Very meager sets

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Definition 1 (Borel). *Let X be a subset of 2^ω. We say that set X is* <u>*strongly null*</u> *($X \in \mathcal{SN}$ for short), if for every sequence of positive real numbers $(\varepsilon_n)_{n\in\omega}$ there exists a sequence of sets of reals $(I_n)_{n\in\omega}$ such that*

$$diam(I_n) < \varepsilon_n$$

and

$$X \subset \bigcup_{n\in\omega} I_n.$$

Proposition 1. *Strongly null sets form a σ-ideal.*

Proposition 2. *Every strongly null subset of 2^ω has Lebesgue measure zero.*

Theorem 1.

- *Continuum Hypothesis implies that there exists a strongly null subset of 2^ω of size continuum.*

- *(Laver) It is relatively consistent with ZFC that every strongly null subset of 2^ω is countable.*

Question 1. *How to define analogous sets for category?*

Theorem 2 (Galvin–Mycielski–Solovay). *A set $X \subset 2^\omega$ is strongly null if and only if for every co-meager set $G \subset 2^\omega$ exists $t \in 2^\omega$ such that*

$$X \subseteq t + G.$$

Definition 2. *A set $X \subset 2^\omega$ is <u>strongly meager</u> if for every full measure set $H \subset 2^\omega$ exists $t \in 2^\omega$ such that*

$$X \subseteq t + H.$$

Proposition 3. *Every countable set is strongly meager.*

Theorem 3.

- *Continuum Hypothesis implies that there exists a strongly meager subset of* 2^ω *of size continuum.*

- *(Carlson) It is relatively consistent with ZFC that every strongly meager subset of* 2^ω *is countable.*

Theorem 4 (Bartoszyński–Shelah). *Under the assumption of CH strongly meager sets are not closed under finite unions.*

Question 2. *What happens if in the Galvin–Mycielski–Solovay theorem and in the definition of strongly meager sets we admit countably many translations instead of one?*

Proposition 4. *A set $X \subset 2^\omega$ is strongly null if and only if for every co-meager set $G \subset 2^\omega$ exists a countable set $T \subset 2^\omega$ such that*

$$X \subseteq T + G.$$

Definition 3. *A set $X \subset 2^\omega$ is* <u>*very meager*</u> *if for every full measure set $H \subset 2^\omega$ exists a countable set $T \subset 2^\omega$ such that*

$$X \subseteq T + H.$$

Proposition 5. *Every strongly meager set is very meager.*

Proposition 6. *Very meager sets form a σ-ideal.*

Proposition 7. *Continuum Hypothesis implies that there exists a very meager subset of 2^ω which is not strongly meager.*

Proof. By the theorem of Bartoszyński and Shelah, under Continuum Hypothesis strongly meager sets do not form an ideal. $\square$

Proposition 8. *Continuum Hypothesis implies that there exists an uncountable very meager set.*

Theorem 5. *It is relatively consistent with ZFC that every very meager set is countable (add ω_2 Cohen reals to a model of CH).*

Corollary 1. *It is relatively consistent with ZFC, that every very meager set is strongly meager.*

Definition 4 (Zakrzewski). *A set $X \subset 2^\omega$ is* <u>*universally meager*</u> *if for every Borel isomorphism $f : 2^\omega \longrightarrow 2^\omega$ the set $f[X]$ is meager.*

Theorem 6. *Every very meager set is universally meager.*

Theorem 7. *There exists a set which is universally meager and is not very meager.*

Problem 1. *Is every very meager set a countable union of strongly meager sets?*

Remark 1. *The positive answer for this question is relatively consistent with ZFC (e.g. if all very meager sets are countable).*