**Problem 1.** Let G be a group and a topological space. Prove that G is a topological group if and only if the following map

$$\phi\colon G \times G \to G \times G, \qquad \phi(g, h) := (g, gh)$$

is a homeomorphism.

*Solution:*

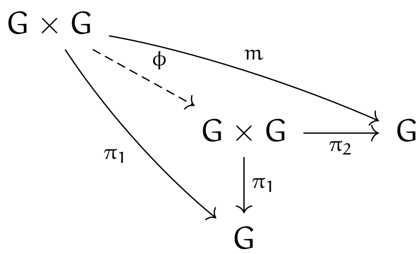
Let $\pi_1, \pi_2 : G \times G \to G$ be the canonical projections from $G \times G$ to the first and second coordinate respectively. By definition of product topology both of these maps are continuous. Denote the multiplication on G by $m : G \times G \to G$ and the inversion by $i : G \times G$. The map $\phi^{-1} : G \times G \to G \times G$ is given by the following formula

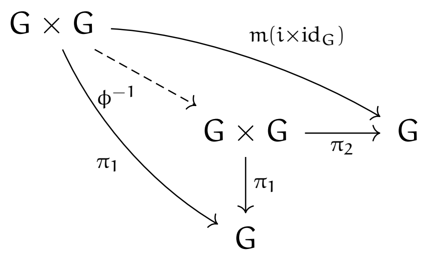
$$\phi^{-1}(g, h) = (g, g^{-1}h).$$

Lastly, we can define a section of $\pi_2$ using the identity element. Denote this section by $\iota : G \to G \times G, \iota(g) := (g, 1)$. It also requires a check that $\iota$ is continuous, but this is true as the product topology is generated using products of two open subsets. Observe that we have the following relations

$$m = \pi_2\phi, \qquad i = \pi_2\phi^{-1}\iota.$$

Thus continuity of $\phi$ and $\phi^{-1}$ implies continuity of $m$ and $i$, i.e. if $\phi$ is a homeomorphism, then G is a topological group. On the other hand, if $m$ and $i$ are continuous, then $\phi$ and $\phi^{-1}$ are the unique maps making the following two diagrams commute






As all arrows in both diagrams are continuous, then by the universal property of products (in topological spaces), both maps $\phi$ and $\phi^{-1}$ are necessarily continuous.

**Problem 2.** Let $H \leq G$ be a subgroup of a topological group G. Recall that we endow $G/H$ with the quotient topology given by the quotient map $\pi\colon G \to G/H$. Prove that

(a) The group action map

$$G \times G/H \to G/H \qquad (g, hH) \mapsto ghH$$

is continuous.

(b) The quotient map $\pi\colon G \to G/H$ is open.

(c) The closure $\overline{H}$ is a subgroup of G.

(d) If $H$ is open in $G$, then $G/H$ is a discrete topological space.

*Solution:*

(a) Let $\rho : G \times G/H \to G/H$ be the action map and $m : G \times G \to G$ be the multiplication. Consider the group $G \times G$ and its subgroup

$$(1, H) := \{(1, h) : h \in H\}.$$

Then $G \times G/H$ is homeomorphic to $(G \times G)/(1, H)$ via the map $(g, hH) \mapsto (g, h)(1, H)$. To prove that $\rho$ is continuous it is enough to prove that $\pi m : G \times G \to G/H$ is continuous (it is) and that $\pi m(1, H) = \{1\}$ and this is also true. Thus $\pi m$ descends to a map $(G \times G)/(1, H) \to G/H$ which we can then turn into a map $G \times G/H \to G/H$.

(b) Let $V \subseteq G$ be open. By definition of the quotient topology on $G/H$, the image $\pi(V)$ would be open if and only if $\pi^{-1}\pi(V)$ would be open. But the former set has the following form

$$\pi^{-1}\pi(V) = \{g \in G : gH \in \pi(V)\} = \bigcup_{h \in H} Vh.$$

From (a) we know that multiplication by $h$ induces a homeomorphism, hence all $Vh$ are open as well. Thus $\pi^{-1}\pi(V)$ is open.

(c) Recall that a function $f : X \to Y$ is continuous if and only if for every $Z \subseteq X$ we have

$$f(\overline{Z}) \subseteq \overline{f(Z)}.$$

In our case we have continuous maps $m : G \times G \to G$ and $i : G \to G$ and they satisfy $m(H \times H) \subseteq H, i(H) \subseteq H$. Thus

$$m(\overline{H} \times \overline{H}) \subseteq m(\overline{H \times H}) \subseteq \overline{m(H \times H)} \subseteq \overline{H}, \qquad i(\overline{H}) \subseteq \overline{i(H)} \subseteq \overline{H}.$$

Meaning that $\overline{H}$ is closed under multiplication and inversion.

