fact.

Lemma 5.3.5. *Let V_1, V_2 be two linear k-planes and π_1, π_2 be the corresponding orthogonal projections. If $d_{0,1}(V_1, V_2) \leqslant \delta$, then $||\pi_1\pi_2 - \mathrm{id}\,||_{V_1\to V_1} \leqslant C(n)\delta^2$.*

Proof. It follows that $\|\pi_1\pi_2^\perp\| \leqslant C\delta$ and $\|\pi_2^\perp\pi_1\| \leqslant C\delta$; in fact one can define the Grassmannian distance this way. Since $\pi_1 = \mathrm{id}$ on V_1, it is enough to estimate the norm of $\pi_1\pi_2\pi_1 - \pi_1$:

$$\begin{aligned}\|\pi_1\pi_2\pi_1 - \pi_1\| &= \|\pi_1(\pi_2 - \mathrm{id})\pi_1\| \\ &= \|\pi_1\pi_2^\perp\pi_1\| \\ &= \|(\pi_1\pi_2^\perp)(\pi_2^\perp\pi_1)\| \\ &\leqslant \|\pi_1\pi_2^\perp\| \cdot \|\pi_2^\perp\pi_1\| \leqslant (C\delta)^2.\end{aligned}$$

□

The following lemma deals with graphs of functions that are C^1 small at scale r. To simplify the notation, we introduce the normalized C^1 norm

$$||g||_{C^1_r} := r^{-1}||g||_\infty + ||\nabla g||_\infty.$$

Lemma 5.3.6 (squash lemma)**.** *Fix some ball $\mathbf{B}_r(y) \subseteq \mathbb{R}^n$ and a k-dimensional affine plane V such that $d(y, V) \leqslant r/2$. Suppose that for all balls $\mathbf{B}_r(x_i) \in \mathsf{J}$ centered in $10\mathbf{B}_r(y)$ we have*

$$d_{x_i,r}(V_i, V) \leqslant \delta_1.$$

Suppose also that $G_0 \subseteq \mathbb{R}^n$ is the graph $G_0 = \{x + g_0(x) : x \in V\} \cap 5\mathbf{B}_r(y)$ of a small function $g_0\colon V \to V^\perp$, i.e. $||g_0||_{C^1_r} \leqslant \delta_0$. If $\delta_0 \leqslant 1$ and $\delta_1 \leqslant \delta(n)$, then

1. *The set $G_1 = \sigma(G_0)$ restricted to $4\mathbf{B}_r(y)$ is a graph of a C^1 function $g_1\colon V \to V^\perp$ with*
$$||g_1||_{C^1_r} \leqslant C(n)(\delta_0 + \delta_1).$$
There is ratio $\theta \geqslant 3 - C(n)(\delta_0 + \delta_1)$ such that on each of the balls $\theta\mathbf{B}_r(x_i)$ the previous bound is actually independent of δ_0, i.e. $||g_1||_{C^1_r} \leqslant C(n)\delta_1$.
2. *The map $\sigma\colon G_0 \to G_1$ is a C^1 diffeomorphism from G_0 to G_1 and*
$$|\sigma(z) - z| \leqslant C(n)(\delta_0 + \delta_1)r \quad \textit{for } z \in G_0.$$
Moreover, its bi-Lipschitz constant does not exceed $1 + C(n)(\delta_0^2 + \delta_1^2)$.

Proof. Note that V_i are also close to V on the larger ball: $d_{y,10r}(V_i, V) \leqslant C\delta_1$ for all i. For $x \in V$ denote $z = x + g(x)$ and

$$h(x) = \sum_i \lambda_i(z)\left(\pi_i(x + g_0(x)) - x\right),$$

so that

$$\begin{aligned}\sigma(x + g_0(x)) &= \psi(z)(x + g_0(x)) + \sum_i \lambda_i(z)\pi_i(x + g_0(x)) \\ &= x + \psi(z)g_0(x) + h(x).\end{aligned}$$

For simplicity, assume that $0 \in V$. Then we can consider the decomposition of σ obtained by projecting onto the linear plane V and its orthogonal complement $V^\perp$:

$$\begin{aligned} \sigma(x + g_0(x)) &= \sigma^T(x) + \sigma^\perp(x), \\ \sigma^T(x) &= x + h^T(x), \\ \sigma^\perp(x) &= \psi(z) g_0(x) + h^\perp(x). \end{aligned}$$

Now we show that $\sigma^T - \mathrm{id}$ and $\sigma^\perp$ are C^1_r-small. Indeed, it is easily checked that $||\pi_i(x + g_0(x)) - x||_{C^1_r} \leqslant C\delta_1$ for all $x \in V \cap 5\mathbf{B}_r(x_i)$ and hence for all x such that $\lambda_i(z) > 0$. Note that this is independent of δ_0, if only $\delta_0 \leqslant 1$. Therefore $||h^T||_{C^1_r}, ||h^\perp||_{C^1_r} \leqslant C\delta_1$.

The remaining term is estimated by $||\psi(z) g_0(x)||_{C^1_r} \leqslant C\delta_0$, but it vanishes for all x such that $z \in \bigcup 3\mathrm{J}$.

Thus we obtained

$$||\sigma^T - \mathrm{id}\,||_{C^1_r} \leqslant C\delta_1, \quad ||\sigma^\perp||_{C^1_r} \leqslant C(\delta_0 + \delta_1)$$

We choose $\delta_1 \leqslant \delta(n)$ small in order to apply the inverse function theorem for $\sigma^T \colon V \to V$. Thus we obtain the inverse function ϕ satisfying $||\phi - \mathrm{id}\,||_{C^1_r} \leqslant C\delta_1$ and $\phi = \mathrm{id}$ outside $\bigcup 4\mathrm{J}$. The inverse enables us to write

$$\sigma(x + g_0(x)) = \sigma^T(x) + g_1(\sigma^T(x)), \quad \text{where } g_1(x) = \sigma^\perp(\phi(x)).$$

This proves point (1) and the first part of point (2).

What is left is the estimate for the bi-Lipschitz constant of σ. To this end, we decompose σ in the following way:

$$G_0 \ni x + g_0(x) \xmapsto{(\mathrm{id} + g_0)^{-1}} x \xmapsto{\sigma^T} \sigma^T(x) \xmapsto{\mathrm{id} + g_1} \sigma^T(x) + g_1(\sigma^T(x)) \in G_1$$

The Lipschitz constant of the map $V \xrightarrow{\mathrm{id} + g_0} G_0$ is bounded by $\sqrt{1 + \delta_0^2}$ and its inverse is a contraction. Similarly, the bi-Lipschitz constant of $V \xrightarrow{\mathrm{id} + g_1} G_1$ is bounded by $\sqrt{1 + C(\delta_0^2 + \delta_1^2)}$.

To obtain a quadratic bound for $V \xrightarrow{\sigma^T} V$, we need to improve the estimate $||\nabla h^T||_\infty \leqslant C\delta_1$ derived before. To this end, compute

$$\nabla h^T(x) = \sum_i \nabla\lambda_i(z)\nabla z\left(\pi_V\pi_i(x+g_0(x)) - x\right) \\ + \sum_i \lambda_i(z)\left(\pi_V\nabla\pi_i(\mathrm{id}+\nabla g_0(x)) - \mathrm{id}\right)$$

In the second sum, the expression in parentheses is $(\pi_V\nabla\pi_i\nabla g_0)+(\pi_V\nabla\pi_i-\mathrm{id})$. The first term is bounded by $C\delta_0\delta_1$, while for the second Lemma 5.3.5 implies the bound $C\delta_1^2$. The estimates for the first sum are obtained analogously. Hence $||\nabla h^T||_\infty \leqslant C(\delta_0^2+\delta_1^2)$ and the bi-Lipschitz constant of σ^T is bounded by $1 + C(\delta_0^2+\delta_1^2)$. In consequence, we obtain the bound for σ as a composition. □

We end with a related lemma, which shows that if G is a graph over V_1 and V_1, V_2 are close, then it is also a graph over V_2.

Lemma 5.3.7. *Let V_1, V_2 be two affine k-planes and $d_{y,r}(V_1,V_2) \leqslant \delta$, and let $G \subseteq \mathbf{B}_r$ be a graph over V_1 of a function g_1 with $||g_1||_{C^1_r} \leqslant \delta$. If $\delta \leqslant \delta(n)$, then $G\cap\theta\mathbf{B}_r$ is also a graph over V_2 of a function g_2, $||g_2||_{C^1_r} \leqslant C\delta$. The ratio θ satisfies $1 - C\delta < \theta < 1$.*

Sketch of proof. We follow the proof of Lemma 5.3.6. The composition

$$V_1 \xmapsto{\mathrm{id}+g_1} G \xmapsto{\pi_{V_2}} V_2$$

is shown to be a diffeomorphism. If we denote its inverse by ϕ, then $G \cap \theta\mathbf{B}_r$ is a graph over V_2 of $g_2(x) = \phi(x) + g_1(\phi(x)) - x$. □

5.4 Proof of the main theorem

Induction upwards

Fix $\tau(n) = 80^{-1}6^{-n}$, then choose $\rho(n,\tau) \in (0,1)$ according to Lemma 5.3.2 applied with the value $2^{-k}\tau$ instead of τ, finally denote $\kappa = \frac{1}{1-\rho}$. Without loss of generality we can assume that each of the balls in S has radius $r_j = \rho^j$ for some natural $j \geqslant 1$. Otherwise we exchange each $\mathbf{B}_r(x) \in \mathsf{S}$ for $\mathbf{B}_{r_j}(x)$, where we take j so that $r_j \leqslant r < r_{j-1}$ if $r < \rho$ and $j = 1$ if $r \geqslant \rho$. This only changes the values in (5.1.3) and (5.1.4) by a multiplicative constant. Similarly, we can assume μ to be supported in $\mathbf{B}_1$, i.e. $\mathsf{Cent}\,\mathsf{S} \subseteq \mathbf{B}_1$ ($\beta_{\mu,q}$ numbers are monotone in μ).

Let S^i denote those balls that have radius $r_i = \rho^i$; we denote $\mathsf{S}^{\leqslant i} = \mathsf{S}^1 \cup \ldots \cup \mathsf{S}^i$ and $\mathsf{S}^{>i}$ etc. analogously. We can further assume S to be finite. Otherwise we proceed with the finite truncated collection $\mathsf{S}^{\leqslant A}$ and its associated measure $\mu^{\leqslant A}$, which also satisfies the assumption (5.1.3):

$$\mu^{\leqslant A} = \sum_{j\colon r_j \geqslant \rho^A} \omega_k r_j^k \delta_{x_j}.$$

If we are able to obtain the claim (5.1.4) for $\mu^{\leqslant A}$ with a constant independent of A, then by passing to the limit $A \to \infty$ we obtain the claim for μ. Thus let us assume that the smallest radius in the collection is r_A.

We focus on proving by induction the following claim:

Claim 5.4.1. *For each $j = A, \ldots, 0$ and any ball $\mathbf{B}_{r_j}(x) \subseteq \mathbf{B}_2$ disjoint from $\mathsf{Cent}\,\mathsf{S}^{\leqslant j}$,*

$$\mu(\mathbf{B}_{r_j}(x)) \leqslant M r_j^k.$$

At the end of the proof, it shall be clear that $M(n,J) = C(n) \cdot \max(1,J)$ works here.

Note that this estimate fails without the additional disjointness assumption, as for any $x \in \mathsf{Cent}\,\mathsf{S}^i$ and arbitrarily large j we have $\mu(\mathbf{B}_{r_j}(x)) = \omega_k r_i^k$. Still, Claim 5.4.1 implies our final claim. Indeed, the collection $\mathsf{S}^{\leqslant 0}$ is empty, thus $\mu(\mathbf{B}_1) \leqslant M$.

On the other hand, for $j = A$ any ball disjoint from $\mathsf{Cent}\,\mathsf{S}^{\leqslant A}$ has measure zero, so the claim is trivial. This is the basis for our upwards induction.

Induction downwards. An outline of the construction

Here we assume that Claim 5.4.1 holds for all $x \in \mathbf{B}_1$ and scales $j+1, \ldots, A$ and consider a ball $\mathbf{B}_{r_j}(x)$. For simplicity let us assume $j = 0$ and work with the ball $\mathbf{B}_1$ (i.e. the last step of the upwards induction).

We proceed with Reifenberg's construction of coverings of $\mathsf{Cent}\,\mathsf{S} \cap \mathbf{B}_1$ at all scales $i = 0, \ldots, A$. A covering at scale i will consist of the excess set $E^{\leqslant i}$ and collections of balls Good^i, Bad^i, Fin^i, each of radius r_i and centered in $\mathsf{Cent}\,\mathsf{S}$. The balls Fin^i will be chosen from the collection S^i (hence $\mu(\mathbf{B}) = \omega_k r_i^k$ for $\mathbf{B} \in \mathsf{Fin}^i$) and the other balls will be separated according to their measure: $\mu(\mathbf{B}) \geqslant \tau M r_i^k$ for good balls and $\mu(\mathbf{B}) < \tau M r_i^k$ for bad balls.

As the first step, we define the approximating surface to be

$$T_0 = V(0, \kappa) \leqslant \mathbb{R}^n.$$

The covering of $\mathsf{Cent}\,\mathsf{S} \cap \mathbf{B}_1$ is obtained by just one good ball $\mathsf{Good}^0 = \{\mathbf{B}_1\}$. Note that if this ball is in fact bad, there is nothing to prove.

