Definition. *A **topological group** is a topological space* G *such that both the multiplication map* $(-)\cdot(-)\colon G\times G\to G$ *and inversion* $(-)^{-1}\colon G\to G$ *are continuous.*

Problem 1. Let G be a group and a topological space. Prove that G is a topological group if and only if the following map

$$\phi\colon G\times G\to G\times G, \qquad \phi(g,h):=(g,gh)$$

is a homeomorphism.

Problem 2. Let $H\leq G$ be a subgroup of a topological group G. Recall that we endow G/H with the quotient topology given by the quotient map $\pi\colon G\to G/H$. Prove that

(a) The group action map

$$G\times G/H\to G/H \qquad (g,hH)\mapsto ghH$$

is continuous.

(b) The quotient map $\pi\colon G\to G/H$ is open.

(c) The closure $\overline{H}$ is a subgroup of G.

(d) If H is open in G, then G/H is a discrete topological space.

Definition. *Let* X *be a topological space. A subset* $Y\subseteq X$ *is called a **connected component** of* X *if* Y *is connected and is maximal with respect to set inclusion.*

Problem 3. Let X be a topological space and $Y\subseteq X$ be a connected component. Prove that Y is closed in X.

Problem 4. Let G be a topological group and $G_0\subseteq G$ be a connected component of the neutral element $1\in G$. Prove that G_0 is a normal subgroup of G.

Problem 5. Let G be a connected topological group. Prove that

(a) Any open neighbourhood of the identity $1\in G$ generates G.

(b) If $H\lhd G$ is a normal subgroup and a discrete topological space (endowed with subspace topology), then H lies in the centre of G.

Definition. *A **regular space** (or a* T_3 *space) is a topological space* X *such that*

(i) *Points of* X *are closed and*

(ii) *For any point* $x\in X$ *and closed subset* $F\subseteq_{cl}X$ *there exist open subsets* $x\in U\subseteq_{op}X$, $F\subseteq V\subseteq_{op}X$ *such that* $U\cap V=\emptyset$.

Problem 6. Let G be a topological group and $H\leq G$ be a subgroup that is closed as a subset of G. Then G/H is a regular space. In particular if G contains a closed point, then G itself is regular.

Problem 7. Let G be a topological group. Prove that $\pi_1(G)$ is abelian.