(d) First, note that $\{H\}$ is an open point in $G/H$ as its preimage under $\pi$ is exactly the open subgroup $H$. From (a) we can therefore conclude that any other point is open as $G$ acts on $G/H$ via homeomorphisms. Thus $G/H$ is necessarily discrete.

**Problem 3.** Let $X$ be a topological space and $Y \subseteq X$ be a connected component. Prove that $Y$ is closed in $X$.

*Solution:*

It is enough to prove that $\overline{Y}$ is connected assuming that $Y$ was. Suppose that $\overline{Y} = U_1 \cup U_2$ with $U_1, U_2 \subseteq_{op} \overline{Y}$ such that $U_1 \cap U_2 = \emptyset$. In particular we get

$$Y = (U_1 \cap Y) \cup (U_2 \cap Y), (U_1 \cap Y) \cap (U_2 \cap Y) = \emptyset,$$

which implies that $U_i \cap Y = Y$ for at least one $i$, we can assume without loss of generality that $U_1 \cap Y = Y, U_2 \cap Y = \emptyset$. But this implies that $U_2$ has no intersection with $\overline{Y}$, since points of $\overline{Y}$ need to have a nonempty intersection with $Y$ on all open neighbourhoods, i.e. $U_2 = \emptyset$.

**Problem 4.** Let $G$ be a topological group and $G_0 \subseteq G$ be a connected component of the neutral element $1 \in G$. Prove that $G_0$ is a normal subgroup of $G$.

*Solution:*

Let us start with the 'subgroup' part. Observe that $G_0 \times G_0$ is connected, so $m(G_0 \times G_0) \subseteq G$ is a connected subset of $G$ that contains $1$, i.e. $m(G_0 \times G_0) \subseteq G_0$. Similarly $i(G_0) \subseteq G_0$. As we already know, multiplication by elements in $G$ is continuous, so the conjugation map (for fixed $g \in G$)

$$G \to G, h \mapsto ghg^{-1}$$

is continuous. Therefore it maps $G_0$ to a connected subset of $G$ containing $1$, hence again $gG_0g^{-1} \subseteq G_0$, proving normality.

**Problem 5.** Let $G$ be a connected topological group. Prove that

(a) Any open neighbourhood of the identity $1 \in G$ generates $G$.

(b) If $H \lhd G$ is a normal subgroup and a discrete topological space (endowed with subspace topology), then $H$ lies in the centre of $G$.

*Solution:*

(a) Let $1 \in U \subseteq_{op} G$ be an open subset that contains $1$. For any $x \notin \langle U \rangle$ we have

$$xU \cap \langle U \rangle = \emptyset.$$

On the other hand, $\langle U \rangle \subseteq_{op} G$ is open, since it is of the form

$$\langle U \rangle = \bigcup_{u \in \langle U \rangle} uU.$$

Thus, we have a decomposition into disjoint opens:

$$G = \langle U \rangle \cup \bigcup_{x \notin \langle U \rangle} xU.$$

As $G$ is connected and $\langle U \rangle \neq \emptyset$, then the other open is necessarily empty.

(b) Since $H$ is normal, then we can define the conjugation action on $H$, i.e. a map $\rho : G \times H \to H$

$$\rho(g, h) = ghg^{-1}.$$

Any fixed $h \in H$ induces a continuous map $G \to G \times H, g \mapsto (g, h)$. Therefore we can compose these to maps to obtain a continuous map $\varphi : G \to G \times H \to H$, where $H$ is discrete and $G$ is continuous, so the image of this map, which is given by

$$\varphi(G) = \{ghg^{-1} : g \in G\},$$

must be a point (as these are the only connected discrete spaces). Thus

$$\{h\} = \{ghg^{-1} : g \in G\},$$

which means that $h$ is central in $G$. This of course holds for any $h \in H$.

**Problem 6.** Let $G$ be a topological group and $H \leq G$ be a subgroup that is closed as a subset of $G$. Then $G/H$ is a regular space. In particular if $G$ contains a closed point, then $G$ itself is regular.

*Solution:*

Notice that all points of $G/H$ are closed as $H$ is closed and $H$ is the preimage of the point $H \in G/H$. Closedness of other points follows from the fact that $G$ acts on $G/H$.

