

Problem 1. Find the image of the exponential map $\exp : \mathfrak{sl}_2(\mathbb{R}) \rightarrow \mathrm{SL}_n(\mathbb{R})$.

Problem 2. (a) Let H be the group of upper-triangular matrices:

$$H = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\},$$

with the group action being matrix multiplication. Compute the Lie algebra T_1H .

(b) Let $\mathfrak{h} \subseteq \mathrm{End}(C^\infty(\mathbb{R}))$ be the Lie algebra generated by the following two operators

$$X(f) = \frac{d}{dt}f, \quad Y(f) = tf.$$

In other words, \mathfrak{h} is the smallest Lie algebra inside $\mathrm{End}(C^\infty(\mathbb{R}))$ that contains both X and Y , with the Lie bracket being the standard commutator:

$$[A, B] = A \circ B - B \circ A.$$

Prove that $\mathfrak{h} \cong T_1H$ as Lie algebras.

(c) Let $\mathrm{Ad} : H \rightarrow \mathrm{GL}(\mathfrak{h})$ be the adjoint representation. Is it a direct sum of its proper subrepresentations?

Definition. Let $k \in \{\mathbb{R}, \mathbb{C}\}$ and V be a vector space over k . Define $k[V]$ to be the algebra of polynomial functions on V , i.e. the algebra generated by linear functions $V \rightarrow k$ inside the algebra of all functions $V \rightarrow k$. Additionally, let $T(V^\vee)$ be the tensor algebra:

$$T(V^\vee) := k \oplus V^\vee \oplus (V^\vee \otimes V^\vee) \oplus (V^\vee \otimes V^\vee \otimes V^\vee) \oplus \cdots = \bigoplus_{n \geq 0} T(V^\vee)_n.$$

If V is a representation of G , then $T(V^\vee)$ has a structure of a G -representation as a direct sum of tensor products of representations. Likewise, $k[V]$ is a graded vector space:

$$k[V] = \bigoplus_{n \geq 0} k[V]_n,$$

where $k[V]_n$ is the linear subspace spanned by products of exactly n linear functionals. Again, if V is a G -representation, then both $k[V]_n$ and $k[V]$ have a natural structure of a G -representation by applying the G -action on each functional individually.

Recall the definitions of $\mathrm{Sym}^n(-)$ and $\Lambda^n(-)$:

$$\mathrm{Sym}^n(V) := (V^{\otimes n})^{\Sigma_n} \subseteq V^{\otimes n},$$

where here Σ_n acts on $V^{\otimes n}$ by permutation of factors and

$$\Lambda^n(V) := (V^{\otimes n})^{\Sigma_n} \subseteq V^{\otimes n},$$

where Σ_n acts by permutation times the sign.

Problem 3. Let V be a representation of a Lie group G over k .

- (a) Prove that $k[V]$ is a quotient of $T(V^\vee)$ with the kernel being the ideal generated by $v \otimes w - w \otimes v$.
- (b) Prove that we have an isomorphism of representations of G

$$k[V]_n \cong \text{Sym}^n(V^\vee),$$

Prove that, in a similar vein, $\Lambda^n(V^\vee)$ is a quotient of $T(V^\vee)$. What is the kernel?

Problem 4. (a) Let V be a representation of a compact Lie group G . Prove that $V \otimes V$ decomposes as a direct sum of $\text{Sym}^2(V)$ and $\Lambda^2 V$.

(b) Can we decompose $V \otimes V \otimes V$ as a sum of $\text{Sym}^3 V$ and $\Lambda^3 V$?

(c) Is this decomposition of $V \otimes V$ a decomposition into irreducible representations?

Problem 5. Let G be compact Lie group. Prove that the category of finite-dimensional representations of G enjoys the following cancellation property:

$$V_1 \oplus W \cong V_2 \oplus W \implies V_1 \cong V_2.$$

Problem 6. Prove that if G is an abelian compact Lie group, then all irreducible representations over \mathbb{C} are one-dimensional. Find the set of irreducible representations for S^1 .