Definition. *Recall the classical matrix groups:*

$$U(n) := \{A \in M_{n\times n}(\mathbb{C}) : A\overline{A^T} = \mathrm{id}\}, \qquad SU(n) := \{A \in U(n) : \det(A) = 1\},$$
$$O(n) := \{A \in M_{n\times n}(\mathbb{R}) : AA^T = \mathrm{id}\}, \qquad SO(n) := \{A \in O(n) : \det(A) = 1\}.$$

Problem 1. Compute the the tangent spaces (at the identity matrix) to the classical matrix groups:

(a) $T_{\mathrm{id}}U(n) =: \mathfrak{u}_n$ and $T_{\mathrm{id}}SU(n) =: \mathfrak{su}_n$,

(b) $T_{\mathrm{id}}O(n) =: \mathfrak{o}_n$ and $T_{\mathrm{id}}SO(n) =: \mathfrak{so}_n$.

Show that in all cases the tangent spaces are closed under the commutator $[A, B] := AB - BA$.

Definition. *We denote by* $\mathbb{H}$ *the quaternion algebra, i.e.* $\mathbb{H} = \mathrm{span}_{\mathbb{R}}(1, i, j, k)$ *as an* $\mathbb{R}$*-vector space and multiplication is given by*

$$i^2 = j^2 = k^2 = ijk = -1,$$
$$ij = k, jk = i, ki = j.$$

The subspace im $\mathbb{H} = \mathrm{span}(i, j, k)$ *is called the space of pure quaternions.*

Problem 2. For any $q \in \mathbb{H} \setminus \{0\}$ we define the conjugation action $\varphi_q : \mathbb{H} \to \mathbb{H}$:

$$\varphi_q(x) = qxq^{-1}.$$

Prove that φ_q is in fact an isometry (under the usual scalar product on $\mathbb{H} \cong \mathbb{R}^4$) and the space of pure quaternions is φ_q-invariant.

Problem 3. Let $\theta \in \mathbb{R}$ be an angle, $v = (x, y, z) \in S^2 \subseteq \mathbb{R}^3$ be a unit vector and define

$$q_{\theta,v} = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)(ix + jy + kz) \in S^3 \subseteq \mathbb{H}.$$

Prove that $\varphi_{q_{\theta,v}} : \mathrm{im}\,\mathbb{H} \to \mathrm{im}\,\mathbb{H}$, after we identify $\mathrm{im}\,\mathbb{H} \cong \mathbb{R}^3$, is a rotation by θ with respect to the axis spanned by v.

Problem 4. (a) Prove that $SU(2) \cong S^3 \subseteq \mathbb{R}^3$ as groups.

(b) Consider the map $S^3 \to SO(3)$ that comes from identifying unit quaternions $q \in S^3 \subseteq \mathbb{H}$ with isometries $\varphi_q : \mathrm{im}\,\mathbb{H} \to \mathrm{im}\,\mathbb{H}$. Prove that this assignment is 2 : 1 and surjective.

(c) Prove that $SO(3) \cong \mathbb{RP}^3$ as topological spaces.

Problem 5. Consider the matrix representations of quaternions, i.e. we set the basis quaternions i, j, k to be the following complex matrices:

$$i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, k = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

We define the quaternion exponential as the matrix exponential

$$\exp(q) = \sum_{n\geq 0} \frac{q^n}{n!},$$

Prove that for a pure, nonzero quaternion $q \in \mathrm{im}\,\mathbb{H}$ the following formula holds

$$\exp(q) = \cos(|q|) + \frac{q}{|q|}\sin(|q|).$$

Problem 6. We define the Pauli matrices as

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Describe the correspondence between Pauli matrices and quaternions. Prove that we have

$$P \oplus \operatorname{im} \mathbb{H} = \mathfrak{sl}_2(\mathbb{C}),$$

where $P = \operatorname{span}_{\mathbb{R}}(\sigma_x, \sigma_y, \sigma_z)$.