On Schmidt's σ-ideal

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joint work with Enrico Zoli

Schmidt's (α, β)-games.

Parameters: $\alpha \in (0, \frac{1}{2}) \cap \mathbb{Q}$, $\beta \in (0, 1) \cap \mathbb{Q}$, $X \subseteq \mathbb{R}$.

Gameplay:

$$P_0 \supseteq S_0 \supseteq P_1 \supseteq S_1 \ldots \supseteq P_k \supseteq S_k \supseteq \ldots,$$

where for all $k \in \omega$

- Adam plays P_k and Eve plays S_k,
- P_k and S_k are compact intervals,
- S_k is α times shorter than P_k,
- P_{k+1} is β times shorter than S_k.

Result of the game:

$$\bigcap_{k\in\omega} P_k = \bigcap_{k\in\omega} S_k = \{x\}.$$

Eve wins, if $x \notin X$.

Definition. *Schmidt's σ-ideal $\mathcal{S}$ consists of $X \subseteq \mathbb{R}$ such that for all $\alpha \in (0, \frac{1}{2}) \cap \mathbb{Q}$, $\beta \in (0,1) \cap \mathbb{Q}$ Eve has a winning strategy in corresponding game.*

Theorem. (Schmidt, 1966) *$\mathcal{S}$ is a translation invariant σ-ideal containing all singletons.*

Definition. *A real number is badly approximable, if there exists a constant* $c > 0$ *such that for all* $\frac{p}{q} \in \mathbb{Q}$

$$\left|x - \frac{p}{q}\right| > \frac{c}{q^2}.$$

Theorem. (Schmidt) *The set of badly approximable numbers is in* $\mathcal{S}^*$.

Corollary.

$$\mathcal{S} \perp (\mathcal{M} \cap \mathcal{N}).$$

Mycielski ideals.

Parameters: $K \subseteq \omega$, $X \subseteq 2^\omega$.

Adam and Eve play terms of an infinite sequence

$$\langle x_0, x_1, x_2, \ldots, x_k, \ldots \rangle \in 2^\omega,$$

When $k \in K$, Adam picks x_k, otherwise Eve does.

For a class $\mathcal{K} \subseteq \mathcal{P}(\omega)$ we define the family $\mathfrak{M}_\mathcal{K}$ consisting of those $X \subseteq 2^\omega$ such that Eve has a winning strategy for all $K \in \mathcal{K}$.

For "suitable" $\mathcal{K}$ the families $\mathfrak{M}_\mathcal{K}$ are σ-ideals.

Why Mycielski's ideals?

- Simpler technical arguments due to "discrete nature" of the game.

- Various set-theoretic properties investigated in the literature.

Why Schmidt's ideal?

- Original motivations in number theory.

- Single natural ideal, in contrast to many possible Mycielski ideals.

- Richer algebraic structure of the space.

- Similar set-theoretic properties to Mycielski ideals.

Theorem. *The ideal $\mathcal{S}$ has a base consisting of G_δ sets.*

Corollary.

$$\mathcal{S} \perp \mathcal{E}.$$

Theorem. *$\mathcal{S}$ is not c.c.c.. In fact, there exists continuum many pairwise disjoint $\mathcal{S}$-positive perfect sets.*

Theorem. *$\mathcal{S}$ does not have Steinhaus property. There exists an $\mathcal{S}$-positive compact perfect set P such that $P - P$ has measure zero.*

Theorem. *$\mathcal{S}$ does not have Ruziewicz property. There exists an $\mathcal{S}$-positive compact perfect set P not containing a similar copy of $\{0, 1, \ldots, 10\}$.*

Theorem. *There exists an analytic set $A \subseteq \mathbb{R}$ not belonging to the σ-field $Bor[\mathcal{S}]$.*

Theorem. (Schmidt) *The ideal $\mathcal{S}$ is invariant under local diffeomorphisms.*

Theorem. *There exists a purely transcendental uncountable subfield of $\mathbb{R}$, whose all irrational numbers are badly approximable.*

Proposition. *There exists a set $X \subseteq \mathbb{R}$ consisting of badly approximable numbers such that $X + X$ is not Lebesgue measurable and does not have the Baire property.*

Proposition. *There exists a set $X \subseteq \mathbb{R}$ consisting of well approximable numbers such that $X + X \notin Bor[\mathcal{S}]$.*

Proposition. *Under CH there exist ,,Erdős – Sierpiński - like'' mappings between $\mathcal{S}$ and $\mathcal{M}$ and $\mathcal{N}$.*

Question. *Can these mappings be additive?*

Definition. *An ideal $\mathcal{I}$ is κ-translatable, when*

$$\forall A \in \mathcal{I} \ \ \exists B \in \mathcal{I} \ \ \forall T \in [\mathbb{R}]^{\kappa} \ \ \exists t \in \mathbb{R} \ \ T + A \subseteq t + B.$$

Question. *What is the minimal number κ such that $\mathcal{S}$ is not κ-translatable?*

Proposition. *Martin's Axiom implies that*

$$\mathrm{non}(\mathcal{S}) = \mathrm{cof}(\mathcal{S}) = \mathfrak{c}.$$

Conjecture.

$$\mathrm{non}(\mathcal{S}) = \mathfrak{c} \quad \& \quad \mathrm{cov}(\mathcal{S}) = \omega_1.$$