Special subsets of the reals and tree forcing notions

Marcin Kysiak (Warsaw University)

(joint work with A. Nowik and T. Weiss)
Forcing notions under consideration:

- Laver forcing $\mathbb{L}$,
- Miller forcing $\mathbb{M}$,
- Mathias forcing $\mathbb{R}$,
- Sacks forcing $\mathbb{S}$,
- Silver forcing $\mathbb{V}$.
Ideals related to forcing

These forcing notions are of the form

$$\langle \mathbb{P}, \subseteq \rangle$$

where

$$\mathbb{P} \subseteq \Pi^0_1.$$ 

A set $X$ is $\mathbb{P}$-null iff the set

$$\{P \in \mathbb{P} : P \cap X = \emptyset \}$$

is dense in $\mathbb{P}$.

So we have the $\sigma$-ideals $(l_0)$, $(m_0)$, $(cr_0)$, $(s_0)$ and $(v_0)$ related to the forcing notions of Laver, Miller, Mathias, Sacks and Silver, respectively.
Motivation for forcing ideals

**Proposition.** A set $X \subseteq \mathbb{R}$ has Lebesgue measure zero if and only if the set of conditions of Solovay forcing disjoint with $X$ is dense.

**Definition.** (Marczewski) A set $X \subseteq \mathbb{R}$ is Marczewski-null (or $X \in (s_0)$), if for every perfect set $P \subseteq \mathbb{R}$ there exists a perfect set $Q \subseteq P$ such that $P \cap X = \emptyset$. 


Underlying spaces

The natural underlying space for \((l_0), (cr_0)\) and \((m_0)\) is \(\omega^\omega\) and \(2^\omega\) for \((v_0)\).

For \((s_0)\) the underlying space \((\omega^\omega\) or \(2^\omega\)) depends on the representation.

We identify \(\omega^\omega\) with \([\omega]^\omega \subseteq 2^\omega\) and define the ideals \((l_0)\) and \((m_0)\) of subsets of \(2^\omega\) in the natural way.
Special sets under consideration

- perfectly meager sets (AFC)
- universally null sets (UMZ)
- strongly null sets (SMZ)
- meager-additive sets
- sets with Rothberger’s property (C’’)
- Lusin sets
- $\gamma$-sets
- $\sigma$-sets
- $\lambda$- and $\lambda'$-sets
General form of a question:

Is a class of special sets included in the ideal of $\mathbb{P}$-null sets?

Two possible interesting answers:

- YES,
- consistently NOT.

How to read these slides?

- Proposition is something which is easy or easily follows from something well-known,
- Theorem requires some arguments (sometimes similar to our previous results, though).
Previous results

**Theorem.** *All classes of special sets considered are included in $s_0$.***

**Theorem.** *(Nowik–Weiss)* *Strongly meager sets have the $l_0$-property.*

**Theorem.** *(Kysiak–Weiss)* *Every perfectly meager set has the $m_0$-property, but under CH there exists one which does not have the $l_0$-property.*

**Theorem.** *(Kysiak–Weiss)* *In the Baire space with the natural metric strongly null sets have the $l_0$- and the $m_0$-property.*

**Theorem.** *(Kysiak–Weiss)* *Under CH in the Cantor set not every strongly null set has the $l_0$-property and not every strongly null set has the $m_0$-property.*
About Silver forcing

**Theorem.** Every perfectly meager set has the $\nu_0$-property.

**Theorem.** Every universally null set in $2^\omega$ has the $\nu_0$-property.

**Corollary.** All other classes of special sets mentioned before are included in $\nu_0$. 
Below strongly null sets

**Proposition.** Every Lusin set (in $2^{\omega}$ and in $\omega^\omega$) has the $l_0$- and the $m_0$-property.

**Proposition.** Under CH there exists a $C'''$ -set in $2^\omega$ which does not have the $l_0$-property. Similarly, under CH there exists a $C''''$ -set in $2^\omega$ which does not have the $m_0$-property.

**Proposition.** Every meager-additive set in $2^\omega$ has the $m_0$-property.

**Theorem.** Under CH there exists a meager-additive set in $2^\omega$ which does not have the $l_0$-property.

**Theorem.** Every $\gamma$-set in $2^\omega$ has the $l_0$- and the $m_0$-property.
Classes related to Borel hierarchy

**Proposition.** Every $\lambda$-set (in $\omega^\omega$ and in $2^\omega$) has the $m_0$- and the $\nu_0$-property. In particular, every $\sigma$-set has these properties.

**Proposition.** Under CH there exists a $\lambda'$-set in $\omega^\omega$ which does not have the $l_0$-property. In particular, this set is a $\lambda$-set in $2^\omega$.

**Proposition.** Every $\lambda'$-set in $2^\omega$ has the $l_0$-property.

**Theorem.** Every $\sigma$-set in $\omega^\omega$ has the $l_0$-property and the $cr_0$-property.
The paper is available at

http://www.mimuw.edu.pl/~mkysiak