Independent families and topology

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Inspiration

Lemma. (Holický–Zajiček–Zelený) * There exists a family $A \subseteq \omega^c$ of size $2^c$ with the property that for every countable $A' \subseteq A$ there exists $\alpha \in c$ such that $f(\alpha) \neq g(\alpha)$, for all distinct $f, g \in A'$.

Question. (Holický–Zajiček–Zelený) Does there exists a family $A \subseteq \omega^{\omega_1}$ of size $2^{\omega_1}$ with the property that for every countable $A' \subseteq A$ there exists $\alpha \in \omega_1$ such that $f(\alpha) \neq g(\alpha)$, for all distinct $f, g \in A'$?

*After my talk, Szymon Plewik pointed to me that this theorem and a proof almost identical to the one below appeared before in R.Engelking, M. Karłowicz - Some theorems of set theory and their topological consequences, Fund. Math. 157(1965), 275–285
Independent families

**Definition.** A family $\mathcal{F} \subseteq \mathcal{P}(X)$ is called independent if for all pairwise distinct $A_0, \ldots, A_n, B_0, \ldots, B_m \in \mathcal{F}$ we have

$$A_0 \cap \ldots \cap A_n \cap B_0^c \cap \ldots \cap B_m^c \neq \emptyset.$$

**Definition.** A family $\mathcal{F} \subseteq \mathcal{P}(X)$ is called $\sigma$-independent if for all pairwise distinct $A_0, \ldots, A_n, \ldots, B_0, \ldots, B_m, \ldots \in \mathcal{F}$ we have

$$\left(A_0 \cap \ldots \cap A_n \ldots\right) \cap \left(B_0^c \cap \ldots \cap B_m^c \ldots\right) \neq \emptyset.$$
Theorem. (Fichtenholz–Kantorovich–Hausdorff) For every cardinal \( \kappa \) there exists an independent family \( \mathcal{F} \subseteq \mathcal{P}(\kappa) \) of size \( 2^\kappa \).

Proof.

- \(|\text{Clop}(2^\kappa)| = \kappa|,

- for \( x \in 2^\kappa \) define \( A_x = \{ F \in \text{Clop}(2^\kappa) : x \in F \} \),

- \( \mathcal{F} = \{ A_x : x \in 2^\kappa \} \) is independent.
Theorem. (Tarski) For every cardinal $\kappa$ such that $\kappa^\omega = \kappa$ there exists a $\sigma$-independent family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ of size $2^\kappa$.

Proof.

• $\text{Baire}(2^\kappa) = \sigma(\text{Clop}(2^\kappa))$,

• $|\text{Baire}(2^\kappa)| = \kappa$,

• for $x \in 2^\kappa$ define $A_x = \{F \in \text{Baire}(2^\kappa) : x \in F\}$,

• $\mathcal{F} = \{A_x : x \in 2^\kappa\}$ is $\sigma$-independent.
Independent partitions

**Definition.** An indexed family \( \langle A_\alpha,n : \alpha \in \lambda, n \in \omega \rangle \) of subsets of a set \( X \) is called a \( \sigma \)-independent family of \( \omega \)-partitions, if

\[
\begin{align*}
\bullet \ & \forall \alpha \in \lambda \forall n, m \in \omega \ (n \neq m \Rightarrow A_\alpha,n \cap A_\alpha,m = \emptyset), \\
\bullet \ & \forall \alpha \in \lambda \bigcup_{n \in \omega} A_\alpha,n = X, \\
\bullet \ & \forall A \in [\lambda]^{\omega} \forall f : A \rightarrow \omega \ \bigcap_{\alpha \in A} A_\alpha,f(\alpha) \neq \emptyset.
\end{align*}
\]
Theorem. For every cardinal $\kappa$ such that $\kappa^\omega = \kappa$ there exists a $\sigma$-independent family $\langle A_{\alpha,n} : \alpha \in 2^\kappa, n \in \omega \rangle$ of $\omega$-partitions.

Proof.

- Part $= \left\{ \langle B_n : n \in \omega \rangle : B_n \in \text{Baire}(2^\kappa) \land 2^\kappa = \bigcup_n B_n \right\}$,

- $|\text{Part}(2^\kappa)| = \kappa$,

- for $x \in 2^\kappa$ and $k \in \omega$ define $A_{x,k} = \left\{ \langle B_n : n \in \omega \rangle \in \text{Part}(2^\kappa) : x \in B_k \right\}$,

- $\langle A_{x,k} : x \in 2^\kappa, k \in \omega \rangle$ is a $\sigma$-independent family of $\omega$-partitions.

$\square$
Lemma. (Holický–Zajiček–Zelený) There exists a family $A \subseteq \omega^c$ of size $2^c$ with the property that for every countable $A' \subseteq A$ there exists $x \in c$ such that $f(x) \neq g(x)$, for all distinct $f, g \in A'$.

Proof.

- Fix a $\sigma$-independent family $\langle A_{\alpha,n} : \alpha \in 2^c, n \in \omega \rangle$ of $\omega$-partitions of $c$,

- define $f_\alpha(x) = n \iff x \in A_{\alpha,n}$, for $\alpha \in 2^c$, $x \in c$, $n \in \omega$,

- let $A = \{f_\alpha : \alpha \in 2^c\}$,
• for $A' = \{ f_{\alpha_0}, \ldots, f_{\alpha_k}, \ldots \} \in [A]^{\omega}$ find $x \in \bigcap_{k \in \omega} A_{\alpha_k, k}$,

• we have $f_{\alpha_k}(x) = k$. 

$\square$
About the problem

Question. (Holický–Zajiček–Zelený) Does there exists a family $A \subseteq \omega^{\omega_1}$ of size $2^{\omega_1}$ with the property that for every countable $A' \subseteq A$ there exists $x \in \omega_1$ such that $f(x) \neq g(x)$, for all distinct $f, g \in A'$?

- CH implies the positive answer,

- CH is equivalent to the existence of the $\sigma$-independent family of subsets (or $\omega$-partitions) of $\omega_1$.

Proposition. (Kubiš) The positive answer to the problem is consistent with $\mathfrak{c} = \aleph_2$. 