Problem 1. Compute the the tangent spaces (at the identity matrix) to the classical matrix groups:

(a) $T_{id}U(n) =: \mathfrak{u}_n$ and $T_{id}SU(n) =: \mathfrak{su}_n$,

(b) $T_{id}O(n) =: \mathfrak{o}_n$ and $T_{id}SO(n) =: \mathfrak{so}_n$.

Show that in all cases the tangent spaces are closed under the commutator $[A, B] := AB - BA$.

Solution:

(a) Consider the exponential $\exp : M_{n\times n}(\mathbb{C}) \to GL_n(\mathbb{C})$. If $X + X^* = 0$, then tX commutes with tX^* and thus
$$I = \exp(tX + tX^*) = \exp(tX)(\exp(tX))^*.$$
So the map $t \mapsto \exp(tX)$ is a curve in $U(n)$ whos derivative at $t = 0$ is X. That is, $X \in T_I U(n)$. Conversely, suppose that we have a tangent vector $X \in T_I U(n)$, i.e. a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to U(n)$ such that $\gamma(0) = I, \gamma'(0) = X$. That is, we have
$$\gamma(t)\gamma(t)^* = I.$$
Differentiating at $t = 0$ we yield
$$0 = \gamma(0)\gamma'(0)^* + \gamma'(0)\gamma(0)^* = X^* + X,$$
so we get the desired description of the tangent space:
$$\mathfrak{u}_n = \{X \in M_{n\times n} : X + X^* = 0\}.$$
As was noted during the lectures, $\det(\exp(tX)) = 1$ if and only if $\mathrm{tr}(X) = 0$, so we also get the tangent space of $SU(n)$:
$$\mathfrak{su}_n = \{A \in M_{n\times n}(\mathbb{C}) : A + A^* = 0, \mathrm{tr}(A) = 0\}.$$
In fact in a matrix group $G \subseteq GL_n(\mathbb{C})$ any tangent vector $X \in T_I G \subseteq M_{n\times n}(\mathbb{C})$ corresponds to a curve of the form $t \mapsto \exp(tX)$. We will prove this in the future.

Alternative solution. Let $F : M_{n\times n}(\mathbb{C}) \to \{A \in M_{n\times n}(\mathbb{C}) : A = A^*\}$ be the map $F(A) = AA^*$, so that $U(n) = F^{-1}(I)$. Observe that, for $A \in U(n)$, $dF_A : M_{n\times n}(\mathbb{C}) \to \{A \in M_{n\times n}(\mathbb{C}) : A = A^*\}$ can be computed by looking at $F(A + H)$ for some small H:
$$F(A + H) = I + HA^* + AH^* + HH^*$$
so $dF_A(H) = HA^* + AH^*$. In particular I is a regular value – for any hermitian symmetric W we can pick $H = WA/2$ and one can check that $dF_A(WA/2) = W$. As I is a regular value for F, then we can compute the tangent space of $U(n)$ as the kernel of the differential:
$$T_I U(n) = \ker dF_I = \{A \in M_{n\times n}(\mathbb{C}) : A + A^* = 0\}.$$
Next, observe that for $\det : U(n) \to S^1$, 1 is a regular value – let $A \in SU(n)$, then multiplication $A\cdot : U(n) \to U(n)$ is a diffeomorphism; coupling this with the fact that $T_I U(n)$ is given by curves $\exp(tX)$ with $X + X^* = 0$ we can compute $T_A U(n)$ – any element of this tangent space is a curve $A\exp(tX)$. Observe that $d_A \det : T_A U(n) \to T_1 S^1$ takes a curve $A\exp(tX)$

and computes its determinant $\det(A)\det(\exp(tX)) = \det(\exp(tX)) \in S^1$. We can therefore compute directly the derivative at $t = 0$:

$$\frac{d}{dt}\det\exp(tX)|_{t=0} = \frac{d}{dt}e^{t\operatorname{tr}X}|_{t=0} = \operatorname{tr}(X).$$

Note that $\operatorname{tr}(X) \in \mathbb{C}$ is purely imaginary from our assumptions, i.e. in T_1S^1. Next also note that we can always find X such that $X + X^* = 0$ and $\operatorname{tr}(X)$ is any purely imaginary number, i.e. $d_A \det$ is surjective. Thus, $T_I SU(n)$ can be computed as the kernel of the derivative of det:

$$T_I SU(n) = \ker\operatorname{tr} \subseteq \mathfrak{u}_n = \{X \in M_{n\times n}(\mathbb{C}) : X + X^* = 0, \operatorname{tr}X = 0\}.$$

