Special subsets of the reals and tree forcing notions

Marcin Kysiak (Warsaw University)

(joint work with A. Nowik and T. Weiss)

Forcing notions under consideration:

- Laver forcing $\mathbb{L}$,
- Miller forcing $\mathbb{M}$,
- Mathias forcing $\mathbb{R}$,
- Sacks forcing $\mathbb{S}$,
- Silver forcing $\mathbb{V}$.

Ideals related to forcing

These forcing notions are of the form

$$\langle \mathbb{P}, \subseteq \rangle$$

where

$$\mathbb{P} \subseteq \mathbf{\Pi}^0_1.$$

A set X is $\mathbb{P}$-null iff the set

$$\{P \in \mathbb{P} : P \cap X = \emptyset\}$$

is dense in $\mathbb{P}$.

So we have the σ-ideals (l_0), (m_0), (cr_0), (s_0) and (v_0) related to the forcing notions of Laver, Miller, Mathias, Sacks and Silver, respectively.

Motivation for forcing ideals

Proposition. *A set $X \subseteq \mathbb{R}$ has Lebesgue measure zero if and only if the set of conditions of Solovay forcing disjoint with X is dense.*

Definition. (Marczewski) *A set $X \subseteq \mathbb{R}$ is Marczewski-null (or $X \in (s_0)$), if for every perfect set $P \subseteq \mathbb{R}$ there exists a perfect set $Q \subseteq P$ such that $P \cap X = \emptyset$.*

Underlying spaces

The natural underlying space for (l_0), $(cr_0)$and (m_0) is ω^ω and 2^ω for (v_0).

For (s_0) the underlying space (ω^ω or 2^ω) depends on the representation.

We identify ω^ω with $[\omega]^\omega \subseteq 2^\omega$ and define the ideals (l_0) and (m_0) of subsets of 2^ω in the natural way.

Special sets under consideration

- perfectly meager sets (AFC)
- universally null sets (UMZ)
- strongly null sets (SMZ)
- meager-additive sets
- sets with Rothberger's property (C'')
- Lusin sets
- γ-sets
- σ-sets
- λ- and λ'-sets

General form of a question:

Is a class of special sets included in the ideal of $\mathbb{P}$-null sets?

Two possible interesting answers:

- YES,
- consistently NOT.

How to read these slides?

- **Proposition** is something which is easy or easily follows from something well-known,
- **Theorem** requires some arguments (sometimes similar to our previous results, though).

Previous results

Theorem. *All classes of special sets considered are included in* (s_0).

Theorem. (Nowik–Weiss) *Strongly meager sets have the* l_0*-property.*

Theorem. (Kysiak-Weiss) *Every perfectly meager set has the* m_0*-property, but under CH there exists one which does not have the* l_0*-property.*

Theorem. (Kysiak–Weiss) *In the Baire space with the natural metric strongly null sets have the* l_0*- and the* m_0*-property.*

Theorem. (Kysiak–Weiss) *Under CH in the Cantor set not every strongly null set has the* l_0*-property and not every strongly null set has the* m_0*-property.*

About Silver forcing

Theorem. *Every perfectly meager set has the v_0-property.*

Theorem. *Every universally null set in 2^ω has the v_0-property.*

Corollary. *All other classes of special sets mentioned before are included in (v_0).*

Below strongly null sets

Proposition. *Every Lusin set (in 2^ω and in ω^ω) has the l_0- and the m_0-property.*

Proposition. *Under CH there exists a C''-set in 2^ω which does not have the l_0-property. Similarly, under CH there exists a C''-set in 2^ω which does not have the m_0-property.*

Proposition. *Every meager-additive set in 2^ω has the m_0-property.*

Theorem. *Under CH there exists a meager-additive set in 2^ω which does not have the l_0-property.*

Theorem. *Every γ-set in 2^ω has the l_0- and the m_0-property.*

Classes related to Borel hierarchy

Proposition. *Every λ-set (in ω^ω and in 2^ω) has the m_0- and the v_0-property. In particular, every σ-set has these properties.*

Proposition. *Under CH there exists a λ'-set in ω^ω which does not have the l_0-property. In particular, this set is a λ-set in 2^ω.*

Proposition. *Every λ'-set in 2^ω has the l_0-property.*

Theorem. *Every σ-set in ω^ω has the l_0-property and the cr_0-property.*

The paper is available at

`http://www.mimuw.edu.pl/~mkysiak`

and will appear in the Proceedings of the American Mathematical Society.