Probability on graphs
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Problem set 5

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Problem 1. Let $G \sim G(n,p)$ with $p = \frac{1-\varepsilon}{n}$, $\varepsilon > 0$. We consider the *strictly subcritical* case, corresponding to $\varepsilon \in (0,1)$ being fixed, and the *barely subcritical* case, corresponding to $\varepsilon = \lambda n^{-1/3}$ with $\varepsilon = o(1)$, but $\lambda \to \infty$ as $n \to \infty$.

Prove that in both cases with high probability all connected components of G contain at most one cycle (in other words, for each component its number of edges $|E|$ and vertices $|V|$ satsifies $|E| - |V| + 1 \leq 1$).

Problem 2. Let $\lambda > 1$ and $G \sim G\left(n, \frac{\lambda}{n}\right)$. Let $\chi_\lambda = \mathbb{E}|\mathcal{C}(v)|$ be the expected size of the connected component of any fixed vertex v in G. Prove that

$$\chi_\lambda = \zeta_\lambda^2 n(1 + o(1)),$$

where ζ_λ is the survival probability of a Poisson branching process with parameter λ.

Problem 3. For a graph G let its 2-*core* $G^{(2)}$ be a graph obtained from G by successively removing vertices of degree 0 or 1 (i.e., at each step we remove a vertex of degree 0 or 1 until all remaining vertices have degree at least 2). Let $|G^{(2)}|$ denote its number of vertices.

Let $\lambda > 1$ and $G \sim G\left(n, \frac{\lambda}{n}\right)$. Let $\eta_\lambda = 1 - \zeta_\lambda$ be the extinction probability of a Poisson branching process with parameter λ.

(a) Prove that $\mathbb{E}|G^{(2)}| = (1 - \lambda\eta_\lambda)\zeta_\lambda n(1 + o(1))$.

(b) (bonus) Prove that in fact $|G^{(2)}| = (1 - \lambda\eta_\lambda)\zeta_\lambda n(1 + o(1))$ with high probability.