Probability on graphs
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Problem set 6

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Problem 1. Let $G \sim G(n,d)$ be a uniformly random d-regular graph with $d \geq 3$. Prove that with high probability G is connected.

Problem 2. Let $G \sim G(n,2)$ be a uniformly random 2-regular graph. Find the asymptotic probability that G is connected.

Problem 3. In this problem we will show that in random d-regular graphs small sets have relatively good edge expansion. Let $d \geq 3$.

(a) Let W be a random configuration with dn half-edges. Fix $\varepsilon > 0$ and let K be set of dk half-edges satisfying $\frac{k}{n} \leq \min\{\frac{\varepsilon}{2d}, \frac{1}{2}\}$. Let $E(K)$ be the set of edges e such that both half-edges forming e are in K. Prove that

$$\mathbb{P}\left(|E(K)| \geq (1+\varepsilon)k\right) \leq e^{-\log\left(\frac{\varepsilon}{2d}\cdot\frac{n}{k}\right)\left(1+\frac{\varepsilon}{2}\right)k}.$$

(b) Let $G \sim G(n,d)$ be a uniformly random d-regular graph. Use the result from (a) to prove the following: for any $\varepsilon > 0$ there exists $\eta > 0$ such that with high probability all subsets $V \subseteq G$ of size at most ηn satisfy

$$\frac{|\partial V|}{|V|} \geq \frac{d}{2} - (1+\varepsilon).$$