Finally, suppose that we have $A, B \in \mathfrak{su}_n$ and notice that

$$\begin{aligned}&[A,B] + [A,B]^* = AB - BA + B^*A^* - A^*B^* = 0,\\ &\operatorname{tr}([A,B]) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = 0.\end{aligned}$$

Thus $\mathfrak{u}_n, \mathfrak{su}_n$ are closed under the commutator.

(b) The case of $O(n)$ is completely analogous to $U(n)$, we just omit putting conjugation bars and the same argument holds, i.e. we have

$$T_I O(n) = \ker(H \mapsto H + H^T) = \{A \in M_{n\times n}(\mathbb{R}) : A + A^T = 0\}.$$

Here the case of $SO(n)$ is even simpler; notice that

$$O(n) = (\det{}^{-1}(-1) \cap O(n)) \sqcup (\det{}^{-1}(1) \cap O(n)),$$

so $SO(n) = \det^{-1}(1) \cap O(n) \subseteq_{op} O(n)$ is an open subset of $O(n)$ which contains I. In particular the tangent space of an open subset is the same as for the ambient space:

$$T_I SO(n) = T_I O(n) = \{A \in M_{n\times n}(\mathbb{R}) : A + A^T = 0\}.$$

The check that such a set is closed under the commutator is also analogous to the (a) case.

Problem 2. For any $q \in \mathbb{H} \setminus \{0\}$ we define the conjugation action $\varphi_q : \mathbb{H} \to \mathbb{H}$:

$$\varphi_q(x) = qxq^{-1}.$$

Prove that φ_q is in fact an isometry (under the usual scalar product on $\mathbb{H} \cong \mathbb{R}^4$) and the space of pure quaternions is φ_q-invariant.

Solution:

We prove both of these facts using properties of quaternions. First, note that

$$|\varphi_q(x)| = |qxq^{-1}| = |q||x||q^{-1}| = |x|,$$

so φ_q is an isometry. Next observe that $x \in \mathbb{H}$ is a pure quaternion if and only if $\bar{x} = -x$. Now observe that for any $q \in \mathbb{H} \setminus \{0\}$ and $x \in \operatorname{im} \mathbb{H}$ we have

$$\overline{qxq^{-1}} = \overline{qx\frac{\bar{q}}{|q|^2}} = q\bar{x}\frac{\bar{q}}{|q|^2} = -qxq^{-1}.$$

Problem 3. Let $\theta \in \mathbb{R}$ be an angle, $v = (x, y, z) \in S^2 \subseteq \mathbb{R}^3$ be a unit vector and define

$$q_{\theta,v} = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)(ix + jy + kz) \in S^3 \subseteq \mathbb{H}.$$

Prove that $\varphi_{q_{\theta,v}} : \operatorname{im}\mathbb{H} \to \operatorname{im}\mathbb{H}$, after we identify $\operatorname{im}\mathbb{H} \cong \mathbb{R}^3$, is a rotation by θ with respect to the axis spanned by v.

Solution:

As we have proven in Problem 2, $\varphi_{q_{\theta,v}}$ is an isometry, so to prove that it is *a* rotation we just have to show that $\det(\varphi_{q_{\theta,v}}) = 1$. To see that it is a rotation around the line spanned by v, observe that the quaternion $v = ix + jy + kz$ is an eigenvector of this map with eigenvalue 1:

$$q_{\theta,v} v \overline{q_{\theta,v}} = \left(\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) v\right) v \left(\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) \overline{v}\right) =$$
$$= \left(\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) v\right)\left(\cos\left(\frac{\theta}{2}\right) v + \sin\left(\frac{\theta}{2}\right)\right) = v.$$

To talk about the determinant, we will observe whether -1 can be an eigenvalue for $\varphi_{q_{\theta,v}}$. A direct computation shows that a quaternion $ia + jb + kc$ is an eigenvector of $\varphi_{q_{\theta,v}}$ with eigenvalue -1 if and only if the following equations hold

$$\begin{cases} ax + by + cz = 0, \\ a\cos\left(\frac{\theta}{2}\right) = 0, \\ b\cos\left(\frac{\theta}{2}\right) = 0, \\ c\cos\left(\frac{\theta}{2}\right) = 0. \end{cases}$$