The covering will satisfy the following properties:

Claim 5.4.2 (properties of the covering)**.** *The support of μ is covered by the collections of balls Good^i, $\mathsf{Bad}^{\leqslant i}$, $\mathsf{Fin}^{\leqslant i}$ and the excess set $E^{<i}$, i.e.*

$$\mathsf{Cent}\,\mathsf{S} \subseteq \bigcup \mathsf{Good}^i \cup \bigcup \mathsf{Bad}^{\leqslant i} \cup \bigcup \mathsf{Fin}^{\leqslant i} \cup E^{<i}.$$

The collections $\frac{1}{2}\mathsf{Good}^i$, $\frac{1}{2}\mathsf{Bad}^{\leqslant i}$, $\frac{1}{2}\mathsf{Fin}^{\leqslant i}$ *taken together are disjoint. Moreover, the collection* Good^i *is disjoint from* $\mathsf{Cent\,S}^{\leqslant i}$*.*

A sequence of surfaces approximating $\mathsf{Cent\,S}$ will also be constructed, but it is not used to obtain Claim 5.4.2.

Excess set

For each good ball $\mathbf{B}_{r_i}(y) \in \mathsf{Good}^i$ we define the excess set

$$E(y, r_i) := \mathbf{B}_{r_i}(y) \setminus \mathbf{B}_{r_{i+1}/4}(V(y, \kappa r_i)).$$

This set is exactly what prevents the set $\mathsf{Cent\,S}$ from satisfying the uniform Reifenberg condition $\beta_\infty(y, r_i) \leqslant \rho/4$. Its measure will be estimated via Chebyshev's inequality later on.

We sum up over all good balls to obtain

$$E^i := \bigcup_{\mathsf{Good}^i} E(y, r_i).$$

We add it to the previous excess sets: $E^{\leqslant i} := E^{\leqslant i-1} \cup E^i$.

Denote the remainder set

$$R^{\leqslant i} := \bigcup \mathsf{Bad}^{\leqslant i} \cup \bigcup \mathsf{Fin}^{\leqslant i} \cup E^{\leqslant i}.$$

The measure of this set can be estimated in a straightforward way, hence we do not need to cover it in the next steps of our inductive construction.

Construction of the covering

In order to cover the set $\bigcup \mathsf{Good}^i \setminus R^{\leqslant i}$ at scale r_i, we first choose the final balls

$$\mathsf{Fin}^{i+1} := \left\{ \mathbf{B}_{r_{i+1}}(z) : z \in \mathsf{Cent}\,\mathsf{S}^{i+1} \cap \left(\bigcup \mathsf{Good}^i \setminus R^{\leqslant i} \right) \right\},$$

so that $\mathsf{Fin}^{i+1} \subseteq \mathsf{S}^{i+1}$. Due to Claim 5.4.2, what is left to cover is the set

$$\mathsf{Cent}\,\mathsf{S}^{>i+1} \cap \left(\bigcup \mathsf{Good}^i \setminus R^{\leqslant i} \right). \tag{$\star$}$$

We choose any maximal r_{i+1}-separated subset $\mathsf{Cent}\,\mathsf{J}^{i+1}$ of the set ($\star$) and consider the collection of balls

$$\mathsf{J}^{i+1} := \{ \mathbf{B}_{r_{i+1}}(z) : z \in \mathsf{Cent}\,\mathsf{J}^{i+1} \}.$$

By maximality, the set ($\star$) is covered by $\bigcup \mathsf{J}^{i+1}$. We divide J^{i+1} into two subcollections:

$$\begin{aligned} \mathsf{Good}^{i+1} &:= \left\{ \mathbf{B} \in \mathsf{J}^{i+1} : \mu(\mathbf{B}) \geqslant \tau M r_{i+1}^k \right\}, \\ \mathsf{Bad}^{i+1} &:= \left\{ \mathbf{B} \in \mathsf{J}^{i+1} : \mu(\mathbf{B}) < \tau M r_{i+1}^k \right\}. \end{aligned}$$

Proof of Claim 5.4.2. By inductive hypothesis, $R^{\leqslant i}$ covers $\mathsf{Cent}\,\mathsf{S}^{\leqslant i}$. We covered the rest of $\mathsf{Cent}\,\mathsf{S}^{i+1}$ by Fin^{i+1} and $\mathsf{Cent}\,\mathsf{S}^{>i+1}$ by Good^{i+1}, Bad^{i+1}, thus we obtained the desired covering. Since the balls in S are disjoint and $\mathsf{Cent}\,\mathsf{J}^{i+1}$ is an r_{i+1}-separated set, the rest of the claim follows. □

Construction of the approximating surface

Here we apply the construction from Definition 5.3.4 for the collection of balls $\mathsf{J} = \mathsf{Good}^{i+1}$. Thus for each ball $\mathbf{B}_{r_{i+1}}(y_s) \in \mathsf{Good}^{i+1}$ there is an associated function λ_s, which together with ψ forms a partition of unity. We choose V_s as the L^2-best plane $V(y_s, \kappa r_{i+1})$ on a slightly enlarged ball. This defines the smooth function

$$\sigma_{i+1}(x) = \psi(x)x + \sum_s \lambda_s(x)\pi_{V_s}(x)$$

and the surface

$$T_{i+1} = \sigma_{i+1}(T_i).$$

The construction is now complete. Our aim is to derive three crucial properties (5.4.1), (5.4.2), (5.4.3). Once these are obtained, the final estimate is an easy consequence. First we need some basic properties of the surfaces constructed above.

Properties of the approximating surface

Proposition 5.4.3. *(a) For $y \in T_i$,*

$$|\sigma_{i+1}(y) - y| \leqslant \frac{1}{10} r_{i+1}.$$

(b) If $\mathbf{B}_{r_{i+1}}(y) \in \mathsf{Good}^{i+1}$, then

$$|T_{i+1} \cap 5\mathbf{B}_{r_{i+1}(y)}| \leqslant 10 \cdot \omega_k (5r_{i+1})^k,$$

(c) $\sigma_{i+1} \colon T_i \to T_{i+1}$ is bi-Lipschitz and for every $\mathbf{B}_{r_{i+1}}(y) \in \mathsf{Good}^{i+1}$ its bi-Lipschitz constant on $5\mathbf{B}_{r_{i+1}}(y)$ is bounded by

$$\mathrm{Lip}_{i+1} \leqslant 1 + C(n,q,\rho,\tau) M^{-\frac{q+2}{q}} \delta_q^2(y, 6r_{i-1}),$$

in particular $\mathrm{Lip}_{i+1} \leqslant 2^{1/k}$.

(d) If $\mathbf{B}_{r_{i+1}}(y) \in \mathsf{Good}^{i+1}$, the surface T_{i+1} is a graph over $V(y, \kappa r_{i+1})$ on $2\mathbf{B}_{r_{i+1}}(y)$ of a C^1 function satisfying

$$||f||_{C^1_{r_{i+1}}} \leqslant C(n,q,\rho,\tau) M^{-\frac{q+2}{q}} \delta_q^2(y, 5r_i).$$

Proof. In order to derive these, we apply the squash lemma (Lemma 5.3.6) for a ball $\mathbf{B}_{r_{i+1}(y)} \in \mathsf{Good}^{i+1}$. Its center y lies in some $\mathbf{B}_{r_i}(z) \in \mathsf{Good}^i$; we let $V := V(z, \kappa r_i)$ be the reference plane. Consider any $y' \in \mathsf{Cent\,Good}^{i+1}$ such that $|y - y'| \leqslant 5r_{i+1}$. Then y' lies in $\mathbf{B}_{2r_i}(z)$ and we may apply Lemma 5.3.3 (with $2^{-k}\tau$ instead of τ) and obtain

$$\begin{aligned} &d^2_{y', r_{i+1}}(V(z, 2\kappa r_i), V(y', \kappa r_{i+1})) \\ &\qquad \leqslant C(n, q, \rho, \tau) M^{-\frac{2}{q}} \left(\beta_q^2(y', \kappa r_{i+1}) + \beta_q^2(z, 2\kappa r_i)\right) \\ &\qquad \leqslant CM^{-\frac{q+2}{q}} \left(\delta_q^2(y', 2\kappa r_{i+1}) + \delta_q^2(z, 4\kappa r_i)\right) \\ &\qquad \leqslant CM^{-\frac{q+2}{q}} \delta_q^2(y, 5r_i) \\ &\qquad \leqslant CM^{-\frac{q+2}{q}} J \end{aligned}$$

Here we used again the pointwise estimate (5.3.3) and a very bad estimate $\delta_q^2(x, r) \leqslant J$ (the latter shall be refined in the next subsection). We can choose $M \geqslant C(\tau) J^{\frac{q}{q+2}}$ large enough so that the right-hand side is small. The planes $V(z, 2\kappa r_i)$ and $V(z, \kappa r_i)$ are compared in the same way:

$$d^2_{z, r_i}(V(z, 2\kappa r_i), V(z, \kappa r_i)) \leqslant CM^{-\frac{q+2}{q}} \delta^2(y, 5r_i) \leqslant CM^{-\frac{q+2}{q}} J.$$

By the inductive assumption, T_i is a graph over $V(z, \kappa r_i)$ on $2\mathbf{B}_{r_i}(z)$ hence we can apply Lemma 5.3.6 with

$$\delta_1 := \left(CM^{-\frac{q+2}{q}} \delta_q^2(y, 5r_i)\right)^{1/2}, \quad \delta_0 := \left(CM^{-\frac{q+2}{q}} \delta_q^2(z, 5r_{i-1})\right)^{1/2}.$$

Thus we obtain (a) and (b), while (c) follows after an additional estimate on δ_0, δ_1.

We also obtain an altered version of (d): T_{i+1} is also a graph over $V(z, \kappa r_i)$ on $\theta \mathbf{B}_{r_{i+1}}(y)$ with the desired C^1 bound (one can take $\theta = 2.5$). By an application of Lemma 5.3.7, one can change the plane: T_{i+1} is a graph over $V(y, \kappa r_{i+1})$ on $2\mathbf{B}_{r_{i+1}}(y)$. This completes the proof of Proposition 5.4.3. □

Estimates on the approximating surfaces T_i

Combining the bound for the bi-Lipschitz constant of $\sigma_{i+1}\colon T_i \to T_{i+1}$ in Proposition 5.4.3c with the elementary estimate $(1+x)^k \leqslant 1 + k2^{k-1}x$ (valid for $x \in [0,1]$), we obtain

$$\mathrm{Lip}^k_{i+1}(x) \leqslant 1 + CM^{-\frac{q+2}{q}}\delta^2(y_s, 6r_{i-1}), \qquad x \in 5\mathbf{B}_{r_{i+1}}(y_s)$$

for each ball $\mathbf{B}_{r_{i+1}}(y_s) \in \mathsf{Good}^{i+1}$. Summing over all balls in Good^{i+1} and noting that $\sigma_{i+1} = \mathrm{id}$ outside $5\mathsf{Good}^{i+1}$,

$$\mathrm{Lip}^k_{i+1}(x) \leqslant 1 + CM^{-\frac{q+2}{q}} \sum_s \delta^2(y_s, 6r_{i-1})\chi_{5\mathbf{B}_{r_{i+1}}(y_s)}(x).$$

The measure of $T_{i+1} = \sigma_{i+1}(T_i)$ can be estimated by

$$|T_{i+1}| \leqslant \int_{T_i} \mathrm{Lip}^k_{i+1}(x)\,\mathrm{d}\mathcal{H}^k(x).$$

Applying the above estimate and Proposition 5.4.3b,

$$\begin{aligned}
|T_{i+1}| &\leqslant |T_i| + CM^{-\frac{q+2}{q}} \sum_s |T_i \cap 5\mathbf{B}_{r_{i+1}}|\delta_q^2(y_s, 6r_{i-1}) \\
&\leqslant |T_i| + CM^{-\frac{q+2}{q}} \sum_s \int_{\mathbf{B}_{6r_{i-1}}(y_s)} \beta_q^2(z, 6r_{i-1})\,\mathrm{d}\mu(z) \\
&\leqslant |T_i| + CM^{-\frac{q+2}{q}} \int_{\mathbf{B}_2} \beta_q^2(z, 6r_{i-1})\,\mathrm{d}\mu(z).
\end{aligned}$$

In the last line we used the fact that any point $z \in \mathbf{B}_2$ belongs to at most $C(n,\rho)$ balls $\mathbf{B}_{6r_{i-1}}(y_s)$, as the balls $\frac{1}{2}\mathsf{Good}^{i+1}$ are disjoint.

Applying this inductively, we arrive at the following bound:

$$\begin{aligned}
|T_i| &\leqslant |T_0| + CM^{-\frac{q+2}{q}} \sum_{l=0}^{i-1} \int_{\mathbf{B}_2} \beta_q^2(z, 6r_l)\,\mathrm{d}\mu(z) \\
&\leqslant \omega_k \left(1 + C_2(n,q,\rho,\tau)M^{-\frac{q+2}{q}}J\right). \qquad (5.4.1)
\end{aligned}$$

Here, the bound on the series follows from Remark 5.3.1, and equality $|T_0| = \omega_k$ comes from the fact that T_0 is a plane.