Now, assume that $F \subseteq_{cl} G/H$ is some closed subset and $xH \in G/H$. Shifting everything by $x^{-1}$ we can assume $xH = H$. Let $\varphi : G \times G \to G$ be the continuous map defined by $\varphi(g, h) := g^{-1}h$. Then $\varphi^{-1}(G \setminus \pi^{-1}(F)) \subseteq_{op} G$ is open and $(1, 1)$ lies inside this set. Therefore there exists a basic open $U \times V \subseteq \varphi^{-1}(G \setminus \pi^{-1}(F))$ such that $(1, 1) \in U \times V$. By definition this implies that if $g \in U$ and $h \in V$, then $g^{-1}h \notin \pi^{-1}(F)$. In particular we get

$$V \cap (U \cdot \pi^{-1}(F)) = V \cap (\bigcup_{g \in \pi^{-1}F} Ug) = \emptyset.$$

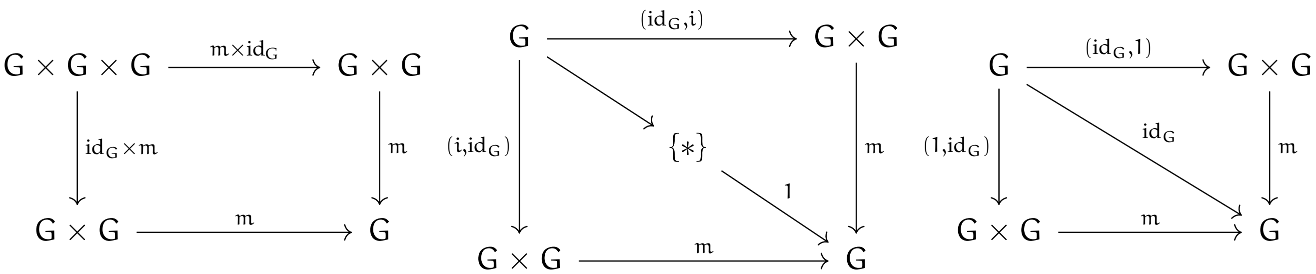
and both sets above are open, note that $1 \in V$. Taking the image of everything by $\pi$, we get opens $H \in \pi(V) \subseteq_{op} G/H$, $F \subseteq \pi(U \cdot \pi^{-1}(F)) \subseteq_{op} G/H$ and the intersection of the opens is empty (note that it is also crucial that $1 \in U$ as well).

**Problem 7.** Let $G$ be a topological group. Prove that $\pi_1(G)$ is abelian.

*Solution:*

Observe that $G$, being a topological group, is a group object in the category of topological spaces. This means that we have continuous maps $m : G \times G \to G, i : G \to G, 1 : \{*\} \to G$, which correspond to multiplication, inversion and inclusion of a neutral element respectively. These maps need to satisfy commutative diagrams that describe associativity, properties of inversion and neutral elements.

In general, a group object $G$ in a category $\mathscr{C}$ with a final object $\{*\}$ and finite products is an object $G$ together with maps $m : G \times G \to G, i : G \to G, 1 : \{*\} \to G$ such that the following diagrams are commutative:




The left diagram translates to associativity, the middle one to the definition of an inverse element and the last one to neutrality of the neutral element.

Now, formally, $\pi_1$ is a functor defined on the pointed category $\mathrm{Top}^*$ of pointed topological spaces. Observe that a topological group $G$ can be turned to a pointed topological group by just choosing the neutral element $(G, 1)$. In this setup all maps $m, i, 1$ descend to morphisms in the pointed category and all commutative diagrams work in the same way. Since $\pi_1$ is a functor and $\pi_1$ commutes with products, i.e. there is a natural isomorphism

$$\pi_1(X \times Y, (x, y)) \cong \pi_1(X, x) \times \pi_1(Y, y),$$

then $\pi_1$ in fact turns our group object $(G, 1)$ into a group object, but in the category Groups. Lastly, we have to understand what are group objects in Groups. Let $A$ be such a group object. Then $A$ is a group, but we also have a multiplication $m : A \times A \to A$ and inversion $i : A \to A$, which are homomorphisms of groups. Note that the neutral elements of both multiplications on $A$ need to agree, since the map $1 : \{*\} \to A$ is a homomorphism from a trivial group to $A$, i.e. it is the inclusion of the usual neutral element of $A$. Next, observe that definition of the neutral element (i.e. the third diagram above) implies that for any $g \in A$ we have

$$m(g, 1) = m(1, g) = g.$$

Therefore $m$ must agree with the usual multiplication of $A$, as it is a homomorphism:

$$m(g, h) = m((g, 1)(1, h)) = gh.$$

But as $(g, 1)(1, h) = (1, h)(g, 1)$, then we in fact obtain commutativity:

$$gh = m((g, 1)(1, h)) = m((1, h)(g, 1)) = hg.$$

Meaning that $\pi_1(G, 1)$ is an abelian group, as a group object in Groups. Note that any other $\pi_1(G, g)$ is isomorphic to $\pi_1(G, 1)$, since $G$ acts on itself via homeomorphisms.