Probability on graphs

summer term 2019/2020

Problem set 8

Michał Kotowski

Problem 1. Prove that for a reversible chain the separation and total variation distances satisfy for any $t \geq 0$ the inequalities:

$$s(2t) \leq 1 - (1 - \bar{d}(t))^2 \leq 2\bar{d}(t) \leq 4d(t).$$

Problem 2. Consider the simple random walk X_t on the n-cycle $\mathbb{Z}_n$, started at $X_0 = x$. Let τ be the first time at which all the vertices of $\mathbb{Z}_n$ have been visited at least once. Prove that X_τ is distributed uniformly on $\mathbb{Z}_n \setminus \{x\}$.

Problem 3. Consider the coupon collector problem with n coupons. Let τ be the first time at which all coupons have been collected.

(a) Prove that $\mathbb{E}\tau = n \sum_{k=1}^{n} \frac{1}{k}$ and for any $c > 0$ we have

$$\mathbb{P}(\tau > \lceil n \log n + cn \rceil) \leq e^{-c}.$$

(b) Let $I_j(t)$ be the indicator of the event that the j-th coupon has not been collected by time t. Let $R(t) = \sum_{j=1}^{n} I_j(t)$. Prove that the random variables $I_j(t)$ are negatively correlated and setting $p_t = \left(1 - \frac{1}{n}\right)^t$ we have

$$\begin{aligned} &\mathbb{E}R_t = np_t, \\ &\mathrm{Var}R_t \leq np_t(1 - p_t) \leq \frac{n}{4}. \end{aligned}$$

Problem 4. Let G be the d-dimensional torus $\mathbb{Z}_n^d$, $d \geq 1$, i.e., the graph where an edge is present between two vertices $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d)$, $x_i, y_i \in \mathbb{Z}_n$, if there exists exactly one coordinate j such that $x_j = y_j \pm 1 \mod n$ and all the other coordinates are equal.

Prove that the mixing time of the lazy random walk on G satisfies $t_{mix} \leq d^2 n^2$.