Comparison of μ and $\mathcal{H}^k \llcorner T_i$

Let $\mathbf{B} \in \mathsf{Bad}^{i+1} \cup \mathsf{Fin}^{i+1}$ be bad or final. In either case, its center y lies in some $\mathbf{B}(z, r_i) \in \mathsf{Good}^i$ and $d(y, V(z, \kappa r_i)) \leqslant r_{i+1}/4$, so T_i is a graph over $V(z, \kappa r_i)$ on $\mathbf{B}$. In particular,

$$|T_i \cap \mathbf{B}/3| \geqslant \frac{1}{10}(r_{i+1}/3)^k.$$

Since $|\sigma_{i+1}(y) - y| \leqslant \frac{1}{10} r_{i+1}$ and σ_{i+1} has a bi-Lipschitz constant $\mathrm{Lip}_{i+1} \leqslant 2^{1/k}$ due to Proposition 5.4.3, we have

$$\begin{aligned}|T_{i+1} \cap \mathbf{B}/2| &\geqslant |T_i \cap \mathbf{B}/3| \cdot \mathrm{Lip}_{i+1}^{-k} \\ &\geqslant 20^{-1} 3^{-k} r_{i+1}^k.\end{aligned}$$

By construction, the centers $\mathsf{Cent}\,\mathsf{Good}^{>i+1}$ lie outside $\mathbf{B}$, hence $\mathbf{B}/2$ is disjoint with $5\mathsf{Good}^{>i+1}$ and $\sigma_s = \mathrm{id}$ on $\mathbf{B}/2$ for $s > i+1$. Therefore

$$|T_s \cap \mathbf{B}/2| \geqslant 20^{-1} 3^{-k} r_{i+1}^k$$

for $s = i, i+1, \ldots$. By definition, $\mu(\mathbf{B}) \leqslant \tau M r_{i+1}^k$ if $\mathbf{B}$ is bad. We choose $M \geqslant \omega_k/\tau$, so that the same holds if $\mathbf{B}$ is final. Thus we obtain the following comparison estimate

$$\mu(\mathbf{B}) \leqslant C_1 \tau M |T_s \cap \mathbf{B}/2| \tag{5.4.2}$$

for $\mathbf{B} \in \mathsf{Bad}^{i+1} \cup \mathsf{Fin}^{i+1}$ and $s = i, i+1, \ldots$. It is essential that the constant $C_1 = 20 \cdot 3^k$ does not depend on ρ, τ.

Estimates on the excess set

Since

$$E(y, r_i) = \{x \in \mathbf{B}_{r_i}(y) : d(x, V(y, \kappa r_i)) \geqslant r_{i+1}/4\},$$

Chebyshev's inequality yields

$$\begin{aligned}\mu(E(y,r_i)) &\leqslant \frac{1}{(r_{i+1}/4)^q}\int_{\mathbf{B}_{r_i}(y)} d^q(x,V(y,\kappa r_i))\,\mathrm{d}\mu\\ &\leqslant C(n,q,\rho)r_i^k\beta_q^q(y,\kappa r_i)\\ &\leqslant C(n,q,\rho,\tau)M^{-\frac{q}{2}}J^{\frac{q-2}{2}}r_i^k\delta_q^2(x,2r)\end{aligned}$$

where in the last line we applied the estimate (5.3.4). By construction, the balls $\frac{1}{2}\mathsf{Good}^i$ are disjoint, hence any point $x \in \mathbb{R}^n$ belongs to at most $C(n)$ of the balls $2\mathsf{Good}^i$. Thus

$$\mu(E^i) \leqslant C(n,q,\rho,\tau)M^{-\frac{q}{2}}J^{\frac{q-2}{2}}\int_{\mathbf{B}_2}\beta_q^2(x,2r_i)\,\mathrm{d}\mu$$

and by summing over $i = 0, 1, \ldots, A$ we obtain the bound

$$\mu(E^{\leqslant A}) \leqslant C_3(n,q,\rho,\tau)M^{-\frac{q}{2}}J^{\frac{q}{2}}. \tag{5.4.3}$$

Here we used again the assumption (5.1.3) together with Remark 5.3.1.

Derivation of the bound

Here we prove Claim 5.4.1 using the estimates (5.4.1), (5.4.2), (5.4.3). By construction, the balls Good^i are disjoint from $\mathsf{Cent}\,\mathsf{S}^{\leqslant i}$. This means that at the A-th step of the construction we have $\mathsf{Good}^A = \emptyset$, as this collection of balls is disjoint with $\mathsf{Cent}\,\mathsf{S}$. Therefore μ is supported in the remainder set:

$$\operatorname{supp}\mu \subseteq \bigcup \mathsf{Bad}^{\leqslant A} \cup \bigcup \mathsf{Fin}^{\leqslant A} \cup E^{\leqslant A}.$$

Recall that the collections $\frac{1}{2}\mathsf{Bad}^{\leqslant A}$, $\frac{1}{2}\mathsf{Fin}^{\leqslant A}$ are disjoint, so we can use (5.4.2) for all bad and final balls with $s = A$ to obtain:

$$\mu\left(\bigcup \mathsf{Bad}^{\leqslant A} \cup \bigcup \mathsf{Fin}^{\leqslant A}\right) \leqslant C_1\tau M|T_A|.$$

Then the surface estimate (5.4.1) yields

$$\mu\left(\mathsf{Bad}^{\leqslant A} \cup \bigcup \mathsf{Fin}^{\leqslant A}\right) \leqslant \omega_k C_1\tau M(1 + C_2 M^{-\frac{q+2}{q}}J).$$

We add it with the estimate for the excess set (5.4.3) and arrive at

$$\mu(\mathbf{B}_1) \leqslant M\left(\omega_k C_1 \tau(1 + C_2 M^{-\frac{q+2}{q}} J) + C_3 M^{-\frac{q+2}{2}} J^{\frac{q}{2}}\right).$$

Note that $\tau(n) = 80^{-1} 6^{-n}$ is chosen so that $\omega_k C_1 \tau \leqslant 1/4$. Now we choose the smallest M satisfying

$$C_2 M^{-\frac{q+2}{q}} J \leqslant 1, \quad C_3 M^{-\frac{q+2}{2}} J^{\frac{q}{2}} \leqslant \frac{1}{2}$$

and other lower bounds of the form $M \geqslant C(n, q)$ imposed during the proof; since $\tau(n)$ is fixed, we see that $M = C(n) \cdot \max\left(1, J^{\frac{q}{q+2}}\right)$. Finally, we are able to estimate

$$\mu(\mathbf{B}_1) \leqslant M\left(\frac{1}{4}(1+1) + \frac{1}{2}\right) = M.$$

This ends the proof of Claim 5.4.1 and Theorem 5.1.1.

5.5 Extentions of the theorem

Generalization to non-discrete measures

We assume that $S \subseteq \mathbf{B}_2$ is a $\mathcal{H}^k$-measurable subset. Here we generalize Theorem 5.1.1 to measures of the form $\mu = \mathcal{H}^k \llcorner S$, i.e. we show that (5.1.3) implies (5.1.4) in this case as well. This was done as a part of an independent theorem in [37, Th. 3.3], but here we show it is a corollary of Theorem 5.1.1.

Theorem 5.5.1. *Let $S \subseteq \mathbf{B}_2$ be a $\mathcal{H}^k$-measurable set and let $\beta_q(x, r)$, $J_q(x, r)$ be defined as in (5.1.1), (5.1.2) corresponding to the measure $\mathcal{H}^k \llcorner S$ and some exponent $2 \leqslant q < \infty$. Assume that for each ball $\mathbf{B}_r(x) \subseteq \mathbf{B}_2$ we have*

$$r^{-k} \int_{S \cap \mathbf{B}_r(x)} J_q(y, r)\, \mathrm{d}\mathcal{H}^k(y) \leqslant J.$$

Then for each ball $\mathbf{B}_r(x) \subseteq \mathbf{B}_1$ the following estimate holds:

$$\mathcal{H}^k(S \cap \mathbf{B}_r(x)) \leqslant C(n, q) \cdot \max\left(1, J^{\frac{q}{q+2}}\right) \cdot r^k.$$

Proof. It is sufficient to show the claim for the ball $\mathbf{B}_1$. Then for any $\mathbf{B}_r(x) \subseteq \mathbf{B}_1$ we can apply the theorem to the scaled set $S' = \frac{1}{r}(S - x)$, which satisfies the assumptions with the same value of J. Thus we obtain

$$\mathcal{H}^k(S \cap \mathbf{B}_r(x)) = \mathcal{H}^k(S' \cap \mathbf{B}_1) \cdot r^k \leqslant C(n) \cdot \max\left(1, J^{\frac{q}{q+2}}\right) \cdot r^k.$$

As a first step we show that $\mu = \mathcal{H}^k \llcorner S$ is σ-finite. Indeed, (5.1.3) yields in particular

$$\int_{\mathbf{B}_2} J_q(y, 2) \, \mathrm{d}\mu(y) \leqslant 2^k \cdot J.$$

For fixed $t > 0$, the measure of the superlevel set $S_t = \{y \in S : J_q(y, 2) \geqslant tJ\}$ can be estimated by $\mu(S_t) \leqslant 2^k/t$ using Chebyshev's inequality. On the other hand, the set $S_0 = \{J_q(y, 2) = 0\}$ is clearly contained in a k-dimensional plane and hence $\mu(S_0) < \infty$. Since

$$S = S_0 \cup \bigcup_{j=1}^{\infty} S_{1/j},$$

μ is σ-finite. We can assume without loss of generality that μ is finite. Indeed, we can first consider the smaller sets $S_0 \cup S_{1/j}$ instead; since the bound (5.1.4) depends on n and q only, in the limit we obtain the bound also for S.

Second, we recall the notion of upper k-dimensional density

$$\Theta^{*k}(S, x) = \limsup_{r \to 0} \frac{\mathcal{H}^k(S \cap \mathbf{B}_r(x))}{\omega_k r^k}$$

and its following property [28]:

Proposition 5.5.2. *Let $S \subseteq \mathbb{R}^n$ be a set with $\mathcal{H}^k(S) < \infty$. Then for $\mathcal{H}^k$-a.e. $x \in S$,*

$$2^{-k} \leqslant \Theta^{*k}(S, x) \leqslant 1. \tag{5.5.1}$$

Consider the set $S^\star$ of all points $x \in S$ satisfying (5.5.1). We can replace S with this possibly smaller set. Since the difference $S \setminus S^\star$ has zero $\mathcal{H}^k$ measure, the obtained bound for $S^\star$ holds also for S. From now on we assume that all points $x \in S$ satisfy (5.5.1).

For every $x \in S$ choose a radius $r_x \in (0, \rho]$ such that

$$\mu\left(\frac{1}{10}\mathbf{B}_{r_x}(x)\right) \geqslant 2^{-k-1}\omega_k(r_x/10)^k,$$
$$\mu(\mathbf{B}_r(x)) \leqslant 2\omega_k r^k \qquad \text{for all } r \leqslant r_x.$$

The set S is covered by balls $\mathbf{B}_{r_x}(x)$ and we can extract a countable Vitali subcovering $\mathbf{B}_j = \mathbf{B}_{r_j}(x_j)$, so that the balls $\frac{1}{5}\mathbf{B}_j$ are disjoint. Choose p_j to be the center of mass of $\frac{1}{10}\mathbf{B}_j$ and define the collection

$$\mathsf{S} := \{\mathbf{B}_{r_j/10}(p_j)\}.$$

Since $p_j \in \frac{1}{10}\mathbf{B}_j$, we have $\mathbf{B}_{r_j/10}(p_j) \subseteq \frac{1}{5}\mathbf{B}_j$, thus the collection S is disjoint. We consider the associated measure

$$\nu := \sum_j \omega_k(r_j/10)^k \delta_{p_j}.$$

Our goal now is to reduce the problem for μ to the already solved problem for the discrete measure ν. We will show that this is possible due to the following comparison estimates:

$$\mu(\mathbf{B}_1) \leqslant 2 \cdot 10^k \nu(\mathbf{B}_{1+2\rho})$$
$$\beta^q_{\nu,q}(x, s) \leqslant 2^{k+1}3^{k+q}\beta^q_{\mu,q}(x, 3s).$$

For the first estimate, we observe that

$$\mu(\mathbf{B}_1) \leqslant \sum_{x_j \in \mathbf{B}_{1+\rho}} \mu(\mathbf{B}_j) \leqslant 2 \cdot 10^k \sum_{x_j \in \mathbf{B}_{1+\rho}} \omega_k(r_j/10)^k \leqslant 2 \cdot 10^k \nu(\mathbf{B}_{1+2\rho}).$$

As for the second, consider a ball $\mathbf{B}_s(x)$ such that $3\mathbf{B}_s(x) \subseteq \mathbf{B}_2$. If there is some $p_j \in \mathbf{B}_s(x)$ with $r_j/10 > 2s$, then by disjointness of S this is the only point from $\operatorname{supp}\nu$ in $\mathbf{B}_s(x)$ and $\beta_{\nu,2}(x,s)=0$. In the other case, $r_j/10 \leqslant 2s$ for all $p_j \in \mathbf{B}_s(x)$. Choose an affine k-plane V. On each $\frac{1}{10}\mathbf{B}_j$ we apply Jensen's inequality for the function $d^q(\cdot, V)$:

$$d^q(p_j, V) \leqslant \fint_{\frac{1}{10}\mathbf{B}_j} d^q(y,V)\,\mathrm{d}\mu.$$

This yields

$$(r_j/10)^k \omega_k d^q(p_j,V) \leqslant 2^{k+1} \int_{\frac{1}{10}\mathbf{B}_j} d^q(y,V)\,\mathrm{d}\mu$$

and hence

$$\int_{\mathbf{B}_s(x)} d^q(y,V)\,\mathrm{d}\nu \leqslant 2^{k+1} \sum_{p_j\in\mathbf{B}_s(x)} \int_{\frac{1}{10}\mathbf{B}_j} d^q(y,V)\,\mathrm{d}\mu \leqslant 2^{k+1} \int_{\mathbf{B}_{3s}(x)} d^q(y,V)\,\mathrm{d}\mu.$$

Taking the infimum on the right-hand side,

$$\beta^q_{\nu,q}(x,s) \leqslant 2^{k+1}3^{k+q}\beta^q_{\mu,q}(x,3s).$$

Therefore the flatness condition (5.1.3) is satisfied also for the measure ν and we obtain our claim by an application of Theorem 5.1.1. To be more precise, one first needs to apply an easy rescaling and covering argument, as one needs to bound $\nu(\mathbf{B}_{1+2\rho})$ instead of $\nu(\mathbf{B}_1)$, and also the obtained estimate works only for balls $\mathbf{B}_s(x)$ such that $3\mathbf{B}_s(x) \subseteq \mathbf{B}_2$. □

Remark 5.5.3. This proof shows that Theorem 5.1.1 actually works for all measures μ with the covering property resulting from Proposition 5.5.2. Consider μ supported in the union of balls $\mathbf{B}_{r_j}(x_j)$, each satisfying

$$\mu\left(\frac{1}{10}\mathbf{B}_{r_j}(x_j)\right) \geqslant c_\mu (r_j/10)^k,$$
$$\mu\left(\mathbf{B}_r(x_j)\right) \leqslant C_\mu r_j^k \qquad \text{for all } r \leqslant r_j.$$

In particular, this is satisfied by any μ such that

$$c_\mu \leqslant \Theta^{*k}(\mu,x) \leqslant C_\mu \quad \text{for } \mu\text{-a.e. } x.$$

If μ satisfies the assumption (5.1.3), then $\mu(\mathbf{B}_1)$ is bounded as in (5.1.4). Naturally, the constant obtained in the final estimate depends on c_μ, C_μ.

