

Definition. Let $f: X \rightarrow Y$ be a surjective continuous map. We say that f is a **covering map** if for every $y \in Y$ there exists an open subset $U \subseteq_{op} Y$ such that the preimage $f^{-1}(U)$ decomposes as a disjoint union of open subsets $\{U_i \subseteq_{op} X\}_{i \in I}$:

$$f^{-1}(U) = \bigsqcup_{i \in I} U_i$$

and for each i the restriction $f|_{U_i}: U_i \rightarrow f^{-1}(U)$ is a homeomorphism.

If for every $y \in Y$ we can find neighbourhoods $U_y \subseteq_{op} Y$ as above with the indexing set I_y being a set of n elements (with n independent of y), then we say that $f: X \rightarrow Y$ is an **n -sheeted covering**.

If f is smooth and the restrictions $f|_{U_i}: U_i \rightarrow f^{-1}(U)$ are diffeomorphism, then we say that f is a **smooth covering**.

Problem 1. Let $f: G_1 \rightarrow G_2$ be a morphism of connected Lie groups such that $d_1 f: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is an isomorphism. Prove that f is a covering with $\ker f$ being discrete and central.

Problem 2. Let $f: G_1 \rightarrow G_2$ be a morphism of connected Lie groups such that f is bijective and $\dim G_1 = \dim G_2$. Prove that f is an isomorphism.

Remark. The assumption $\dim G_1 = \dim G_2$ is *not* necessary.

Problem 3. We define the following complex matrix

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \in GL_{2n}(\mathbb{C}).$$

There is a \mathbb{C} -action on \mathbb{H} where we multiply a quaternion q by a complex number z on the right, identifying the complex i with the quaternion i . This gives rise to a \mathbb{C} -linear isomorphism $\mathbb{H} \cong \mathbb{C}^2$ which maps

$$\mathbb{C}^2 \ni (z_1, z_2) \mapsto z_1 + jz_2.$$

This isomorphism lifts to an isomorphism $\mathbb{H}^n \cong \mathbb{C}^{2n}$ and to matrices:

$$M_{n \times n}(\mathbb{H}) = M_{n \times n}(\mathbb{C}) \oplus jM_{n \times n}(\mathbb{C}).$$

(a) Let $A \in GL_n(\mathbb{H})$ be an invertible matrix with quaternion entries. Find the matrix that corresponds to A considered as a \mathbb{C} -linear map $\mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$, $v \mapsto Av$.

(b) Prove that the same isomorphism lifts to an isomorphism of Lie groups

$$GL_n(\mathbb{H}) \cong GL_{2n}^J(\mathbb{C}) := \{A \in GL_{2n}(\mathbb{C}) : JAJ^{-1} = \bar{A}\}.$$

(c) Let $Sp_n(\mathbb{C}) \subseteq GL_{2n}(\mathbb{C})$ be the subgroup of matrices such that

$$Sp_n(\mathbb{C}) := \{A \in GL_{2n}(\mathbb{C}) : A^T J A = J\}.$$

Prove that $GL_{2n}^J(\mathbb{C}) \cap U(2n) = Sp_n(\mathbb{C}) \cap U(2n)$.

Definition. Let M be a smooth manifold. A **flow** on M is a smooth map $F: \mathbb{R} \times M \rightarrow M$ such that

(i) For any $m \in M$ we have $F(0, m) = m$.

(ii) For all $m \in M, s, t \in \mathbb{R}$ we have $F(s + t, m) = F(s, F(t, m))$.

A smooth map $F: U \rightarrow M$ defined on $\{0\} \times M \subseteq U \subseteq_{op} \mathbb{R} \times M$ such that the above conditions hold whenever everything is well-defined, will be called a **local flow**.

For any local flow $F: D \rightarrow M$ we can associate a vector field $X_F \in \mathfrak{X}(M)$ defined by

$$X_F(m) := d_{(0,m)}F(1, 0) \in T_m M.$$

Problem 4. (a) Let $k \in \{\mathbb{R}, \mathbb{C}\}$ be one of our main fields of interest and $x \in \mathfrak{gl}_n(k)$ be a matrix over k . Prove that the map $\phi: \mathbb{R} \times GL_n(k) \rightarrow GL_n(k)$ defined by

$$\phi(t, A) = A \exp(tx)$$

is a flow on $GL_n(k)$. What is the associated vector field?

(b) Let $X \in \mathfrak{X}(GL_n(k))$ be the vector field defined by x , i.e. $X(A) = (L_A)_* x := d_L L_A x$, where $L_A: GL_n(k) \rightarrow GL_n(k)$ is multiplication by A on the left hand side. Prove that this vector field is left-invariant, i.e. for all $A, B \in GL_n(k)$ we have

$$(L_A^! X)(B) := d_{L_A B} L_A^{-1} X(L_A B) = X(B).$$

What is the exact formula for X ?

(c) Let $F: \mathbb{R} \times GL_n(k) \rightarrow GL_n(k)$ be a flow such that $X_F = X$ with X coming from (b). Prove that

$$F(t, A) = A \exp(tx).$$

Hint. Try to produce differential equations.