Thus either -1 is not an eigenvalue, or we have two distinct eigenvectors which correspond to the basis of $v^\perp$. This should be expected as $\cos\left(\frac{\theta}{2}\right) = 0$ should be the case of flipping the space $v^\perp$ around the origin, i.e. we should get two eigenvectors with eigenvalue -1. In both cases we get that $\det(\varphi_{q_{\theta,v}}) = 1$, so $\varphi_{q_{\theta,v}}$ is a rotation around the axis spanned by v. What about the angle? For a fixed $v \in S^2$ we have a map $\mathbb{R} \to SO(3)$ that maps $\theta \mapsto \varphi_{q_{\theta,v}}$. It is quite easy to see that the resulting matrix of $\varphi_{q_{\theta,v}}$ has entries which are polynomials in $\cos(\theta/2), \sin(\theta/2)$ and entries of v, i.e. the assignment $\mathbb{R} \to SO(3)$ is continuous and in fact it lands in $S^1 \subseteq SO(3)$ that is the subgroup of rotations around the span of v. It has period at most 2π, since $\varphi_{2k\pi,v} = \mathrm{id}$. In particular it descends to a continuous map $S^1 \to S^1$ and even more is true – note that if $\theta, \theta' \in \mathbb{R}$ are two angles, then we have

$$q_{\theta,v} q_{\theta',v} = q_{\theta+\theta',v}.$$

This relation lifts to a relation $\varphi_{q_{\theta,v}} \varphi_{q_{\theta',v}} = \varphi_{q_{\theta+\theta',v}}$ which translates to saying that our continuous map $S^1 \to S^1$ is a homomorphism of groups. But a continuous automorphism of S^1 is necessarily of the form $\mathbb{R}/2\pi\mathbb{Z} \ni \theta \mapsto k\theta \in \mathbb{R}/2\pi\mathbb{Z}$ for some integer $k \in \mathbb{Z}$. Similar as for finding eigenvectors with eigenvalue -1, we can look at eigenvectors with eigenvalue 1 and a direct computation shows that $ia + jb + kc$ is an eigenvector with eigenvalue 1 if and only if the following system of equations is satisfied

$$\begin{cases} \sin\left(\frac{\theta}{2}\right)(cy - bz) = 0, \\ \sin\left(\frac{\theta}{2}\right)(az - cx) = 0, \\ \sin\left(\frac{\theta}{2}\right)(bx - ay) = 0. \end{cases}$$

Thus, either $\theta = 2k\pi$ for some $k \in \mathbb{Z}$, and we trivially get $\varphi_{q_{2k\pi,v}} = \mathrm{id}$, or θ is not of the form $2k\pi$ and then we get a system of equations that is satisfied if and only if $(a, b, c) \in \mathrm{span}(v)$, i.e. we will have only one eigenvector with eigenvalue 1. Therefore $\theta \mapsto \varphi_{q_{\theta,v}} : S^1 \to S^1$ is injective, meaning that it must be a rotation around v by $\pm\theta$. The choice of the sign must be independent of v as $\mathbb{R}^3 \times S^1 \ni (v, \theta) \mapsto \varphi_{q_{\theta,v}} \in SO(3)$ is continuous, so it induces a continuous map $\mathbb{R}^3 \to \mathrm{Aut}(S^1)$ with image inside the discrete space $\{-1, 1\}$.

Alternative solution. Consider the map $\mathbb{R} \times S^2 \ni (\theta, v) \mapsto \varphi_{q_{\theta,v}} \in O(3)$ – we know that $\varphi_{q_{\theta,v}}$ is an isometry, so it lands in $O(3)$. This map is continuous as entries of $\varphi_{q_{\theta,v}}$ are continuous in θ and entries of v. Taking the determinant we therefore get a continuous map $\mathbb{R} \times S^2 \ni (\theta, v) \mapsto \det(\varphi_{q_{\theta,v}}) \in \{-1, 1\}$, so the determinant is either only -1 or 1 independently of θ, v, but $\varphi_{q_{0,v}} = \mathrm{id}$, so the determinant is always 1. By a direct computation we can see the following three properties:

$$\begin{aligned}
&\varphi_{q_{\theta,v}}(v) = v, \\
&\varphi_{q_{\theta,v}}\varphi_{q_{\theta',v}} = \varphi_{q_{\theta+\theta',v}}, \\
&\varphi_{q_{2\pi,v}} = \varphi_{q_{0,v}} = \mathrm{id}.
\end{aligned}$$