Weakened assumptions

The proof of Theorem 5.1.1 applies also with the assumption (5.1.3) replaced by $\fint_{\mathbf{B}} J_2 \leqslant J$. This means that we consider the integral divided by $\mu(\mathbf{B}_r(x))$ instead of r^k. Since there is no a priori upper bound for μ, this assumption is weaker.

Theorem 5.5.4. *Let $\mathsf{S} = \{\mathbf{B}_{r_j}(x_j)\}$ be a collection of disjoint balls in $\mathbf{B}_2$ and $\mu = \sum_j \omega_k r_j^k \delta_{x_j}$ be its associated measure and let $\beta_q(x,r)$, $J_q(x,r)$ be defined as in (5.1.1), (5.1.2), where $2 \leqslant q < \infty$. Assume that for each ball $\mathbf{B}_r(x) \subseteq \mathbf{B}_2$ we have*

$$\fint_{\mathbf{B}_r(x)} J_q(y,r)\,\mathrm{d}\mu(y) \leqslant J.$$

Then the following estimate holds:

$$\mu(\mathbf{B}_1) = \sum_{x_j \in B_1} \omega_k r_j^k \leqslant C(n,q) \cdot \max\left(1, J^{\frac{q}{2}}\right).$$

Sketch of proof. Proceeding as in the proof of Theorem 5.1.1, one obtains the following counterparts of estimates (5.4.3), (5.4.1):

$$\mu(E^{\leqslant A}) \leqslant C_3 J^{\frac{q}{2}},$$
$$|T_A| \leqslant \omega_k \left(1 + C_2 M^{-\frac{2}{q}} J\right).$$

The main difference lies in the last step of each estimate, where one needs to bound the integral $\int_{\mathbf{B}_2} J_q(x,r)\,\mathrm{d}\mu(x)$. A closer look at the proof shows that in fact an integral over $\mathbf{B}_{1.5}$ is sufficient to bound these quantities (actually, any ball larger than $\mathbf{B}_1$ is sufficient if ρ is small enough). In the case considered in Theorem 5.1.1, this is bounded by J; in this case, one has to use the rough estimate $\mu(\mathbf{B}_{1.5}) \leqslant C(n)M$ to obtain

$$\int_{\mathbf{B}_{1.5}} J_q(x,r)\,\mathrm{d}\mu(x) = \mu(\mathbf{B}_{1.5}) \fint_{\mathbf{B}_{1.5}} J_q(x,r)\,\mathrm{d}\mu \leqslant C(n)MJ.$$

This rough estimate can be derived as follows. Since the collection $\mathsf{S} = \mathsf{S}^{\geqslant 1}$ is disjoint, there are at most $C(n,\rho)$ ball centers $\mathbf{B}_{1.5} \cap \operatorname{Cent} \mathsf{S}^1$ and each has measure $\omega_k \rho^k$. The rest of $\mathbf{B}_{1.5}$ can be covered by $C(n, \rho)$ balls of radius ρ disjoint from $\mathsf{Cent}\,\mathsf{S}^{\leqslant 1}$. By the inductive assumption of Claim 5.4.1, each has measure bounded by $M\rho^k$. This yields

$$\mu(\mathbf{B}_{1.5}) \leqslant C(n, \rho)\omega_k \rho^k + C(n, \rho) M \rho^k \leqslant C(n, \rho) M.$$

The proof of the estimate (5.4.2) carries over without changes:

$$\mu(\mathbf{B}) \leqslant C_1 \tau M |T_s \cap \mathbf{B}/2| \quad \text{for } \mathbf{B} \in \mathsf{Bad}^{i+1} \cup \mathsf{Fin}^{i+1} \text{ and } s \geqslant i.$$

Similarly, these three estimates combined yield

$$\mu(\mathbf{B}_1) \leqslant M \left(\omega_k C_1 \tau \left(1 + C_2 M^{-\frac{2}{q}} J \right) + C_3 M^{-1} J^{\frac{q}{2}} \right)$$

and the proof works for $M = C(n) \cdot \max\left(1, J^{\frac{q}{2}}\right)$. □

Chapter 6

Linear bound on the measure of singularities

6.1 Linear law

This chapter is devoted to the proof of Theorem 6.1.1, a higher-dimensional counterpart for Almgren–Lieb's linear estimate on the number of singularities.

Theorem 6.1.1. *Let $u \in W^{1,2}(\Omega, \mathbb{S}^2)$ be a minimizing harmonic map in a smooth bounded domain $\Omega \subseteq \mathbb{R}^n$, and let $\varphi := u|_{\partial\Omega}$ be its trace. Then*

$$\mathcal{H}^{n-3}(\operatorname{sing} u) \leqslant C(\Omega) \int_{\partial\Omega} |\nabla\varphi|^{n-1} \, \mathrm{d}\mathcal{H}^{n-1}. \tag{6.1.1}$$

In contrast to Almgren and Lieb's original proof, the one presented here does not depend on the classification of singularities (Theorem 2.4.1), and the only necessary special property of $\mathbb{S}^2$ is the extension property (Theorem 2.5.1). As already noted in Remark 2.5.2, it holds for all closed simply connected manifolds (see [17]). Thus, we obtain the following more general result.

Corollary 6.1.2. *Assume that $\mathcal{N}$ is a smooth closed simply-connected manifold. Let $u \in W^{1,2}(\Omega, \mathcal{N})$ be a minimizing harmonic map in a smooth bounded domain*

$\Omega \subseteq \mathbb{R}^n$, and let $\varphi := u|_{\partial\Omega}$ be its trace. Then

$$\mathcal{H}^{n-3}(\operatorname{sing} u) \leqslant C(\Omega, \mathcal{N}) \int_{\partial\Omega} |\nabla\varphi|^{n-1} \,\mathrm{d}\mathcal{H}^{n-1}. \tag{6.1.2}$$

Let us discuss why such an estimate is noteworthy. First, a non-linear estimate on the measure of singularities follows easily from Naber and Valtorta's interior bounds. As an example, consider the following very simple result.

Theorem 6.1.3. *Let Ω, u and φ be as before. Then*

$$\mathcal{H}^{n-3}(\operatorname{sing} u) \leqslant C(\Omega, \operatorname{Lip}(\varphi)).$$

Proof. If the boundary map φ is Lipschitz continuous, there exists $\sigma > 0$ (depending on the geometry of Ω and the Lipschitz constant of φ) such that any minimizer u is smooth in the region $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \leqslant \sigma\}$; this was proved in [18], but also follows from more general Corollary 2.6.5. For any ball $\mathbf{B}_{\sigma/2}(p)$ centered outside this region, Corollary 4.1.2 implies an upper bound $\mathcal{H}^{n-3}(\operatorname{sing} u \cap \mathbf{B}_{\sigma/2}(p)) \leqslant C(n, \sigma)$. Since $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geqslant \sigma\}$ can be covered by finitely many such balls, we obtain an upper bound depending on Ω and $\operatorname{Lip}(\varphi)$. □

The second improvement lies in the norm used for estimating $\operatorname{sing} u$. A closer look at the argument above shows that in fact $\operatorname{Lip}(\varphi)$ can be replaced by the norm $\|\varphi\|_{W^{1,p}(\partial\Omega)}$ for any $p > n-1$. However, even a non-linear estimate in terms of $\|\varphi\|_{W^{1,n-1}(\partial\Omega)}$ has to involve some more sophisticated geometric considerations – the singularities still have positive distance to the boundary, but this distance cannot be estimated in terms of the norm alone.

As already mentioned in the introduction, one can hope for a further refinement – replacing $W^{1,n-1}(\partial\Omega)$ by $W^{1,p}(\partial\Omega)$ with any $p > 2$ – but the analysis becomes more challenging, as the singularities may approach the boundary.

Examples in [31] show, the $W^{1,p}(\partial\Omega)$-norm with $p < 2$ cannot control the singularities, so Corollary 6.1.2 is sharp in dimension $n = 3$.

6.2 Hot spots – refined boundary regularity

We start by refining further the boundary regularity theorems from Chapter 2.

Recall that Theorem 2.5.4 gives a bound on the energy of $u\colon \mathbf{B}_1^+ \to \mathbb{S}^2$ in terms of the energy $\int_{T_1} |\nabla\varphi|^2$ of its boundary map. In this section, we are considering possible *hot spots* on the boundary. That is, we assume $\nabla\varphi$ is controlled on most of T_1 except for a small ball, on which the integral may be arbitrarily large.

The first result states that in this case a uniform bound on the energy is also available away from the hot spot (see [1, Thm. 2.3.]).

Theorem 6.2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary. There exists a number $r_0 = r_0(\Omega) > 0$ with the following property.*

For $p \in \partial\Omega$ let $A_{(r,s)}(p) := \{x \in \mathbb{R}^n : r < \operatorname{dist}(x,p) < s\}$. Suppose also that u is a minimizer in Ω having boundary map φ. Then, whenever $0 < r < r_0$,

$$r^{2-n} \int_{\Omega\cap A_{(r,2r)}(p)} |\nabla u|^2 \,\mathrm{d}x \leqslant C + Cr^{3-n} \int_{\partial\Omega\cap A_{(r/2,5r/2)}(p)} |\nabla\varphi|^2 \,\mathrm{d}\mathcal{H}^{n-1},$$

where $C(n) > 0$ is a dimensional constant.

Proof. We choose r_0 so that balls $\mathbf{B}_{r_0}(q) \cap \Omega$ with $q \in \partial\Omega$ are C^2-close to $\mathbf{B}_1^+$ after rescaling (see the remark preceding Corollary 2.5.3).

On any ball $\mathbf{B}_{r/4}(q)$ with $q \in \partial\Omega \cap A_{(r,2r)}(p)$ we have the estimate

$$\int_{\Omega\cap\mathbf{B}_{r/4}(q)} |\nabla u|^2 \,\mathrm{d}x \lesssim r^{n-2} + r \int_{\partial\Omega\cap A_{(r/2,5r/2)}(p)} |\nabla\varphi|^2 \,\mathrm{d}\mathcal{H}^{n-1}$$

by Theorem 2.5.4. On the other hand, by the same theorem

$$\int_{\mathbf{B}_{r/8}(q)} |\nabla u|^2 \,\mathrm{d}x \lesssim r^{n-2}$$

for any q such that $\mathbf{B}_{r/4}(q) \subseteq \Omega$. We can cover the annulus $A_{(r,2r)}(p)$ by finitely many balls of these two types, the number of balls depending only on the dimension. Summing up, we conclude the final estimate (compare [29, Theorem 6.1]). □

With this uniform energy bound, we can actually show that *boundary energy in small balls cannot induce distant singularities* [1, Thm. 2.4]. In the contradiction argument, the hot spot tends to zero in size and disappears completely in the limit.

Theorem 6.2.2 (regularity away from *hot spots*)**.** *There is* $\varepsilon(n) > 0$ *such that the following holds. Suppose* $u \in W^{1,2}(\mathbf{B}_1^+, \mathbb{S}^2)$ *is a minimizer with trace* φ *on* T_1*, and*

$$\int_{T_1 \setminus \mathbf{B}_\varepsilon(p)} |\nabla \varphi|^2 \, d\mathcal{H}^{n-1} \leqslant \varepsilon$$

for some ball $\mathbf{B}_\varepsilon(p)$*. Then* u *is smooth in*

$$T_{1/2} \times (\lambda/2, \lambda),$$

where $\lambda(n) > 0$ *is a small dimensional constant.*

Proof. We argue by contradiction. Assume that $u_i \colon \mathbf{B}_1^+ \to \mathbb{S}^2$ is a sequence of minimizers with boundary maps φ_i such that

$$\int_{T_1 \setminus \mathbf{B}_{\varepsilon_i}(p_i)} |\nabla \varphi_i|^2 \, \mathrm{d}\mathcal{H}^{n-1} \leqslant \varepsilon_i$$

for a sequence of balls $\mathbf{B}_{\varepsilon_i}(p_i)$ and $\varepsilon_i \xrightarrow{i \to \infty} 0$. Setting $r_i := (\varepsilon_i)^{\frac{1}{n-2}} > \varepsilon_i$, we obtain

$$(r_i)^{3-n} \int_{T_1 \setminus \mathbf{B}_{r_i}(p_i)} |\nabla \varphi_i|^2 \, \mathrm{d}\mathcal{H}^{n-1} < r_i, \tag{6.2.1}$$

where $r_i \xrightarrow{i \to \infty} 0$, and up to taking a subsequence, $r_i < 2^{-i}$.

Now, we assume (by contradiction) that each u_i has at least one singularity $y_i \in T_{1/2} \times (\lambda/2, \lambda)$.

