Probability on graphs

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Problem set 3

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Problem 1. Let X be the offspring distribution of a branching process (as usual we assume $\mathbb{P}(X = 1) < 1$) and let η be the extinction probability of the process.

(a) Prove that if $\mathbb{E}X \leq 1$, then $\eta = 1$, while for $\mathbb{E}X > 1$ we have $\eta < 1$.

(b) Prove that η is given by the smallest solution in $[0, 1]$ of the equation

$$\eta = G_X(\eta),$$

where $G_X(s) = \mathbb{E}s^X$.

Problem 2. Consider a branching process with offspring distribution $X \sim \text{Poiss}(\lambda)$. Let T^* denote its total progeny and let $\zeta_\lambda = 1 - \eta_\lambda$ denote its survival probability. Let $I_\lambda = \lambda - 1 - \log \lambda$.

(a) Prove that for $n \geq 1$ we have

$$\mathbb{P}(T^* = n) = \frac{(\lambda n)^{n-1}}{n!} e^{-\lambda n}$$

and deduce for $n \to \infty$ the asymptotic formula

$$\mathbb{P}(T^* = n) = \frac{1}{\sqrt{2\pi}\lambda n^{3/2}} e^{-I_\lambda n}(1 + o(1)).$$

(b) Deduce that the total progeny is typically either small or infinite: for any $\lambda > 0$ there exists n_0 such that for $n \geq n_0$ we have

$$\mathbb{P}(n \leq T^* < \infty) \leq e^{-I_\lambda n}.$$

(c) Prove that the function $\lambda \mapsto \zeta_\lambda$ is differentiable for any $\lambda > 1$ and we have

$$\zeta_\lambda = 2(\lambda - 1)(1 + o(1))$$

as $\lambda \searrow 1$.

Hint: for part (a) use the hitting time theorem we proved in the lecture. For part (c) use the formula for the extinction probability η_λ from the previous problem.

Problem 3. Consider a random walk with i.i.d. steps X_i taking nonnegative integer values. Let $S_n = X_1 + \ldots + X_n$ with $S_0 = 0$. Prove the following identity

$$\mathbb{P}\left(S_m < m \text{ for all } 1 \leq m \leq n | S_n = n - k\right) = \frac{k}{n}.$$

Hint: use the hitting time theorem proved in the lecture.