On Schmidt’s $\sigma$-ideal

Marcin Kysiak

joint work with Enrico Zoli
Schmidt’s \((\alpha, \beta)\)-games.

Parameters: \(\alpha \in (0, \frac{1}{2}) \cap \mathbb{Q}, \ \beta \in (0, 1) \cap \mathbb{Q}, \ X \subseteq \mathbb{R}\).

Gameplay:
\[
P_0 \supseteq S_0 \supseteq P_1 \supseteq S_1 \ldots \supseteq P_k \supseteq S_k \supseteq \ldots,
\]
where for all \(k \in \omega\)

- Adam plays \(P_k\) and Eve plays \(S_k\),

- \(P_k\) and \(S_k\) are compact intervals,

- \(S_k\) is \(\alpha\) times shorter than \(P_k\),

- \(P_{k+1}\) is \(\beta\) times shorter than \(S_k\).
Result of the game:

\[ \bigcap_{k \in \omega} P_k = \bigcap_{k \in \omega} S_k = \{x\}. \]

Eve wins, if \( x \not\in X \).

**Definition.** Schmidt’s \( \sigma \)-ideal \( S \) consists of \( X \subseteq \mathbb{R} \) such that for all \( \alpha \in (0, \frac{1}{2}) \cap \mathbb{Q}, \beta \in (0, 1) \cap \mathbb{Q} \) Eve has a winning strategy in corresponding game.

**Theorem.** (Schmidt, 1966) \( S \) is a translation invariant \( \sigma \)-ideal containing all singletons.
Definition. A real number is badly approximable, if there exists a constant $c > 0$ such that for all $\frac{p}{q} \in \mathbb{Q}$

$$\left| x - \frac{p}{q} \right| > \frac{c}{q^2}.$$ 

Theorem. (Schmidt) The set of badly approximable numbers is in $S^*$. 

Corollary.

$$S \perp (\mathcal{M} \cap \mathcal{N}).$$
Mycielski ideals.

Parameters: $K \subseteq \omega$, $X \subseteq 2^\omega$.

Adam and Eve play terms of an infinite sequence

$$\langle x_0, x_1, x_2, \ldots, x_k, \ldots \rangle \in 2^\omega,$$

When $k \in K$, Adam picks $x_k$, otherwise Eve does.

For a class $\mathcal{K} \subseteq \mathcal{P}(\omega)$ we define the family $\mathcal{M}_\mathcal{K}$ consisting of those $X \subseteq 2^\omega$ such that Eve has a winning strategy for all $K \in \mathcal{K}$.

For “suitable” $\mathcal{K}$ the families $\mathcal{M}_\mathcal{K}$ are $\sigma$-ideals.
Why Mycielski’s ideals?

- Simpler technical arguments due to “discrete nature” of the game.
- Various set-theoretic properties investigated in the literature.

Why Schmidt’s ideal?

- Original motivations in number theory.
- Single natural ideal, in contrast to many possible Mycielski ideals.
- Richer algebraic structure of the space.
- Similar set-theoretic properties to Mycielski ideals.
Theorem. The ideal $S$ has a base consisting of $G_\delta$ sets.

Corollary.

$S \perp \mathcal{E}$. 


Theorem. $S$ is not c.c.c.. In fact, there exists continuum many pairwise disjoint $S$-positive perfect sets.

Theorem. $S$ does not have Steinhaus property. There exists an $S$-positive compact perfect set $P$ such that $P - P$ has measure zero.

Theorem. $S$ does not have Ruziewicz property. There exists an $S$-positive compact perfect set $P$ not containing a similar copy of \{0, 1, \ldots, 10\}.

Theorem. There exists an analytic set $A \subseteq \mathbb{R}$ not belonging to the $\sigma$-field $\text{Bor}[S]$. 
Theorem. (Schmidt) The ideal $S$ is invariant under local diffeomorphisms.

Theorem. There exists a purely transcendental uncountable subfield of $\mathbb{R}$, whose all irrational numbers are badly approximable.
Proposition. There exists a set $X \subseteq \mathbb{R}$ consisting of badly approximable numbers such that $X + X$ is not Lebesgue measurable and does not have the Baire property.

Proposition. There exists a set $X \subseteq \mathbb{R}$ consisting of well approximable numbers such that $X + X \not\in \text{Bor}[\mathcal{S}]$. 
**Proposition.** Under CH there exist „Erdős – Sierpiński - like” mappings between $S$ and $M$ and $N$.

**Question.** Can these mappings be additive?

**Definition.** An ideal $\mathcal{I}$ is $\kappa$-translatable, when
\[
\forall A \in \mathcal{I} \; \exists B \in \mathcal{I} \; \forall T \in [\mathbb{R}]^\kappa \; \exists t \in \mathbb{R} \; T + A \subseteq t + B.
\]

**Question.** What is the minimal number $\kappa$ such that $S$ is not $\kappa$-translatable?
Proposition. *Martin’s Axiom implies that*

\[ \text{non}(S) = \text{cof}(S) = \mathfrak{c}. \]

**Conjecture.**

\[ \text{non}(S) = \mathfrak{c} \quad \& \quad \text{cov}(S) = \omega_1. \]