By Theorem 6.2.1, for large enough i and for any $r \geqslant 2^{-i}$

$$r^{2-n} \int_{\mathbf{B}_1^+ \cap A_{(r,2r)}(p_i)} |\nabla u_i|^2 \, \mathrm{d}x \leqslant C.$$

Thus, for every $1 \leqslant k \leqslant i$,

$$\int_{\mathbf{B}_1^+ \cap A_{(2^{-k}, 2^{-k+1})}(p_i)} |\nabla u_i|^2 \, \mathrm{d}x \leqslant C \, 2^{-k(n-2)}.$$

Up to taking another subsequence we can assume that $p_i \to p_0$, and for convenience also $|p_i - p_0| \leqslant 2^{-i}$. Then, from the above estimate we have

$$\int_{\mathbf{B}_{4/5}^+ \setminus \mathbf{B}_{2^{-i+13}}(p_0)} |\nabla u_i|^2 \, \mathrm{d}x \leqslant C \sum_{k=1}^{i} 2^{-k(n-2)} \leqslant C. \tag{6.2.2}$$

In particular by a diagonal argument and the strong convergence of minimizers, Theorem 2.6.1, we obtain a minimizer u in $W^{1,2}(\mathbf{B}_{3/4}^+ \setminus \mathbf{B}_r(p_0))$ for any $r > 0$. Moreover, its trace, which we shall call $\varphi \in W^{1,2}_{\mathrm{loc}}(T_1 \setminus \{p_0\}, \mathbb{S}^2)$ is the limit of φ_i. Observe that φ is constant on T_1 by (6.2.1).

Moreover, by (2.1.3) the sequence of singular points y_i can be assumed to converge to a singular point of u, which we call $y \in T_{1/2} \times [\lambda/2, \lambda]$.

To reach a contradiction with Theorem 2.6.4, one needs to solve the subtle issue of minimality around p_0. To this end, we note that by (6.2.2) the energy $\int_{\mathbf{B}_{4/5}^+ \setminus \mathbf{B}_r(p_0)} |\nabla u_i|^2$ is uniformly bounded for all $r > 0$, and hence by monotone convergence $u \in W^{1,2}(\mathbf{B}_{3/4}^+)$. In view of Lemma 6.2.3 below, the singularity p_0 is removable, and so u is a minimizing harmonic map in $\mathbf{B}_{3/4}^+$) with a constant boundary map φ. This contradicts the singularity at y. □

To complete the proof of Theorem 6.2.2, we need the following removability lemma.

Lemma 6.2.3 (Removability of points for minimizing harmonic maps)**.** *Assume that $u \in W^{1,2}(\mathbf{B}_1^+, \mathbb{S}^2)$ is a minimizer away from the origin, i.e., assume that for* *any* $\delta > 0$ *and any* $v \in W^{1,2}(\mathbf{B}_1^+, \mathbb{S}^2)$ *satisfying* $v = u$ *on* $\partial \mathbf{B}_1^+$ *and* $v \equiv u$ *on* $\mathbf{B}_\delta^+$ *we have*

$$\int_{\mathbf{B}_1^+ \setminus \mathbf{B}_\delta^+} |\nabla u|^2 \,\mathrm{d}x \leqslant \int_{\mathbf{B}_1^+ \setminus \mathbf{B}_\delta^+} |\nabla v|^2 \,\mathrm{d}x. \tag{6.2.3}$$

Then u *is a minimizing harmonic map in all of* $\mathbf{B}_1^+$*.*

Proof. Let $w \in W^{1,2}(\mathbf{B}_1^+, \mathbb{S}^2)$ with $u \equiv w$ on $\partial \mathbf{B}_1^+$ be a competitor. We need to show that

$$\int_{\mathbf{B}_1^+} |\nabla u|^2 \,\mathrm{d}x \leqslant \int_{\mathbf{B}_1^+} |\nabla w|^2 \,\mathrm{d}x. \tag{6.2.4}$$

For $\delta > 0$, let $\eta_\delta \in C_c^\infty(\mathbf{B}_{2\delta})$ be a standard cut-off function satisfying $\eta_\delta \equiv 1$ in $\mathbf{B}_\delta$ and $|\nabla \eta_\delta| \lesssim 1/\delta$. We define $\tilde{w}_\delta \in W^{1,2}(\mathbf{B}_1^+, \mathbb{R}^3)$ as

$$\tilde{w}_\delta := (1 - \eta_\delta) w + \eta_\delta u;$$

this function satisfies $\tilde{w}_\delta = u$ on $\partial \mathbf{B}_1^+$, $\tilde{w}_\delta \equiv u$ in $\mathbf{B}_\delta^+$ and $\tilde{w}_\delta \equiv w$ in $\mathbf{B}_1^+ \setminus \mathbf{B}_{2\delta}$. By the extension property (Theorem 2.5.1) applied in $\mathbf{B}_{2\delta}^+ \setminus \mathbf{B}_\delta$ we can correct $\tilde{w}_\delta$ to a map $w_\delta \in W^{1,2}(\mathbf{B}_1^+, \mathbb{S}^2)$ such that

$$w_\delta = \begin{cases} u & \text{in } \mathbf{B}_\delta^+ \\ w & \text{in } \mathbf{B}_1^+ \setminus \mathbf{B}_{2\delta} \\ u & \text{on } \partial \mathbf{B}_1 \end{cases}$$

and

$$\int_{\mathbf{B}_{2\delta}^+ \setminus \mathbf{B}_\delta} |\nabla w_\delta|^2 \,\mathrm{d}x \leqslant \int_{\mathbf{B}_{2\delta}^+ \setminus \mathbf{B}_\delta} |\nabla \tilde{w}_\delta|^2 \,\mathrm{d}x.$$

In particular, $\tilde{w}_\delta$ is a competitor in the sense of (6.2.3), and we have

$$\begin{aligned} \int_{\mathbf{B}_1^+ \setminus \mathbf{B}_\delta} |\nabla u|^2 \,\mathrm{d}x &\leqslant \int_{\mathbf{B}_1^+ \setminus \mathbf{B}_\delta} |\nabla w_\delta|^2 \,\mathrm{d}x \\ &= \int_{\mathbf{B}_1^+ \setminus \mathbf{B}_{2\delta}} |\nabla w_\delta|^2 \,\mathrm{d}x + \int_{\mathbf{B}_{2\delta}^+ \setminus \mathbf{B}_\delta} |\nabla w_\delta|^2 \,\mathrm{d}x \\ &\leqslant \int_{\mathbf{B}_1^+ \setminus \mathbf{B}_{2\delta}} |\nabla w|^2 \,\mathrm{d}x + C \int_{\mathbf{B}_{2\delta}^+} |\nabla \tilde{w}_\delta|^2 \,\mathrm{d}x. \end{aligned}$$

Since u, and $w \in W^{1,2}(\mathbf{B}_1^+)$ using the absolute continuity of the integral we find that

$$\int_{\mathbf{B}_1^+} |\nabla u|^2 \,\mathrm{d}x \leqslant \int_{\mathbf{B}_1^+} |\nabla w|^2 \,\mathrm{d}x + C \liminf_{\delta\to 0} \int_{\mathbf{B}_{2\delta}^+} |\nabla \tilde{w}_\delta|^2 \,\mathrm{d}x. \tag{6.2.5}$$

Now

$$\int_{\mathbf{B}_{2\delta}^+} |\nabla \tilde{w}_\delta|^2 \,\mathrm{d}x \leqslant \frac{1}{\delta^2} \int_{\mathbf{B}_{2\delta}^+} |u-v|^2 \,\mathrm{d}x + \int_{\mathbf{B}_{2\delta}^+} |\nabla u|^2 \,\mathrm{d}x + \int_{\mathbf{B}_{2\delta}^+} |\nabla v|^2 \,\mathrm{d}x.$$

Observe that we are in dimension $n \geqslant 3$ and $\mathbb{S}^2$ is compact, so

$$\frac{1}{\delta^2} \int_{\mathbf{B}_{2\delta}^+} |u-v|^2 \,\mathrm{d}x \leqslant \delta.$$

Thus, using again the absolute continuity of the integral and that $u, w \in W^{1,2}$ we find

$$\lim_{\delta\to 0} \int_{\mathbf{B}_{2\delta}^+} |\nabla \tilde{w}_\delta|^2 \,\mathrm{d}x = 0.$$

Plugging this into (6.2.5) we conclude. □

In the applications, we will use the following global version of Theorem 6.2.2 (see [1, Cor. 2.7]).

Theorem 6.2.4 (boundary regularity with hot spots)**.** *For each bounded smooth domain $\Omega \subseteq \mathbb{R}^n$, there are small constants $\sigma, \varepsilon, \lambda, \Lambda > 0$ (σ depending on the geometry of Ω, the others only on the dimension) so that the following statement holds true for any minimizer $u \in W^{1,2}(\Omega, \mathbb{S}^2)$ with trace $\varphi := u\big|_{\partial\Omega}$.*

For any singular point $p \in \operatorname{sing} u$ with $r := \operatorname{dist}(p, \partial\Omega) < \sigma$ and for any ball $\mathbf{B}_{\lambda r}(q) \subseteq \mathbb{R}^n$, we have

$$r^{3-n} \int_{\partial\Omega \cap (\mathbf{B}_{\Lambda r}(p) \setminus \mathbf{B}_{\lambda r}(q))} |\nabla \varphi|^2 \,\mathrm{d}\mathcal{H}^{n-1} \geqslant \varepsilon.$$

Proof. In principle, this is a rescaled version of Theorem 6.2.2, only with non-flat boundary – with $\Lambda = 1/\lambda'$ and $\lambda = \varepsilon/\lambda'$, where ε, λ' are the values from Theorem 6.2.2. By choosing $\sigma > 0$ small enough, we can ensure that after rescaling the balls to unit size, the boundary is arbitrarily close to flat. To consider this more general case, one needs a bit different contradiction argument based on Theorem 2.6.1 (see Remark 2.6.2). □

By Hölder's inequality, the conclusion can be replaced by

$$\int_{\partial\Omega\cap(\mathbf{B}_{\Lambda r}(p)\setminus\mathbf{B}_{\lambda r}(q))} |\nabla\varphi|^{n-1}\,\mathrm{d}\mathcal{H}^{n-1} \geqslant \varepsilon$$

and this is the formulation actually used in the sequel. However, it is important to note that all boundary regularity theorems work with $W^{1,2}$, and the scale-invariance of $W^{1,n-1}$-norm on the boundary is only needed for the final covering argument.

6.3 Covering argument

As in the case $n = 3$, the study of singularities near the boundary involves the following covering lemma, which we here cite from [1, Theorem 2.8, 2.9].

Theorem 6.3.1 (Covering lemma)**.** *Let $\mathcal{B}$ be a family of closed balls in $\mathbb{R}^n$, μ be a Borel measure over $\mathbb{R}^n$, and let $\tau, \omega \in (0,1)$. Moreover, assume that the following two hypotheses hold:*

1. *For any two different $\mathbf{B}_r(p), \mathbf{B}_s(q) \in \mathcal{B}$ we have*

$$|p-q| \geqslant \omega \min(r,s).$$

2. *Suppose that $\mathbf{B}_r(p) \in \mathcal{B}$ and $q \in \mathbb{R}^n$ is an arbitrary point, then*

$$\mu\left(\mathbf{B}_r(p) \setminus \mathbf{B}_{\tau r}(q)\right) \geqslant 1.$$

Then

$$\#\textit{balls in } \mathcal{B} \leqslant C\mu(\mathbb{R}^n),$$

for a constant $C(\omega,\tau,n) > 0$.

Proof of Theorem 6.1.1. Choose $\sigma > 0$ (depending on the geometry of $\partial\Omega$) according to Theorems 2.6.3, 6.2.4. We first estimate the measure of the set

$$A_1 := \{p \in \operatorname{sing} u : r(p) \leqslant \sigma\}, \quad \text{where } r(p) = \tfrac{1}{2}\operatorname{dist}(p, \partial\Omega),$$

which is covered by balls $\mathbf{B}_{r(p)}(p)$. Then we choose a Vitali subcovering such that the balls $\mathbf{B}_{r_j}(p_j)$ cover A_1 and the balls $\mathbf{B}_{r_j/5}(p_j)$ are disjoint; let $\mathcal{B}$ be the family of balls $\mathbf{B}_{r_j/\lambda}(p_j)$ with λ as in Theorem 6.2.4. The first condition from Theorem 6.3.1 with $\omega = \lambda/5$ follows: for any two distinct balls in our collection we have

$$|p_i - p_j| \geqslant \tfrac{1}{5}(r_i + r_j) \geqslant \tfrac{\lambda}{5}\max(r_i/\lambda, r_j/\lambda).$$

Now let μ be the measure

$$\mu = \frac{1}{\varepsilon}|\nabla\varphi|^{n-1}\,\mathcal{H}^{n-1}\llcorner\partial\Omega, \quad \text{i.e. } \mu(U) = \frac{1}{\varepsilon}\int_{\partial\Omega\cap U}|\nabla\varphi|^{n-1}\,\mathrm{d}\mathcal{H}^{n-1},$$

where $\varepsilon > 0$ is the constant from Theorem 6.2.4. If we set $\tau = \lambda^2$, then the second condition of Theorem 6.3.1 follows from Theorem 6.2.4 and we infer that

$$\#\mathcal{B} \leqslant C\int_{\partial\Omega}|\nabla\varphi|^{n-1}\,\mathrm{d}\mathcal{H}^{n-1}.$$

On each ball $\mathbf{B}_{r_j}(p_j)$, Corollary 4.1.2 implies $\mathcal{H}^{n-3}(\operatorname{sing} u \cap \mathbf{B}_{r_j}(p_j)) \leqslant C r_j^{n-3}$. Summing over all balls, we obtain

$$\mathcal{H}^{n-3}(A_1) \leqslant C\int_{\partial\Omega}|\nabla\varphi|^{n-1}\,\mathrm{d}\mathcal{H}^{n-1}.$$

Next we estimate the set

$$A_2 := \{p \in \operatorname{sing} u : r(p) \geqslant \sigma\}.$$

For each ball $\mathbf{B}_\sigma(y)$ with $\operatorname{dist}(y, \partial\Omega) \geqslant 2\sigma$, Corollary 4.1.2 yields an upper bound $\mathcal{H}^{n-3}(\operatorname{sing} u \cap \mathbf{B}_\sigma(y)) \leqslant C\sigma^{n-3}$. The set A_2 can be covered by finitely many such balls (the number of balls depending only on σ and the geometry of Ω), which gives us an estimate

$$\mathcal{H}^{n-3}(A_2) \leqslant C_0.$$

Taking C_0 as above and ε as in Theorem 2.6.3, we have two possibilities. Either the smallness condition $\int_{\partial\Omega} |\nabla\varphi|^{n-1} \, \mathrm{d}\mathcal{H}^{n-1} \leqslant \varepsilon$ is satisfied and $\mathcal{H}^{n-3}(A_2) = 0$ follows, or

$$\mathcal{H}^{n-3}(A_2) \leqslant C_0 \leqslant \frac{C_0}{\varepsilon} \int_{\partial\Omega} |\nabla\varphi|^{n-1} \, \mathrm{d}\mathcal{H}^{n-1}.$$

In both cases, combining the estimates for A_1 and A_2 ends the proof. □

Chapter 7

Stability of singularities

7.1 Statement of results

This chapter is concerned with stability of singularities. By this we mean that if two boundary maps $\varphi, \varphi' \colon \partial\Omega \to \mathbb{S}^2$ are *close* in the right Sobolev norm, then the singularities of their corresponding minimizers $u, u' \colon \Omega \to \mathbb{S}^2$ are *close* as well. Since minimizers are in general non-unique, the precise statement is a little more subtle – e.g. by assuming uniqueness a priori.

