Very meager sets

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Definition 1 (Borel). Let $X$ be a subset of $2^\omega$. We say that set $X$ is strongly null ($X \in SN$ for short), if for every sequence of positive real numbers $(\varepsilon_n)_{n \in \omega}$ there exists a sequence of sets of reals $(I_n)_{n \in \omega}$ such that

$$diam(I_n) < \varepsilon_n$$

and

$$X \subset \bigcup_{n \in \omega} I_n.$$
Proposition 1. Strongly null sets form a $\sigma$-ideal.

Proposition 2. Every strongly null subset of $2^\omega$ has Lebesgue measure zero.

Theorem 1.

- Continuum Hypothesis implies that there exists a strongly null subset of $2^\omega$ of size continuum.

- (Laver) It is relatively consistent with ZFC that every strongly null subset of $2^\omega$ is countable.
Question 1. How to define analogous sets for category?

Theorem 2 (Galvin–Mycielski–Solovay). A set $X \subset 2^\omega$ is strongly null if and only if for every co-meager set $G \subset 2^\omega$ exists $t \in 2^\omega$ such that

$$X \subseteq t + G.$$ 

Definition 2. A set $X \subset 2^\omega$ is strongly meager if for every full measure set $H \subset 2^\omega$ exists $t \in 2^\omega$ such that

$$X \subseteq t + H.$$
Proposition 3. Every countable set is strongly meager.

Theorem 3.

• Continuum Hypothesis implies that there exists a strongly meager subset of $2^\omega$ of size continuum.

• (Carlson) It is relatively consistent with ZFC that every strongly meager subset of $2^\omega$ is countable.

Theorem 4 (Bartoszyński–Shelah). Under the assumption of CH strongly meager sets are not closed under finite unions.
Question 2. What happens if in the Galvin–Mycielski–Solovay theorem and in the definition of strongly meager sets we admit countably many translations instead of one?

Proposition 4. A set $X \subset 2^\omega$ is strongly null if and only if for every co-meager set $G \subset 2^\omega$ exists a countable set $T \subset 2^\omega$ such that

$$X \subseteq T + G.$$ 

Definition 3. A set $X \subset 2^\omega$ is very meager if for every full measure set $H \subset 2^\omega$ exists a countable set $T \subset 2^\omega$ such that

$$X \subseteq T + H.$$
Proposition 5. Every strongly meager set is very meager.

Proposition 6. Very meager sets form a \( \sigma \)-ideal.

Proposition 7. Continuum Hypothesis implies that there exists a very meager subset of \( 2^\omega \) which is not strongly meager.

Proof. By the theorem of Bartoszyński and Shelah, under Continuum Hypothesis strongly meager sets do not form an ideal. \( \square \)
Proposition 8. Continuum Hypothesis implies that there exists an uncountable very meager set.

Theorem 5. It is relatively consistent with ZFC that every very meager set is countable (add $\omega_2$ Cohen reals to a model of CH).

Corollary 1. It is relatively consistent with ZFC, that every very meager set is strongly meager.
Definition 4 (Zakrzewski). A set $X \subset 2^\omega$ is universally meager if for every Borel isomorphism $f : 2^\omega \rightarrow 2^\omega$ the set $f[X]$ is meager.

Theorem 6. Every very meager set is universally meager.

Theorem 7. There exists a set which is universally meager and is not very meager.
Problem 1. Is every very meager set a countable union of strongly meager sets?

Remark 1. The positive answer for this question is relatively consistent with ZFC (e.g. if all very meager sets are countable).