The first property implies that if we fix v, then the map $\mathbb{R} \ni \theta \mapsto \varphi_{q_{\theta,v}} \in SO(3)$ lands in a circle of rotations around v and second and third properties mean that the same map is a homomorphism of groups $\mathbb{R}/2\pi\mathbb{Z} = S^1 \to S^1 \subseteq SO(3)$. Note that this homomorphism is necessarily continuous as our initial mapping was. We will not prove this here, but the set of continuous homomorphisms $S^1 \to S^1$ is the discrete space $\mathbb{Z}$ of homomorphisms of the form $\mathbb{R}/2\pi\mathbb{Z} \ni \theta \mapsto k\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Thus we have a continuous map

$$S^2 \ni v \mapsto \mathrm{Hom}_{\mathrm{TopGroups}}(S^1, S^1) \cong \mathbb{Z}.$$

In other words, the degree of the the resulting map $S^1 \to S^1$ is independent of v. Thus we can check what is the degree on an easy example – let $v = i$. Then

$$\varphi_{q_{\theta,i}}(ia + jb + kc) = ia + (b\cos\theta - c\sin\theta)j + (c\cos\theta + b\sin\theta)k$$

and this is exactly the image of rotation of a vector (a, b, c) around $(1, 0, 0)$ by the angle θ. Thus any other pair (θ, v) produces a rotation which is around v and the same angle, i.e. by θ.

Problem 4. (a) Prove that $SU(2) \cong S^3 \subseteq \mathbb{R}^4$ as groups.

(b) Consider the map $S^3 \to SO(3)$ that comes from identifying unit quaternions $q \in S^3 \subseteq \mathbb{H}$ with isometries $\varphi_q : \operatorname{im}\mathbb{H} \to \operatorname{im}\mathbb{H}$. Prove that this assignment is $2:1$ and surjective.

(c) Prove that $SO(3) \cong \mathbb{RP}^3$ as topological spaces.

Solution:

(a) Recall the complex matrix representation of quaternions. It is an injection (of $\mathbb{R}$-algebras) $\mathbb{H} \to M_{2\times 2}(\mathbb{C})$ defined on the $\mathbb{R}$-basis of $\mathbb{H}$ as

$$1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, i \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, j \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, k \mapsto \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

The image of this map can be described as follows

$$\mathbb{H} \cong \left\{ \begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix} : z, w \in \mathbb{C} \right\}$$

and we can identify the unit sphere $S^3 \subseteq \mathbb{H}$ with the subgroup (of $GL_2(\mathbb{C})$):

$$S^3 \cong \left\{ \begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix} : z, w \in \mathbb{C}, |z|^2 + |w|^2 = 1 \right\}.$$

This isomorphism is a group isomorphism as our initial isomorphism $\mathbb{H} \to M_{2\times 2}(\mathbb{C})$ was multiplicative. The newly acquired equation implies that both the determinant of the matrix is 1 and the matrix is unitary. In fact, any matrix $A \in SU(2)$ is of this form; suppose that our matrix A has entries

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then stating $A \in SU(2)$ in particular means stating that

$$\begin{cases} ad - bc = 1, \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{bmatrix} = \begin{bmatrix} |a|^2 + |b|^2 & a\overline{c} + b\overline{d} \\ \overline{a}c + \overline{b}d & |c|^2 + |d|^2 \end{bmatrix} \end{cases}$$

The first row is already as we want it to be, we just have to show that $c = -\overline{b}$ and $d = \overline{a}$. We turn to the following equations

$$\begin{cases} ad - bc = 1, \\ \overline{a}c + \overline{b}d = 0. \end{cases}$$

Some routine row reduction yields the desired result. Thus our map $S^3 \to SU(2)$ is a group isomorphism.

