Nonmeasurable algebraic sums of sets of reals

Marcin Kysiak

Doksy, February 2004
Theorem. (Sierpiński, 1920) There exist measure zero sets $X, Y \subseteq \mathbb{R}$ such that the set $X + Y = \{x + y : x \in X, y \in Y\}$ is nonmeasurable.

Theorem. (Cichoń–Jasiński, 2003) Let $\mathcal{I}$ be a $\sigma$-ideal with coanalytic base and let $A = \text{Bor}[\mathcal{I}]$. Then the following conditions are equivalent

1. $\exists X, Y \in \mathcal{I} \; X + Y \notin \mathcal{I}$

2. $\exists X, Y \in \mathcal{I} \; X + Y \notin A$

Theorem. (Ciesielski–Fejzić–Freiling, 2001) For every $A \subseteq \mathbb{R}$ such that $A + A \notin \mathcal{N}$ there exists $X \subseteq A$ such that $X + X$ is nonmeasurable.
Definition. Let \( I \) be a \( \sigma \)-ideal of subsets of \( \mathbb{R} \) and let \( A \supseteq I \) be any family of subsets of \( \mathbb{R} \). The pair \( \langle A, I \rangle \) has the Perfect Set Property, if
\[
\forall X \in A \setminus I \; \exists P \in \text{Perf} \quad P \subseteq X.
\]

Theorem. If the pair \( \langle A, I \rangle \) has the Perfect Set Property then

- for every set \( A \) such that \( A + A \notin I \) there exists \( X \subseteq A \) such that \( X + X \notin A \),

- for every pair of sets \( A, B \) such that \( A + B \notin I \) there exist \( X \subseteq A \) and \( Y \subseteq B \) such that \( X + Y \notin A \).
Corollary. If the pair $\langle A,I \rangle$ has the Perfect Set Property then the following conditions are equivalent

1. $\exists X \in I \ X + X \not\in I,$

2. $\exists X \in I \ X + X \not\in A,$

3. $\exists X,Y \in I \ X + Y \not\in I,$

4. $\exists X,Y \in I \ X + Y \not\in A.$
Some applications:

1. $I = \mathcal{N}$ and $A = \mathcal{LM}$.

2. $I = \mathcal{M}$ and $A = B\mathcal{P}$.

3. $I$ - any $\sigma$-ideal with coanalytic base and $A = Bor[I]$.

4. $I = [\mathbb{R}]^\omega$ and $A = Bor(\mathbb{R})$.

5. $I = (s_0)$ and $A = (s)$.

6. analogous algebras and ideals for other forcing notions.
Lemma. There exists a subgroup $G \subseteq \mathbb{R}$ such that $G$ is a Bernstein set in $\mathbb{R}$ and $|\mathbb{R}/G| = \omega$.

Proof. Straightforward inductive construction. □

Take $A + A \not\in \mathcal{I}$. We will find $X \subseteq A$ such that $X + X \not\in A$.

- Put $\{T_n : n \in \omega\} = \mathbb{R}/G$ and $A_n = A \cap T_n$.

- $A + A = \bigcup_{n,m \in \omega} A_n + A_m$.

- $\exists n, m \in \omega \ A_n + A_m \not\in \mathcal{I}$.

- Let $X = A_n \cup A_m$. Then $X \subseteq A$ and $X + X \not\in \mathcal{I}$. 
• $X + X$ intersects at most three cosets of $G$.

• the union of finitely many cosets is a Bernstein set.

• $X + X \notin \mathcal{I}$ and $X + X$ does not contain a perfect set.

• $X + X \notin \mathcal{A}$.
We have shown that

$$A + A \notin \mathcal{N} \implies \exists X \subseteq A \quad X + X \notin \mathcal{LM}$$

Can we always get such a set $X \in \mathcal{N}$?

No! A Sierpiński set is an obvious counterexample.

**Definition.** A set $A \subseteq \mathbb{R}$ is called a Sierpiński set, if $|A| > \omega$ but

$$\forall X \subseteq A \quad X \in \mathcal{N} \implies |X| \leq \omega.$$ 

**Theorem.** If $A \subseteq \mathbb{R}$ is a measurable set and $A + A \notin \mathcal{N}$ then there exists a measure zero set $X \subseteq A$ such that $X + X$ is not measurable.
Sketch of the proof.

• If $A \in \mathcal{N}$ we are already done, so assume that $\lambda(A) > 0$.

• We can assume that $A$ is Borel and has full measure.

• We find a countable transitive model $M \models \text{ZFC}^*$ such that $A$ is coded in $M$.

• If $c$ is a Cohen real over $M$ then in $M[c]$ there exists a Borel measure zero set $H \subseteq A$ such that $M[c] \models H + H \in \mathcal{N}^*$.

• $V \models \left( H \in \mathcal{N} \land H + H \in \mathcal{N}^* \right)$.

• Using previously proved general theorem we obtain $X \subseteq H$ such that $X + X$ is non-measurable. \hfill \Box
Similarly, we can get

**Theorem.** If $A, B \subseteq \mathbb{R}$ are measurable sets of positive measure then there exist measure zero sets $X \subseteq A$ and $Y \subseteq B$ such that $X + Y$ is nonmeasurable.

**Question.** Is it true that if $P$ is perfect and $A$ has positive measure then there exist measure zero sets $X \subseteq P$ and $Y \subseteq A$ such that $X + Y$ is nonmeasurable?
Theorem. (Erdős–Stone, Rogers) There exists a Borel set $A \subseteq \mathbb{R}$ such that $A + A$ is not Borel.

Proposition. For every uncountable set $A \subseteq \mathbb{R}$ there exists $X \subseteq A$ such that $X + X$ is not Borel.

Can we get $X$ Borel, if $A$ is?

No!

Theorem. There exists a compact set $P \subseteq \mathbb{R}$ such that for every Borel sets $A, B \subseteq P$ the set $A + B$ is Borel.