In any case, let us discuss the right notions of *closeness*. In dimension $n = 3$, when the singular set consists of finitely many points, Hardt and Lin [18] proved Theorem 1.4.2. They considered the Lipschitz norm for boundary data, and showed that small perturbations do not change the number of singularities. Moreover, they constructed a bi-Lipschitz diffeomorphism $\eta \colon \Omega \to \Omega$ (close to identity in Lipschitz norm) such that u is close to $u' \circ \eta$ in some C^β norm. These results were recently extended to the case of $W^{1,2}$-perturbations of boundary data by Li [24].

In higher dimension $n \geqslant 3$, we consider perturbations in the $W^{1,n-1}$ norm. Since the singular set is a rectifiable set of codimension 3, we prove its stability with respect to Wasserstein metric (see [51])

$$d_W(\mu,\nu) = \sup_{\substack{h\colon \mathbb{R}^n\to[-1,1]\\ |\nabla h|\leqslant 1}} \left\{ \int_{\mathbb{R}^n} h \,\mathrm{d}\mu - \int_{\mathbb{R}^n} h\,\mathrm{d}\nu \right\}, \tag{7.1.1}$$

i.e., we show that the distance between measures $\mathcal{H}^{n-3} \llcorner \operatorname{sing} u$ and $\mathcal{H}^{n-3} \llcorner \operatorname{sing} u'$ is small. Since taking $h \equiv 1$ in the definition yields

$$|\mu(\mathbb{R}^n) - \nu(\mathbb{R}^n)| \leqslant d_W(\mu,\nu),$$

we obtain in particular that the size of the singular set $\mathcal{H}^{n-3}(\operatorname{sing} u)$ is also stable under $W^{1,n-1}$-perturbations of boundary data.

Theorem 7.1.1 (stability of singularities)**.** *Let $u \in W^{1,2}(\Omega,\mathbb{S}^2)$ be a minimizer in a bounded smooth domain $\Omega \subseteq \mathbb{R}^n$ with boundary data $\varphi \in W^{1,n-1}(\partial\Omega,\mathbb{S}^2)$. If u_k is a sequence of minimizers with boundary data φ_k and*

$$u_k \to u \text{ in } W^{1,2}(\Omega), \quad \varphi_k \to \varphi \text{ in } W^{1,n-1}(\partial\Omega), \tag{7.1.2}$$

then

$$\mathcal{H}^{n-3} \llcorner \operatorname{sing} u_k \xrightarrow{d_W} \mathcal{H}^{n-3} \llcorner \operatorname{sing} u,$$

in particular $\mathcal{H}^{n-3}(\operatorname{sing} u_k) \to \mathcal{H}^{n-3}(\operatorname{sing} u)$.

For $n = 3$, we recover most of Hardt and Lin's Theorem 1.4.2. Indeed, we see that $\#\operatorname{sing} u_k = \#\operatorname{sing} u$ for large k (as $\mathcal{H}^0$ is simply the counting measure) and that $\operatorname{sing} u_k$ converges to $\operatorname{sing} u$ with respect to Hausdorff distance. However, generalizing the diffeomorphism statement to higher dimensions seems very challenging – note that bi-Lipschitz regularity of $\operatorname{sing}_* u$ is an open problem for $n > 3$.

If one assumes uniqueness, the statement becomes slightly simpler:

Corollary 7.1.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded smooth domain, and assume that for boundary data $\varphi \in W^{1,n-1}(\partial\Omega,\mathbb{S}^2)$ there is a unique minimizer $u \in W^{1,2}(\Omega,\mathbb{S}^2)$. Then for each $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\|\varphi' - \varphi\| \leqslant \delta \text{ in } W^{1,n-1} \quad \Rightarrow \quad d_W\left(\mathcal{H}^{n-3} \llcorner \operatorname{sing} u', \mathcal{H}^{n-3} \llcorner \operatorname{sing} u\right) \leqslant \varepsilon$$

for any minimizer u' with boundary data φ'.

Proof. For the sake of contradiction, let u_k be a sequence of minimizers with boundary data φ_k, with $\varphi_k \to \varphi$ in $W^{1,n-1}(\partial\Omega, \mathbb{S}^2)$. Taking a subsequence, by Theorem 2.6.1 we may assume that u_k converges in $W^{1,2}(\Omega, \mathbb{S}^2)$ to a minimizer $\overline{u}$ with boundary data φ. By uniqueness, $\overline{u} = u$ and Theorem 7.1.1 implies that $\mathcal{H}^{n-3} \llcorner \operatorname{sing} u_k$ tends to $\mathcal{H}^{n-3} \llcorner \operatorname{sing} u$. Thus, we obtain a contradiction for large enough k. □

7.2 Outline

In analogy to the original argument of Hardt and Lin [18], the heart of the argument lies in the special case when u is the tangent map Ψ as in (2.4.1) given by

$$\mathbb{R}^3 \times \mathbb{R}^{n-3} \ni (x, y) \xmapsto{\Psi} \frac{x}{|x|} \in \mathbb{S}^2.$$

Establishing a stability result for the singular set (which for Ψ is an $(n-3)$-dimensional plane) requires some care. Here we adopt the notion of δ-flatness introduced in Chapter 3, which combines topological and analytic conditions for a minimizer to be *close* to Ψ. In Section 7.3 we cite some of the necessary results in our case.

With this in hand, we are able to modify the original arguments of Naber and Valtorta [37] and improve their measure estimates in the special case of maps into $\mathbb{S}^2$. In result, we obtain the stability result for Ψ mentioned earlier (Lemma 7.4.1).

Since around $\mathcal{H}^{n-3}$-almost every singular point, any energy minimizer is close to the map Ψ (composed with an isometry), this stability result can be seen as a local case for Theorem 7.1.1. Indeed, in Section 7.5 we cover most of the singular set of u by balls on which Lemma 7.4.1 can be applied. An argument based on Proposition 3.5.1 then shows that the same covering works for both $\operatorname{sing} u$ and $\operatorname{sing} u_k$, and the global estimate follows.

7.3 Behavior of top-dimensional singularities

This section recapitulates the results of Chapter 3 in the special case of the target manifold $\mathcal{N} = \mathbb{S}^2$. These results will allow us to study further the top-dimensional part of the singular set.

Recall that by Theorem 2.4.2, the map $\Psi\colon \mathbb{R}^3 \times \mathbb{R}^{n-3} \to \mathbb{S}^2$ given by $\Psi(x,y) = x/|x|$ (2.4.1) is the only locally minimizing $(n-3)$-symmetric harmonic map from $\mathbb{R}^n$ to $\mathbb{S}^2$ (up to linear isometries of $\mathbb{R}^n$). In particular, its energy density $\Theta = \int_{\mathbf{B}_1} |\nabla\Psi|^2$ from (2.4.2) is the only energy density on the top-dimensional part of the singular set – i.e., $\theta_u(x,0) = \Theta$ for each $x \in \operatorname{sing}_* u$. As already noted in Chapter 3, this implies that the homotopy class $[\mathrm{id}]$ is indecomposable in the sense of Definition 3.2.4.

From now on, we shall use the notion of δ-flatness (see Definition 3.3.3) with this fixed homotopy class and its (lowest) energy level Θ.

Below we summarize the main consequences of δ-flatness from Chapter 3 in the special case $\mathcal{N} = \mathbb{S}^2$. For simplicity, we only deal with the ball $\mathbf{B}_2$, but one can easily obtain the corresponding statement for any ball using the scale-invariance.

Theorem 7.3.1. *For each $\varepsilon > 0$ there is $\delta > 0$ such that the following holds. If u is δ-flat in $\mathbf{B}_2$, then*

1. *for some tangent map of the form $\overline{\Psi} = \Psi \circ q$ (with Ψ as in (2.4.1) and some linear isometry q) we have*
$$\|u - \overline{\Psi}\|^2_{W^{1,2}(\mathbf{B}_1)} \leqslant \varepsilon,$$
2. *for the $(n-3)$-dimensional linear plane $L := \operatorname{sing}\overline{\Psi}$,*
$$\operatorname{sing} u \cap \mathbf{B}_1 \subseteq \mathbf{B}_\varepsilon(L) \quad \textit{and} \quad L \cap \mathbf{B}_{1-\varepsilon} \subseteq \pi_L(\operatorname{sing} u \cap \mathbf{B}_1),$$
3. *all singular points in $\mathbf{B}_1$ lie in the top-dimensional part $\operatorname{sing}_* u$, and u is ε-flat in each of the balls $\mathbf{B}_r(z)$ with $z \in \operatorname{sing} u \cap \mathbf{B}_1$ and $0 < r \leqslant 1/2$.*

Proof. Points (1) and (2) are essentially the content of Lemma 3.4.3, except for the condition $L \cap \mathbf{B}_{1-\varepsilon} \subseteq \pi_L(\operatorname{sing} u \cap \mathbf{B}_1)$, which follows from Lemma 3.4.1. Point (3) comes from combining Proposition 3.4.7 and Corollary 3.4.8. □

7.4 Local case

The lemma below can be thought of as a local version of the stability theorem. It says that perturbing the tangent map Ψ a little does not change the size of the singular set much.

Lemma 7.4.1. *For each $\varepsilon > 0$ there is $\delta > 0$ such that the following is true. If $u\colon \mathbf{B}_{80} \to \mathbb{S}^2$ is energy minimizing and δ-flat in $\mathbf{B}_{80}$ (see Definition 3.3.3), then*

$$(1-\varepsilon)\omega_{n-3} \leqslant \mathcal{H}^{n-3}(\operatorname{sing} u \cap \mathbf{B}_1) \leqslant (1+\varepsilon)\omega_{n-3}.$$

Here $\omega_{n-3} = \mathcal{H}^{n-3}(\operatorname{sing} \Psi \cap \mathbf{B}_1)$ is the volume of the $(n-3)$-dimensional ball.

It is natural that in order to conclude the right estimate on $\mathbf{B}_1$, one needs to make assumptions on a larger ball. The ball $\mathbf{B}_2$ would be enough here, but working with $\mathbf{B}_{80}$ saves us from an additional covering argument.

Proof. The lower bound follows from a simple topological argument (compare with [37, Lemma 6.1]). Fix $\varepsilon' = \frac{\varepsilon}{n-2}$, then apply Theorem 7.3.1 to find that there is an $(n-3)$-dimensional linear plane L such that

$$L \cap \mathbf{B}_{1-\varepsilon'} \subseteq \pi_L(\operatorname{sing} u \cap \mathbf{B}_1),$$

provided δ is small enough. Since the orthogonal projection π_L is 1-Lipschitz, this shows

$$\mathcal{H}^{n-3}(\operatorname{sing} u \cap \mathbf{B}_1) \geqslant \mathcal{H}^{n-3}(L \cap \mathbf{B}_{1-\varepsilon'}) = (1-\varepsilon')^{n-3}\omega_{n-3} \geqslant (1-\varepsilon)\omega_{n-3}.$$

A rough upper bound follows from Naber and Valtorta's work [37], namely Corollary 4.1.2:

$$\mathcal{H}^{n-3}(\operatorname{sing} u \cap \mathbf{B}_r(z)) \leqslant C(n) r^{n-3} \tag{7.4.1}$$

for each ball $\mathbf{B}_{2r}(z) \subseteq \mathbf{B}_2$.

To obtain the sharp upper bound, we will follow the general outline of Naber and Valtorta's work [37, Sec. 1.4]. When the target manifold is $\mathbb{S}^2$, the original reasoning can be made significantly easier due to topological control of singularities (analyzed in Chapter 3). In particular, we will be able to apply Rectifiable Reifenberg Theorem 4.1.4 to the whole singular set in $\mathbf{B}_1$, without decomposing it into many pieces.

With $\delta_1 > 0$ to be fixed later, by Theorem 7.3.1 we can choose δ small enough so that all singular points in $\mathbf{B}_{40}$ lie in the top-dimensional part $\operatorname{sing}_* u$, moreover u is also δ_1-flat in each ball $\mathbf{B}_r(z)$ with $z \in \operatorname{sing} u \cap \mathbf{B}_{40}$ and $0 < r \leqslant 20$.

