Probability on graphs

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Problem set 8

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Problem 1. Consider the simple random walk X_t on the n-cycle $\mathbb{Z}_n$, started at $X_0 = x$. Let τ be the first time at which all the vertices of $\mathbb{Z}_n$ have been visited at least once. Prove that X_τ is distributed uniformly on $\mathbb{Z}_n \setminus \{x\}$.

Problem 2. Consider the random transpositions chain on the symmetric group S_n – we start from the identity permutation $\sigma_0 = id$ and at each step perform a transposition $\tau_{i,j}$ of two randomly chosen elements i, j (we allow the elements to be equal, in which case nothing happens). Formally, the transition matrix is given by

$$P(\sigma, \sigma') = \begin{cases} \frac{1}{n}, & \sigma = \sigma', \\ \frac{2}{n^2}, & \sigma' = \sigma \circ \tau_{i,j}, \\ 0, & \text{otherwise.} \end{cases}$$

Show that the mixing time satisfies $t_{mix} = O(n^2)$. Prove that for any $\varepsilon > 0$ and sufficiently large n we have $t_{mix} \geq \left(\frac{1}{2} - \varepsilon\right) n \log n$ (hint: consider the number of fixed points as a distinguishing statistic).

Problem 3. Let $S = \{0,1\}^n$ and consider the following transition matrix on S:

$$P(x, y) = \begin{cases} \frac{1}{2}, & (y_1, \ldots, y_{n-1}) = (x_2, \ldots, x_n), \\ 0, & \text{otherwise.} \end{cases}$$

One can imagine the chain as a "sliding window" of length n moving over an infinite sequence of independent bits. Show that the chain is ergodic with the uniform distribution as the stationary distribution and determine its mixing time $t_{mix}(\varepsilon)$.