Probability on graphs
winter term 2024/2025
Problem set 11

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Problem 1. Consider the lazy random walk on the bipartite complete graph $K_{2,n}$ with two vertices on the one side and n vertices on the other side. Show that if the canonical paths Γ_{xy} in the definition of congestion B are chosen deterministically, then necessarily $B = \Omega(n)$. On the other hand, show that a randomized choice of Γ_{xy} can achieve $B = \Theta(1)$.

Problem 2. Let G be a graph constructed by joining two complete graphs on n vertices by a single edge. Using the method of canonical paths show thatthe mixing time of the simple random walk on G satsifies $t_{mix} = O(n^2 \log n)$.

Problem 3. Consider a reversible Markov chain $(X_t)_{t=0}^{\infty}$ on a state space S with stationary distribution π. Let γ^* be the absolute spectral gap of the chain and let $t = \lceil \frac{1}{\gamma^*} \rceil$. Prove that there exists a constant $C > 0$ such that for any function $f : S \to \mathbb{R}$ and any $m \geq 1$ we have

$$\mathrm{Var}_\pi \left(\sum_{i=0}^{m-1} f(X_{it}) \right) \leq Cm \mathrm{Var}_\pi f,$$

where $X_0 \sim \pi$. *Hint:* first prove the statement for f which is an eigenfunction of the chain.

Problem 4 (bonus)**.** Create a small work of art (picture, drawing, cartoon, sculpture, song, poem, movie etc.) illustrating your favorite concept or result from the course. Here artistic value and creative expression will be more important than the mathematical content, but your work should be related to the course material at least in *some* sense.