Nonmeasurable algebraic sums of sets of reals

Marcin Kysiak

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Theorem. (Sierpiński, 1920) *There exist measure zero sets $X, Y \subseteq \mathbb{R}$ such that the set*

$$X + Y = \{x + y : x \in X, y \in Y\}$$

is nonmeasurable.

Theorem. (Cichoń–Jasiński, 2003) *Let $\mathcal{I}$ be a σ-ideal with coanalytic base and let $\mathcal{A} = Bor[\mathcal{I}]$. Then the following conditions are equivalent*

1. $\exists X, Y \in \mathcal{I} \;\; X + Y \notin \mathcal{I}$

2. $\exists X, Y \in \mathcal{I} \;\; X + Y \notin \mathcal{A}$

Theorem. (Ciesielski–Fejzić–Freiling, 2001) *For every $A \subseteq \mathbb{R}$ such that $A + A \notin \mathcal{N}$ there exists $X \subseteq A$ such that $X + X$ is nonmeasurable.*

Definition. *Let $\mathcal{I}$ be a σ-ideal of subsets of $\mathbb{R}$ and let $\mathcal{A} \supseteq \mathcal{I}$ be any family of subsets of $\mathbb{R}$. The pair $\langle \mathcal{A}, \mathcal{I} \rangle$ has the* <u>*Perfect Set Property*</u>*, if*

$$\forall X \in \mathcal{A} \setminus \mathcal{I} \quad \exists P \in Perf \quad P \subseteq X.$$

Theorem. *If the pair $\langle \mathcal{A}, \mathcal{I} \rangle$ has the Perfect Set Property then*

- *for every set A such that $A + A \notin \mathcal{I}$ there exists $X \subseteq A$ such that $X + X \notin \mathcal{A}$,*

- *for every pair of sets A, B such that $A + B \notin \mathcal{I}$ there exist $X \subseteq A$ and $Y \subseteq B$ such that $X + Y \notin \mathcal{A}$.*

Corollary. *If the pair* $\langle \mathcal{A}, \mathcal{I} \rangle$ *has the Perfect Set Property then the following conditions are equivalent*

1. $\exists X \in \mathcal{I} \;\; X + X \notin \mathcal{I}$,

2. $\exists X \in \mathcal{I} \;\; X + X \notin \mathcal{A}$,

3. $\exists X, Y \in \mathcal{I} \;\; X + Y \notin \mathcal{I}$,

4. $\exists X, Y \in \mathcal{I} \;\; X + Y \notin \mathcal{A}$.

Some applications:

1. $\mathcal{I} = \mathcal{N}$ and $\mathcal{A} = \mathcal{LM}$.

2. $\mathcal{I} = \mathcal{M}$ and $\mathcal{A} = \mathcal{BP}$.

3. $\mathcal{I}$ - any σ-ideal with coanalytic base and $\mathcal{A} = Bor[\mathcal{I}]$.

4. $\mathcal{I} = [\mathbb{R}]^\omega$ and $\mathcal{A} = Bor(\mathbb{R})$.

5. $\mathcal{I} = (s_0)$ and $\mathcal{A} = (s)$.

6. analogous algebras and ideals for other forcing notions.

Lemma. *There exists a subgroup $G \subseteq \mathbb{R}$ such that G is a Bernstein set in $\mathbb{R}$ and $|\mathbb{R}/G| = \omega$.*

Proof. Straightforward inductive construction. $\square$

Take $A + A \notin \mathcal{I}$. We will find $X \subseteq A$ such that $X + X \notin \mathcal{A}$.

- Put $\{T_n : n \in \omega\} = \mathbb{R}/G$ and $A_n = A \cap T_n$.
- $A + A = \bigcup_{n,m \in \omega} A_n + A_m$.
- $\exists n, m \in \omega \;\; A_n + A_m \notin \mathcal{I}$.
- Let $X = A_n \cup A_m$. Then $X \subseteq A$ and $X + X \notin \mathcal{I}$.

- $X + X$ intersects at most three cosets of G.

- the union of finitely many cosets is a Bernstein set.

- $X + X \notin \mathcal{I}$ and $X + X$ does not contain a perfect set.

- $X + X \notin \mathcal{A}$.

$\square$

We have shown that

$$A + A \notin \mathcal{N} \longrightarrow \exists X \subseteq A \;\; X + X \notin \mathcal{LM}$$

Can we always get such a set $X \in \mathcal{N}$?

No! A Sierpiński set is an obvious counterexample.

Definition. *A set $A \subseteq \mathbb{R}$ is called a Sierpiński set, if $|A| > \omega$ but*

$$\forall X \subseteq A \;\; X \in \mathcal{N} \longrightarrow |X| \leqslant \omega.$$

Theorem. *If $A \subseteq \mathbb{R}$ is a measurable set and $A + A \notin \mathcal{N}$ then there exists a <u>measure zero set</u> $X \subseteq A$ such that $X + X$ is not measurable.*

Sketch of the proof.

- If $A \in \mathcal{N}$ we are already done, so assume that $\lambda(A) > 0$.

- We can assume that A is Borel and has full measure.

- We find a countable transitive model $M \models \mathsf{ZFC}^*$ such that A is coded in M.

- If c is a Cohen real over M then in $M[c]$ there exists a Borel measure zero set $H \subseteq A$ such that $M[c] \models H + H \in \mathcal{N}^*$.

- $\mathbf{V} \models \Big(H \in \mathcal{N} \ \ \& \ \ H + H \in \mathcal{N}^*\Big)$.

- Using previously proved general theorem we obtain $X \subseteq H$ such that $X + X$ is non-measurable. $\square$

Similarly, we can get

Theorem. *If $A, B \subseteq \mathbb{R}$ are measurable sets of positive measure then there exist measure zero sets $X \subseteq A$ and $Y \subseteq B$ such that $X + Y$ is nonmeasurable.*

Question. *Is it true that if P is perfect and A has positive measure then there exist measure zero sets $X \subseteq P$ and $Y \subseteq A$ such that $X + Y$ is nonmeasurable?*

Theorem. (Erdős–Stone, Rogers) *There exists a Borel set $A \subseteq \mathbb{R}$ such that $A + A$ is not Borel.*

Proposition. *For every uncountable set $A \subseteq \mathbb{R}$ there exists $X \subseteq A$ such that $X + X$ is not Borel.*

Can we get X Borel, if A is?

No!

Theorem. *There exists a compact set $P \subseteq \mathbb{R}$ such that for every Borel sets $A, B \subseteq P$ the set $A + B$ is Borel.*