(b) First notice that any unit quaternion $q \in S^3$ is of the form $q = q_{\theta,\nu}$ for some angle θ and unit vector $\nu \in S^2$. Let $\mathfrak{Im}(q)$ be the pure part of q and $\mathfrak{Re}(q)$ be the real part. We then have $(\mathfrak{Im}(q))^2 = -|\mathfrak{Im}(q)|^2$ and

$$1 = q\overline{q} = \mathfrak{Re}(q)^2 + |\mathfrak{Im}(q)|^2$$

So $\nu = \mathfrak{Im}(q)/|\mathfrak{Im}(q)|$ is our unit vector of choice and the angle is $\theta = 2\arccos(\mathfrak{Re}(q))$. So we can use Problem 3 and claim that our map $S^3 \to SO(3)$ has all rotations in its image, which is exactly $SO(3)$. Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be the clockwise rotation around ν (oriented to point 'up') by the angle θ. Then R is equal to only one other clockwise rotation – we could have rotated around $-\nu$ by the angle $2\pi - \theta$. Thus $\varphi_q = \varphi_{q'}$ implies that if

$$q = \cos(\theta/2) + \sin(\theta/2)(ix + jy + kz),$$

then the only possible choice for q' is by taking

$$q' = \cos\left(\frac{2\pi - \theta}{2}\right) + \sin\left(\frac{2\pi - \theta}{2}\right)(-ix - jy - kz) = -q,$$

which proves that the map is 2-to-1.

(c) Note that our map $S^3 \to SO(3)$ is a surjective group homomorphism. Being 2-to-1 means that the kernel is a group of order two (the kernel is just the quaternions ± 1), hence

$$SO(3) \cong S^3/(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{RP}^3.$$

Small caveat: through this method we only see that $SO(3) \cong \mathbb{RP}^3$ as groups without mentioning topology, but the map $S^3 \to SO(3)$ was smooth, hence the isomorphism $\mathbb{RP}^3 \to SO(3)$ is smooth. This is a general fact that in this scenario the inverse needs to be smooth as well.

Problem 5. Consider the matrix representations of quaternions, i.e. we set the basis quaternions i, j, k to be the following complex matrices:

$$i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, k = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

We define the quaternion exponential as the matrix exponential

$$\exp(q) = \sum_{n \geq 0} \frac{q^n}{n!},$$

Prove that for a pure, nonzero quaternion $q \in \operatorname{im} \mathbb{H}$ the following formula holds

$$\exp(q) = \cos(|q|) + \frac{q}{|q|} \sin(|q|).$$

Solution:

Let $q = ix + jy + kz$ be a pure quaternion. Its matrix form is the following

$$q \leftrightarrow \begin{bmatrix} ix & y + iz \\ -y + iz & -ix \end{bmatrix}$$

Notice that $q^2 = -|q|^2$ corresponds to the matrix $-|q|^2 I$. Therefore we can observe that

$$q^{2n} = (-1)^n |q|^{2n} I,$$
$$q^{2n+1} = (-1)^n |q|^{2n} q = (-1)^n |q|^{2n+1} \frac{q}{|q|}.$$

In the exponential $\exp(q)$ we can thus distinguish terms with q and without q, yielding the desired result (after recalling what are the Taylor series for sin and cos).

Problem 6. We define the Pauli matrices as

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Describe the correspondence between Pauli matrices and quaternions. Prove that we have

$$P \oplus \operatorname{im} \mathbb{H} = \mathfrak{sl}_2(\mathbb{C}),$$

where $P = \operatorname{span}_{\mathbb{R}}(\sigma_x, \sigma_y, \sigma_z)$.

Solution:

Note that the Pauli matrices correspond to 'rotated' quaternions. Namely, we have the following correspondences

$$i \leftrightarrow i\sigma_z, j \leftrightarrow i\sigma_y, k \leftrightarrow i\sigma_x.$$

Notice that $\sigma_x, \sigma_y, \sigma_z$ and the complex matrices coming from quaternions i, j, k are traceless, hence we have $P + \operatorname{im} \mathbb{H} \subseteq \mathfrak{sl}_2(\mathbb{C})$. One can check directly that $i, j, k, \sigma_x, \sigma_y, \sigma_z$ forms an independent set over $\mathbb{R}$. Thus $P \oplus \operatorname{im} \mathbb{H} \subseteq \mathfrak{sl}_2(\mathbb{C})$ and computing dimensions of both spaces we get equality.