We can now apply the L^2-best approximation Theorem 4.1.5 on these balls; for simplicity, we consider the ball $\mathbf{B}_{10}$ first. By Theorem 7.3.1, u is $W^{1,2}$-close to a map of the form $\overline{\Psi} = \Psi \circ q$ (with Ψ as in (2.4.1) and some linear isometry q). Note that $\overline{\Psi}$ lies in $\operatorname{sym}_{n,0}$ and the value

$$\varepsilon_0 := \operatorname{dist}_{L^2(\mathbf{B}_{10})}(\overline{\Psi},\ \operatorname{sym}_{n,k+1}) > 0$$

depends only on the dimension n (not on the choice of q). Hence, by taking δ_1 small enough we can ensure that

$$\operatorname{dist}_{L^2(\mathbf{B}_{10})}(u,\ \operatorname{sym}_{n,0}) \leqslant \delta,$$
$$\operatorname{dist}_{L^2(\mathbf{B}_{10})}(u,\ \operatorname{sym}_{n,k+1}) \geqslant 2\varepsilon_0$$

with $\delta = \delta(2\varepsilon_0)$ chosen according to Theorem 4.1.5. Then we obtain

$$\beta^2(0,1) \leqslant C(n) \int_{\mathbf{B}_1} (\theta_u(y,8) - \theta_u(y,1))\, \mathrm{d}\mu(y),$$

where $\mu := \mathcal{H}^{n-3} \llcorner (\operatorname{sing} u \cap \mathbf{B}_2)$ and $\beta = \beta_{\mu,n-3,2}$. Similarly,

$$\beta^2(z,s) \leqslant C(n) s^{-(n-3)} \int_{\mathbf{B}_s(z)} (\theta_u(y,8s) - \theta_u(y,s))\, \mathrm{d}\mu(y) \tag{7.4.2}$$

for each ball $\mathbf{B}_s(z) \subseteq \mathbf{B}_2$ with $z \in \operatorname{sing} u$. To see this, one simply needs to consider the rescaled map $\overline{u}(x) = u(z + rx)$ and apply scaling-invariance of δ-flatness and β-numbers.

Now we verify the hypotheses of Rectifiable Reifenberg Theorem 4.1.4. Fix a ball $\mathbf{B}_r(x) \subseteq \mathbf{B}_2$; we only need to check that

$$\int_{\mathbf{B}_r(x)} \int_0^r \beta^2(z,s) \frac{\mathrm{d}s}{s} \, \mathrm{d}\mu(z) \leqslant \delta_2 r^{n-3} \tag{7.4.3}$$

with $\delta_2(\varepsilon) > 0$ chosen according to Theorem 4.1.4,

First, we integrate the estimate (7.4.2) over $\mathbf{B}_r(x)$ and exchange the order of summation:

$$\begin{aligned}\int_{\mathbf{B}_r(x)} \beta^2(z,s) \, \mathrm{d}\mu(z) &\lesssim s^{-(n-3)} \int_{\mathbf{B}_r(x)} \int_{\mathbf{B}_s(z)} (\theta_u(y,8s) - \theta_u(y,s)) \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(z) \\ &\leqslant s^{-(n-3)} \int_{\mathbf{B}_{2r}(x)} \int_{\mathbf{B}_s(y)} (\theta_u(y,8s) - \theta_u(y,s)) \, \mathrm{d}\mu(z) \, \mathrm{d}\mu(y) \\ &\lesssim \int_{\mathbf{B}_{2r}(x)} (\theta_u(y,8s) - \theta_u(y,s)) \, \mathrm{d}\mu(y).\end{aligned}$$

Note that in the last step we used the weak upper bound (7.4.1) on the ball $\mathbf{B}_s(y)$.

When the above is integrated with respect to s, we obtain a telescopic sum. In order to estimate it, first recall that u is δ_1-flat in each ball $\mathbf{B}_{8r}(y)$ such that $y \in \operatorname{sing} u$ and $\mathbf{B}_r(y) \subseteq \mathbf{B}_2$, in particular

$$\theta_u(y,8r) - \theta_u(y,0) \leqslant \delta_1$$

on the support of μ. Thus, the substitution $s \mapsto 8s$ together with monotone convergence $\theta_u(y,s) \searrow \theta_u(y,0)$ give us

$$\begin{aligned}\int_0^r (\theta_u(y,8s) - \theta_u(y,s)) \frac{\mathrm{d}s}{s} &= \int_r^{8r} (\theta_u(y,s) - \theta_u(y,0)) \frac{\mathrm{d}s}{s} \\ &\leqslant \ln(8)\delta_1.\end{aligned}$$

Now we are ready to combine the above estimates:

$$\begin{aligned}\int_{\mathbf{B}_r(x)} \int_0^r \beta^2(z,s) \frac{\mathrm{d}s}{s} \,\mathrm{d}\mu(z) &\lesssim \int_0^r \int_{\mathbf{B}_{2r}(x)} (\theta_u(y,8s) - \theta_u(y,s)) \,\mathrm{d}\mu(y) \frac{\mathrm{d}s}{s} \\ &\leqslant \int_{\mathbf{B}_{2r}(x)} \ln(8)\delta_1 \,\mathrm{d}\mu(y) \\ &\lesssim \delta_1 r^{n-3},\end{aligned}$$

where we used (7.4.1) again in the last line. Assuming $\delta_1 \leqslant \delta_2(\varepsilon)/C(n)$, we have verified the assumption (7.4.3) and we infer the upper estimate

$$\mathcal{H}^{n-3}(\operatorname{sing} u \cap \mathbf{B}_1) = \mu(\mathbf{B}_1) \leqslant (1+\varepsilon)\omega_{n-3}.$$

□

7.5 Global case

The idea of the proof is to cover most of $\operatorname{sing} u$ by good balls, on which u is δ-flat and thus the measure of $\operatorname{sing} u$ is controlled by Lemma 7.4.1. The rest of the singular set is to be covered by bad balls, whose total mass is small. To achieve this, we will need the following simple covering lemma.

Lemma 7.5.1. *Let $S \subseteq \mathbb{R}^n$ be a compact set of finite $\mathcal{H}^k$-measure and let $\mathcal{B}$ be a family of open balls such that for each point $p \in S$, all small enough balls around p belong to $\mathcal{B}$. Then, given any $\varepsilon > 0$, S can be covered by the union of two finite families of open balls* Good, Bad, *where* Good $\subseteq \mathcal{B}$ *consists of pairwise disjoint balls and* Bad $= \mathbf{B}_{r_j}(p_j)$ *is a small family in the sense that*

$$\sum_j r_j^k \leqslant \varepsilon. \tag{7.5.1}$$

Proof. One way to construct this covering is by using Vitali's covering theorem for Radon measures (see e.g., [28, Theorem 2.8]). Applying it to the measure

$\mu := \mathcal{H}^k \llcorner S$, we obtain a countable family of pairwise disjoint closed balls $\mathcal{A} = \left\{ \overline{\mathbf{B}_{r_s}(p_s)} \right\}$, covering μ-almost all S and satisfying $\mathbf{B}_{2r_s}(p_s) \in \mathcal{B}$ for each s. Since the measure μ is finite, we can divide $\mathcal{A}$ into two subfamilies Good', Bad', where Good' is finite and Bad' is small, i.e., $\mu\left(\bigcup \mathsf{Bad}'\right) \leqslant \varepsilon$. To obtain the desired properties, we still need to alter these families a little.

First, we define Good to be the balls of Good' slightly enlarged to open balls, but still pairwise disjoint and still belonging to $\mathcal{B}$.

Now, the remaining part $S \setminus \bigcup \mathsf{Good}$ is a compact set and

$$\mu\left(S \setminus \bigcup \mathsf{Good}\right) \leqslant \mu\left(\bigcup \mathsf{Bad}'\right) \leqslant \varepsilon.$$

By definition of Hausdorff measure, this set can be covered by a finite family of open balls Bad satisfying the smallness condition (7.5.1). □

Proof of Theorem 7.1.1. Fix $\varepsilon > 0$. For the sake of clarity, we focus on showing that the difference $|\mathcal{H}^{n-3}(\operatorname{sing} u_k) - \mathcal{H}^{n-3}(\operatorname{sing} u)|$ is controlled by ε for k large enough. The estimate for Wasserstein distance follows the same lines; it is briefly discussed at the end of the proof.

Step 1 (boundary regularity). Choose $\varepsilon_0 > 0$ according to the boundary regularity theorem (Theorem 2.6.6). Fix $r > 0$ such that

$$\int_{\mathbf{B}_r(x)} |\nabla \varphi|^{n-1} \leqslant \varepsilon_0/2$$

for every ball $\mathbf{B}_r(x)$ centered at $\partial\Omega$. Then u is smooth in a λr-neighborhood of $\partial\Omega$ (with $\lambda(n) > 0$ as in Theorem 2.6.6). By strong convergence of φ_k to φ in $W^{1,n-1}(\partial\Omega)$, we may assume that $\int_{\partial\Omega} |\nabla\varphi_k|^{n-1} \leqslant \frac{\varepsilon_0}{2} + \int_{\partial\Omega} |\nabla\varphi|^{n-1}$ and thus

$$\int_{\mathbf{B}_r(x)} |\nabla \varphi|^{n-1} \leqslant \varepsilon_0$$

for every ball $\mathbf{B}_r(x)$ centered at $\partial\Omega$. As a consequence, we may assume each u_k is also smooth in the same fixed neighborhood of $\partial\Omega$.

Step 2 (covering the low-dimensional part). Recall the stratification from Section 2.3

$$S_0 \subseteq \ldots \subseteq S_{n-4} \subseteq S_{n-3} = \operatorname{sing} u,$$

in which the k-th stratum S_k has Hausdorff dimension k or smaller. We will consider separately the set S_{n-4} and the top-dimensional part

$$\operatorname{sing}_* u := S_{n-3} \setminus S_{n-4}.$$

Since $\operatorname{sing} u$ is compact and $\operatorname{sing}_* u$ is an open subset of $\operatorname{sing} u$ by Theorem 7.3.1, S_{n-4} is also compact. At the same time, it has a uniform distance from $\partial\Omega$ and $\mathcal{H}^{n-3}(S_{n-4}) = 0$, so it can be covered by a finite family $\mathsf{Bad}_1 = \{\mathbf{B}_{r_i}(p_i)\}$ of open balls satisfying the smallness condition (7.5.1)

$$\sum_i r_i^{n-3} \leqslant \varepsilon$$

and such that $\mathbf{B}_{2r_i}(p_i) \subseteq \Omega$ for each i.

On each such ball Corollary 4.1.2 yields $\mathcal{H}^{n-3}(\operatorname{sing} u \cap \mathbf{B}_{r_i}(p_i)) \leqslant C r_i^{n-3}$, with C depending only on the dimension n. Summing over all balls, we obtain

$$\mathcal{H}^{n-3}\left(\operatorname{sing} u \cap \bigcup \mathsf{Bad}_1\right) \leqslant C\varepsilon.$$

The same estimate holds verbatim for each u_k, by the same application of Corollary 4.1.2.

Step 3 (covering the top-dimensional part and estimating $\mathcal{H}^{n-3}(\operatorname{sing} u)$). Here, we use the covering lemma (Lemma 7.5.1) for the set $S := \operatorname{sing} u \setminus \bigcup \mathsf{Bad}_1$. Thanks to Step 1, $\operatorname{sing} u$ has positive distance from the boundary, so it is a compact set of finite $\mathcal{H}^{n-3}$-measure due to Corollary 4.1.2. We choose $\mathcal{B}$ to be

$$\mathcal{B} = \left\{ \mathbf{B}_r(p) \colon \quad p \in \operatorname{sing}_* u,\ u \text{ is } \delta\text{-flat in } \mathbf{B}_{81r}(p) \right\},$$

where $\delta(\varepsilon) > 0$ is chosen according to Lemma 7.4.1. Since S_{n-4} is already covered by Bad_1, we know that $S \subseteq \operatorname{sing}_* u$ and hence small enough balls around each point in S lie in $\mathcal{B}$ by Lemma 3.4.9 and Corollary 3.4.5.

Having checked the properties required by Lemma 7.5.1, we can cover S by the union of a finite disjoint family $\mathsf{Good} \subseteq \mathcal{B}$ and another finite family Bad_2 satisfying (7.5.1). We add the latter to Bad_1 to obtain the family of bad balls $\mathsf{Bad} := \mathsf{Bad}_1 \cup \mathsf{Bad}_2$, which still satisfies the smallness condition (7.5.1).

Repeating the reasoning from Step 2, we have again via Corollary 4.1.2,

$$\mathcal{H}^{n-3}\left(\operatorname{sing} u \cap \bigcup \mathsf{Bad}\right) \leqslant 2C\varepsilon, \tag{7.5.2}$$
$$\mathcal{H}^{n-3}\left(\operatorname{sing} u_k \cap \bigcup \mathsf{Bad}\right) \leqslant 2C\varepsilon \quad \text{for all } k.$$

By assumption, the map u is δ-flat in $\mathbf{B}_{80r_s}(p_s)$ for each ball $\mathbf{B}_{r_s}(p_s) \in \mathsf{Good}$. By Lemma 7.4.1, we now obtain

$$(1-\varepsilon)\omega_{n-3}r_s^{n-3} \leqslant \mathcal{H}^{n-3}(\operatorname{sing} u \cap \mathbf{B}_{r_s}(p_s)) \leqslant (1+\varepsilon)\omega_{n-3}r_s^{n-3}$$

for each s. To finish the proof, we need to show that a similar comparison holds for u_k if k is large.

Step 4 (estimating $\mathcal{H}^{n-3}(\operatorname{sing} u_k)$). Since $u_k \to u$ in $W^{1,2}(\Omega)$ and $\operatorname{sing} u$ is covered by the open families $\mathsf{Good}, \mathsf{Bad}$, by (2.1.3) the same holds for u_k if k is large enough (from now on we assume it is). For bad balls, the rough estimate (7.5.2) will be enough, so we focus on good balls.

