Probability on graphs

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Problem set 2

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Problem 1. Let $G \sim G(n,p)$ where $p = \frac{\log n + t}{n}$ for some constant $t \in \mathbb{R}$. Prove that the number of isolated vertices in G converges in distribution to a Poisson random variable with parameter e^{-t}.

Problem 2. Let $G = (V, E)$ be a graph with a vertex set V and edge set E. For any subset of vertices $S \subseteq V$ let ∂S denote the set of edges in E which have one endpoint in S and the other one in $V \setminus S$. Let

$$\mathrm{ex}(S) = \frac{|\partial S|}{\min\{|S|, |V \setminus S|\}}.$$

Suppose that $G \sim G(n,p)$ for $p = \frac{100 \log n}{n}$. Prove that there exist constants $\alpha > 0$, $\beta < 1$ such that

$$\mathbb{P}\left(\min_{S \subseteq V} \mathrm{ex}(S) < \alpha\right) < \beta.$$

Problem 3. For a graph G let X denote the number of isolated edges in G (i.e., edges whose both endpoints have degree one). Let $G \sim G(n,p)$ with $p = \frac{\lambda}{n}$.

(a) Let $\lambda = a \log n$ for fixed $a \in \mathbb{R}_+$. Prove that $X \to \infty$ in probability if $a < 1/2$, while $X \to 0$ in probability if $a > 1/2$.

(b) Prove that X converges in distribution if $\lambda = \frac{1}{2} \log n + \frac{1}{2} \log \log n + t$, with $t \in \mathbb{R}$ fixed, and identify the limiting distribution.