Some remarks on Marczewski-measurable sets and functions

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Marczewski measurable sets and functions

Definition

- a set $X \subseteq \mathbb{R}$ is Marczewski measurable ($X \in (s)$ for short) if
  \[ \forall P \in \text{Perf} \ \exists Q \subseteq P \ Q \in \text{Perf} \land (Q \subseteq X \lor Q \cap X = \emptyset), \]
- a set $X \subseteq \mathbb{R}$ is Marczewski null ($X \in (s_0)$ for short) if
  \[ \forall P \in \text{Perf} \ \exists Q \subseteq P \ Q \in \text{Perf} \land Q \cap X = \emptyset, \]
- a function $f : \mathbb{R} \to \mathbb{R}$ is Marczewski measurable, if it is measurable with respect to the $\sigma$-field $(s)$.

Theorem

A function $f : \mathbb{R} \to \mathbb{R}$ is Marczewski measurable if, and only if, for every $P \in \text{Perf}$ $\exists Q \subseteq P \ Q \in \text{Perf} \land f \upharpoonright Q$ is continuous.
**Definition**

Let \( f : X \to Y \) be a function. The **indicatrix** \( s(f) : Y \to \text{Card} \) of the function \( f \) is defined by

\[
s(f)(y) = |f^{-1}\{y\}|.
\]

We say that \( f, g : [0, 1] \to [0, 1] \) are equivalent (\( f \sim g \), for short), if \( s(f) = s(g) \).

**Remark**

The functions \( f, g : [0, 1] \to [0, 1] \) are equivalent if, and only if, there exists a bijection \( \varphi : [0, 1] \to [0, 1] \) such that \( f \circ \varphi = g \).
Let $\mathcal{F} \subseteq [0, 1]^{[0,1]}$ be a class of functions. Can we characterize functions equivalent to a member of $\mathcal{F}$?

In other words, can we characterize indicatrices of members of $\mathcal{F}$?
Theorem (Morayne–Ryll-Nardzewski)

A function $f : [0, 1] \rightarrow [0, 1]$ is equivalent to a Lebesgue measurable one (and equivalently, to a Baire-measurable one) if, and only if, $s(f) > 0$ on a perfect set or there exists $y \in [0, 1]$ such that $s(f)(y) = c$.

Moreover:

- Komisarski, Michalewski and Milewski characterized indicatrices of Borel functions (under Projective Determinacy),
- Kwiatkowska characterized indicatrices of continuous functions (in ZFC).
What about Markiewski-measurable functions?

**Theorem**

A function \( f : [0, 1] \to [0, 1] \) is equivalent to a Markiewski-measurable one if, and only if, \( s(f) > 0 \) on a perfect set or there exists \( y \in [0, 1] \) such that \( s(f)(y) = c \).

**Corollary**

Every Markiewski measurable function is equivalent to a Lebesgue measurable one (and to a Baire measurable one) and vice versa.
Definition
A $\sigma$-algebra $\mathcal{A}$ has the weak Continuous Restriction Property if for every $\mathcal{A}$-measurable function $f : [0, 1] \to [0, 1]$ there exists a perfect set $P \subseteq [0, 1]$ such that $f \upharpoonright P$ is continuous.

Lemma
If a $\sigma$-algebra $\mathcal{A}$ has the weak Continuous Restriction Property then for each $\mathcal{A}$-measurable function $f : [0, 1] \to [0, 1]$ either $s(f) > 0$ on a perfect set or $s(f)$ takes value $c$. 
Lemma

Assume that a $\sigma$-algebra $\mathcal{A}$ contains all Borel sets and that $\mathcal{H}(\mathcal{A})$ contains a set of size $\mathfrak{c}$. Then

- if a function $f : [0, 1] \to [0, 1]$ is constant on a set of cardinality $\mathfrak{c}$ then it is equivalent to an $\mathcal{A}$-measurable function,
- if a function $f : [0, 1] \to [0, 1]$ contains a perfect set in its range then it is equivalent to an $\mathcal{A}$-measurable function.
Remark

The theorem is also true for other algebras, e.g. for algebras associated with “tree forcing notions”.

http://www.mimuw.edu.pl/~mkysiak/