By Proposition 3.5.1, we can assume (by taking k large and δ small) that for each $\mathbf{B}_{r_s}(p_s) \in \mathsf{Good}$ there is $p_s^k \in \operatorname{sing} u_k$ such that $|p_s^k - p_s| \leqslant \varepsilon r_s$ and u_k is δ'-flat in the ball $\mathbf{B}_{80(1+\varepsilon)r_s}(p_s^k)$. Here, the value of δ' is chosen to be $\delta(\varepsilon)$ from Lemma 7.4.1.

Applying Lemma 7.4.1 to u_k on balls $\mathbf{B}_{(1-\varepsilon)r_s}(p_s^k)$ and $\mathbf{B}_{(1+\varepsilon)r}(p_s^k)$, we obtain

$$\begin{aligned}(1-\varepsilon)^{n-2}\omega_{n-3}r_s^{n-3} &\leqslant \mathcal{H}^{n-3}(\operatorname{sing} u_k \cap \mathbf{B}_{(1-\varepsilon)r_s}(p_s^k)) \\ &\leqslant \mathcal{H}^{n-3}(\operatorname{sing} u_k \cap \mathbf{B}_{r_s}(p_s)) \\ &\leqslant \mathcal{H}^{n-3}(\operatorname{sing} u_k \cap \mathbf{B}_{(1+\varepsilon)r_s}(p_s^k)) \\ &\leqslant (1+\varepsilon)^{n-2}\omega_{n-3}r_s^{n-3},\end{aligned}$$

which is only slightly worse that the estimate for $\mathcal{H}^{n-3}(\operatorname{sing} u)$.

Step 5 (comparison). Recalling that Good is a disjoint family, we can sum the above estimate over all s to obtain

$$(1-\varepsilon)^{n-2}A \leqslant \mathcal{H}^{n-3}(\operatorname{sing} u_k \cap \bigcup \mathsf{Good}) \leqslant (1+\varepsilon)^{n-2}A,$$

where $A := \sum_s \omega_{n-3} r_s^{n-3}$. Combining it with the estimate for bad balls (7.5.2), we finally obtain

$$(1-\varepsilon)^{n-2}A \leqslant \mathcal{H}^{n-3}(\operatorname{sing} u_k) \leqslant (1+\varepsilon)^{n-2}A + 2C\varepsilon.$$

Exactly the same estimate is true for u. Combining these two yields

$$\begin{aligned} \left|\mathcal{H}^{n-3}(\operatorname{sing} u_k) - \mathcal{H}^{n-3}(\operatorname{sing} u)\right| &\leqslant \left((1+\varepsilon)^{n-2} - (1-\varepsilon)^{n-2}\right) A + 2C\varepsilon \\ &\leqslant \left(\frac{(1+\varepsilon)^{n-2}}{(1-\varepsilon)^{n-2}} - 1\right) \mathcal{H}^{n-3}(\operatorname{sing} u) + 2C\varepsilon. \end{aligned}$$

Evidently the right-hand side tends to zero when $\varepsilon \to 0$, which ends the proof of stability of $\mathcal{H}^{n-3}(\operatorname{sing} u)$.

Step 6 (Wasserstein distance estimate). With just a little bit more care, the reasoning above leads to the Wasserstein distance estimate. Let us consider the measure $\mu = \mathcal{H}^{n-3} \llcorner \operatorname{sing} u$ and decompose it into $\mu = \mu_b + \sum_s \mu_s$, where

$$\begin{aligned} &\mu_b = \mu \llcorner \left(\bigcup \mathsf{Bad} \setminus \bigcup \mathsf{Good}\right), \\ &\mu_s = \mu \llcorner \mathbf{B}_{r_s}(p_s) \quad \text{for each ball } \mathbf{B}_{r_s}(p_s) \in \mathsf{Good}. \end{aligned}$$

The estimate for μ_b is simply $d_W(\mu_b, 0) \leqslant \mu\left(\bigcup \mathsf{Bad}\right) \leqslant 2C\varepsilon$, whereas on each good ball $\mathbf{B}_{r_s}(p_s)$ we have the inequalities

$$\begin{aligned} &\int_{\mathbb{R}^n} h \, \mathrm{d}\mu_s - \omega_{n-3} r_s^{n-3} h(p_s) \\ &\quad = \int_{\mathbf{B}_{r_s}(p_s)} (h - h(p_s)) \, \mathrm{d}\mu + \left(\mu(\mathbf{B}_{r_s}(p_s)) - \omega_{n-3} r_s^{n-3}\right) h(p_s) \\ &\quad \leqslant r_s \mu(\mathbf{B}_{r_s}(p_s)) + |\mu(\mathbf{B}_{r_s}(p_s)) - \omega_{n-3} r_s^{n-3}| \\ &\quad \leqslant (r_s + 2\varepsilon)\omega_{n-3} r_s^{n-3} \end{aligned}$$

for any function $h\colon \mathbb{R}^n \to \mathbb{R}$ satisfying $|h| \leqslant 1$ and $|\nabla h| \leqslant 1$. If only each radius is smaller than ε, it follows that $d_W(\mu_k, \omega_{n-3} r_s^{n-3} \delta_{p_s}) \leqslant 3\varepsilon\omega_{n-3} r_s^{n-3}$. By triangle inequality, $d_W(\mu, \nu) \leqslant 3\varepsilon A + 2C\varepsilon$, where $\nu = \sum_s \omega_{n-3} r_s^{n-3} \delta_{p_s}$ is the packing measure associated to Good and once again $A = \nu(\mathbb{R}^n)$. Applying the same reasoning to u_k, we conclude as before.

$\square$

Bibliography

[1] Almgren, Jr., F. J., and Lieb, E. H. Singularities of energy minimizing maps from the ball to the sphere: examples, counterexamples, and bounds. *Ann. of Math. (2) 128*, 3 (1988), 483–530.

[2] Azzam, J., and Schul, R. An Analyst's Traveling Salesman Theorem for sets of dimension larger than one. *arXiv preprint 1609.02892* (2016).

[3] Azzam, J., and Tolsa, X. Characterization of n-rectifiability in terms of Jones' square function: Part II. *Geom. Funct. Anal. 25*, 5 (2015), 1371–1412.

[4] Brezis, H., Coron, J.-M., and Lieb, E. H. Harmonic maps with defects. *Comm. Math. Phys. 107*, 4 (1986), 649–705.

[5] Cheeger, J., Jiang, W., and Naber, A. Rectifiability of singular sets in noncollapsed spaces with Ricci curvature bounded below. *arXiv preprint 1805.07988* (2018).

[6] Cheeger, J., and Naber, A. Quantitative stratification and the regularity of harmonic maps and minimal currents. *Comm. Pure Appl. Math. 66*, 6 (2013), 965–990.

[7] David, G., and Toro, T. Reifenberg parameterizations for sets with holes. *Mem. Amer. Math. Soc. 215*, 1012 (2012), vi+102.

[8] Edelen, N. A global bound for the singular set of area-minimizing hypersurfaces. *arXiv preprint 1901.02837* (2019).

[9] Edelen, N., Naber, A., and Valtorta, D. Quantitative Reifenberg theorem for measures. *arXiv preprint 1612.08052* (2016).

[10] Edelen, N., Naber, A., and Valtorta, D. Effective Reifenberg theorems in Hilbert and Banach spaces. *arXiv preprint 1806.01250* (2018).

[11] Eells, J., and Lemaire, L. A report on harmonic maps. *Bull. London Math. Soc. 10*, 1 (1978), 1–68.

[12] Eells, J., and Lemaire, L. Another report on harmonic maps. *Bull. London Math. Soc. 20*, 5 (1988), 385–524.

[13] Federer, H. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[14] Hardt, R., Kinderlehrer, D., and Lin, F.-H. Existence and partial regularity of static liquid crystal configurations. *Comm. Math. Phys. 105*, 4 (1986), 547–570.

[15] Hardt, R., Kinderlehrer, D., and Lin, F.-H. Stable defects of minimizers of constrained variational principles. *Ann. Inst. H. Poincaré Anal. Non Linéaire 5*, 4 (1988), 297–322.

[16] Hardt, R., and Lin, F.-H. A remark on H^1 mappings. *Manuscripta Math. 56*, 1 (1986), 1–10.

[17] Hardt, R., and Lin, F.-H. Mappings minimizing the L^p norm of the gradient. *Comm. Pure Appl. Math. 40*, 5 (1987), 555–588.

[18] Hardt, R., and Lin, F.-H. Stability of singularities of minimizing harmonic maps. *J. Differential Geom. 29*, 1 (1989), 113–123.

[19] Hardt, R., and Lin, F.-H. The singular set of an energy minimizing map from B^4 to S^2. *Manuscripta Math. 69*, 3 (1990), 275–289.

[20] Hélein, F. Regularity of weakly harmonic maps from a surface into a manifold with symmetries. *Manuscripta Math. 70*, 2 (1991), 203–218.

[21] Hélein, F. *Harmonic maps, conservation laws and moving frames*, second ed., vol. 150 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 2002. Translated from the 1996 French original, With a foreword by James Eells.

[22] Jiang, W., and Naber, A. L^2 curvature bounds on manifolds with bounded Ricci curvature. *arXiv preprint 1605.05583* (2016).

[23] Jones, P. W. Rectifiable sets and the traveling salesman problem. *Invent. Math. 102*, 1 (1990), 1–15.

[24] Li, S. Stability of minimising harmonic maps under $W^{1,p}$ perturbations of boundary data: $p \geq 2$. *arXiv preprint 1810.10599* (2018).

[25] Lin, F.-H. A remark on the map $x/|x|$. *C. R. Acad. Sci. Paris Sér. I Math. 305*, 12 (1987), 529–531.

[26] Lin, F. H., and Wang, C. Y. Stable stationary harmonic maps to spheres. *Acta Math. Sin. (Engl. Ser.) 22*, 2 (2006), 319–330.

[27] Luckhaus, S. Partial Hölder continuity for minima of certain energies among maps into a Riemannian manifold. *Indiana Univ. Math. J. 37*, 2 (1988), 349–367.

[28] Mattila, P. *Geometry of sets and measures in Euclidean spaces*, vol. 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[29] Mazowiecka, K., Miśkiewicz, M., and Schikorra, A. On the size of the singular set of minimizing harmonic maps into the sphere in dimension three. *arXiv preprint 1811.00515* (2018).

[30] Mazowiecka, K., Miśkiewicz, M., and Schikorra, A. On the size of the singular set of minimizing harmonic maps into the 2-sphere in dimension four and higher. *arXiv preprint 1902.03161* (2019).

[31] Mazowiecka, K., and Strzelecki, P. The Lavrentiev gap phenomenon for harmonic maps into spheres holds on a dense set of zero degree boundary data. *Adv. Calc. Var.* (2016).

[32] Miśkiewicz, M. Discrete Reifenberg-type theorem. *Ann. Acad. Sci. Fenn. Math. 43*, 1 (2018), 3–19.

[33] Miśkiewicz, M. On Hölder regularity of the singular set of energy minimizing harmonic maps into closed manifolds. *arXiv preprint 1806.08607* (2018).

[34] Mou, L. H. Uniqueness of energy minimizing maps for almost all smooth boundary data. *Indiana Univ. Math. J. 40*, 1 (1991), 363–392.

[35] Naber, A. Lecture notes on Rectifiable Reifenberg for measures. *arXiv preprint 1812.07564* (2018).

[36] Naber, A., and Valtorta, D. The singular structure and regularity of stationary and minimizing varifolds. *arXiv preprint 1505.03428, to appear in J. Eur. Math. Soc.* (2015).

[37] Naber, A., and Valtorta, D. Rectifiable-Reifenberg and the regularity of stationary and minimizing harmonic maps. *Ann. of Math. (2) 185*, 1 (2017), 131–227.

[38] Naber, A., and Valtorta, D. Stratification for the singular set of approximate harmonic maps. *Math. Z. 290*, 3-4 (2018), 1415–1455.

[39] Nakajima, T. Singular points of harmonic maps from 4-dimensional domains into 3-spheres. *Duke Math. J. 132*, 3 (2006), 531–543.

[40] Reifenberg, E. R. Solution of the Plateau Problem for m-dimensional surfaces of varying topological type. *Acta Math. 104* (1960), 1–92.

[41] Rivière, T. Everywhere discontinuous harmonic maps into spheres. *Acta Math. 175*, 2 (1995), 197–226.

[42] Sacks, J., and Uhlenbeck, K. The existence of minimal immersions of 2-spheres. *Ann. of Math. (2) 113*, 1 (1981), 1–24.

[43] Schoen, R., and Uhlenbeck, K. A regularity theory for harmonic maps. *J. Differential Geom. 17*, 2 (1982), 307–335.

[44] Schoen, R., and Uhlenbeck, K. Boundary regularity and the Dirichlet problem for harmonic maps. *J. Differential Geom. 18*, 2 (1983), 253–268.

[45] Schoen, R., and Uhlenbeck, K. Correction to: "A regularity theory for harmonic maps". *J. Differential Geom. 18*, 2 (1983), 329.

[46] Simon, L. Reifenberg's Topological Disc Theorem. Available at math.uni-tuebingen.de/ab/analysis/pub/leon/reifenberg/reifenberg.html.

[47] Simon, L. Rectifiability of the singular set of energy minimizing maps. *Calc. Var. Partial Differential Equations 3*, 1 (1995), 1–65.

[48] Simon, L. *Theorems on regularity and singularity of energy minimizing maps.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1996. Based on lecture notes by Norbert Hungerbühler.

[49] Struwe, M. On the evolution of harmonic mappings of Riemannian surfaces. *Comment. Math. Helv. 60*, 4 (1985), 558–581.

[50] Tolsa, X. Characterization of n-rectifiability in terms of Jones' square function: part I. *Calc. Var. Partial Differential Equations 54*, 4 (2015), 3643–3665.

[51] Villani, C. *Optimal transport*, vol. 338 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 2009. Old and new.