Independent families and topology

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Inspiration

Lemma. (Holický–Zajiček–Zelený) * *There exists a family* $\mathcal{A} \subseteq \omega^{\mathfrak{c}}$ *of size* $2^{\mathfrak{c}}$ *with the property that for every countable* $\mathcal{A}' \subseteq \mathcal{A}$ *there exists* $\alpha \in \mathfrak{c}$ *such that* $f(\alpha) \neq g(\alpha)$*, for all distinct* $f, g \in \mathcal{A}'$*.*

Question. (Holický–Zajiček–Zelený) *Does there exists a family* $\mathcal{A} \subseteq \omega^{\omega_1}$ *of size* 2^{ω_1} *with the property that for every countable* $\mathcal{A}' \subseteq \mathcal{A}$ *there exists* $\alpha \in \omega_1$ *such that* $f(\alpha) \neq g(\alpha)$*, for all distinct* $f, g \in \mathcal{A}'$*?*

*After my talk, Szymon Plewik pointed to me that this theorem and a proof almost identical to the one below appeared before in R.Engelking, M. Karłowicz - *Some theorems of set theory and their topological consequences*, Fund. Math. 157(1965), 275–285

Independent families

Definition. *A family $\mathcal{F} \subseteq \mathcal{P}(X)$ is called independent if for all pairwise distinct $A_0, \ldots, A_n, B_0, \ldots, B_m \in \mathcal{F}$ we have*

$$A_0 \cap \ldots \cap A_n \cap B_0^c \cap \ldots \cap B_m^c \neq \emptyset.$$

Definition. *A family $\mathcal{F} \subseteq \mathcal{P}(X)$ is called σ-independent if for all pairwise distinct $A_0, \ldots, A_n, \ldots, B_0, \ldots, B_m, \ldots \in \mathcal{F}$ we have*

$$\Big(A_0 \cap \ldots \cap A_n \ldots\Big) \cap \Big(B_0^c \cap \ldots \cap B_m^c \ldots\Big) \neq \emptyset.$$

Theorem. (Fichtencholz–Kantorovich–Hausdorff) *For every cardinal κ there exists an independent family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ of size 2^κ.*

Proof.

- $|\mathrm{Clop}(2^\kappa)| = \kappa$,
- for $x \in 2^\kappa$ define $A_x = \{F \in \mathrm{Clop}(2^\kappa) : x \in F\}$,
- $\mathcal{F} = \{A_x : x \in 2^\kappa\}$ is independent.

$\square$

Theorem. (Tarski) *For every cardinal κ such that $\kappa^\omega = \kappa$ there exists a σ-independent family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ of size 2^κ.*

Proof.

- $\mathrm{Baire}(2^\kappa) = \sigma\Big(\mathrm{Clop}(2^\kappa)\Big)$,
- $|\mathrm{Baire}(2^\kappa)| = \kappa$,
- for $x \in 2^\kappa$ define $A_x = \{F \in \mathrm{Baire}(2^\kappa) : x \in F\}$,
- $\mathcal{F} = \{A_x : x \in 2^\kappa\}$ is σ-independent.

$\square$

Independent partitions

Definition. *An indexed family* $\langle A_{\alpha,n} : \alpha \in \lambda, n \in \omega \rangle$ *of subsets of a set* X *is called a* σ*-independent family of* ω*-partitions, if*

- $\forall \alpha \in \lambda \, \forall n, m \in \omega \, (n \neq m \Rightarrow A_{\alpha,n} \cap A_{\alpha,m} = \emptyset)$,
- $\forall \alpha \in \lambda \, \bigcup_{n \in \omega} A_{\alpha,n} = X$,
- $\forall A \in [\lambda]^{\omega} \, \forall f : A \to \omega \, \bigcap_{\alpha \in A} A_{\alpha,f(\alpha)} \neq \emptyset$.

Theorem. *For every cardinal κ such that $\kappa^\omega = \kappa$ there exists a σ-independent family $\langle A_{\alpha,n} : \alpha \in 2^\kappa, n \in \omega\rangle$ of ω-partitions.*

Proof.

- $\mathsf{Part} = \Big\{\langle B_n : n \in \omega\rangle : B_n \in \mathsf{Baire}(2^\kappa) \wedge 2^\kappa = \dot{\bigcup}_n B_n\Big\}$,
- $|\mathsf{Part}(2^\kappa)| = \kappa$,
- for $x \in 2^\kappa$ and $k \in \omega$ define $A_{x,k} = \{\langle B_n : n \in \omega\rangle \in \mathsf{Part}(2^\kappa) : x \in B_k\}$,
- $\langle A_{x,k} : x \in 2^\kappa, k \in \omega\rangle$ is a σ-independent family of ω-partitions.

$\square$

Lemma. (Holický–Zajiček–Zelený) *There exists a family $\mathcal{A} \subseteq \omega^{\mathfrak{c}}$ of size $2^{\mathfrak{c}}$ with the property that for every countable $\mathcal{A}' \subseteq \mathcal{A}$ there exists $x \in \mathfrak{c}$ such that $f(x) \neq g(x)$, for all distinct $f, g \in \mathcal{A}'$.*

Proof.

- Fix a σ-independent family $\langle A_{\alpha,n} : \alpha \in 2^{\mathfrak{c}}, n \in \omega \rangle$ of ω-partitions of $\mathfrak{c}$,

- define $f_\alpha(x) = n \iff x \in A_{\alpha,n}$, for $\alpha \in 2^{\mathfrak{c}}$, $x \in \mathfrak{c}$, $n \in \omega$,

- let $\mathcal{A} = \{f_\alpha : \alpha \in 2^{\mathfrak{c}}\}$,

- for $\mathcal{A}' = \{f_{\alpha_0}, \ldots, f_{\alpha_k}, \ldots\} \in [\mathcal{A}]^\omega$ find $x \in \bigcap_{k\in\omega} A_{\alpha_k,k}$,

- we have $f_{\alpha_k}(x) = k$.

$\square$

About the problem

Question. (Holický–Zajiček–Zelený) *Does there exists a family $\mathcal{A} \subseteq \omega^{\omega_1}$ of size 2^{ω_1} with the property that for every countable $\mathcal{A}' \subseteq \mathcal{A}$ there exists $x \in \omega_1$ such that $f(x) \neq g(x)$, for all distinct $f, g \in \mathcal{A}'$?*

- CH implies the positive answer,

- CH is equivalent to the existence of the σ-independent family of subsets (or ω-partitions) of ω_1.

Proposition. (Kubiś) *The positive answer to the problem is consistent with $\mathfrak{c} = \aleph_2$.*