Probability on graphs

summer term 2019/2020

Problem set 5

Michał Kotowski

Problem 1. Let X be the offspring distribution of a branching process (we will always assume that $\mathbb{P}(X = 1) < 1$) and let η be the extinction probability of the process. Prove the theorem on the phase transition in branching processes: if $\mathbb{E}X \le 1$, then $\eta = 1$, while for $\mathbb{E}X > 1$ we have $\eta < 1$. Prove that η is given by the smallest solution in $[0, 1]$ of the equation

$$\eta = G_X(\eta),$$

where $G_X(s) = \mathbb{E}s^X$.

Hint: if Z_n is the size of the n-th generation, find a recursion for $G_n(s) = \mathbb{E}s^{Z_n}$.

Problem 2. Consider a branching process with offspring distribution $X \sim \mathrm{Poiss}(\lambda)$. Let T^* denote its total progeny and let ζ_λ denote its survival probability. Let $I_\lambda = \lambda - 1 - \log \lambda$.

(a) Prove that for $n \ge 1$ we have

$$\mathbb{P}(T^* = n) = \frac{(\lambda n)^{n-1}}{n!} e^{-\lambda n}$$

and deduce that as $n \to \infty$ we have the asymptotic formula

$$\mathbb{P}(T^* = n) = \frac{1}{\sqrt{2\pi}\lambda n^{3/2}} e^{-I_\lambda n}(1 + o(1)).$$

(b) Deduce that the total progeny is typically either small or infinite: for any $\lambda > 0$ there exists n_0 such that for $n \ge n_0$ we have

$$\mathbb{P}(n \le T^* < \infty) \le e^{-I_\lambda n}.$$

(c) Prove that the function $\lambda \mapsto \zeta_\lambda$ is differentiable for any $\lambda > 1$ and we have

$$\zeta_\lambda = 2(\lambda - 1)(1 + o(1))$$

as $\lambda \searrow 1$.

Hint: for part (a) use the hitting time theorem. For part (c) use the formula for the extinction probability η_λ from the previous problem.

Problem 3. Consider a branching process with offspring distribution $X \sim \text{Bin}(n, p)$ and another branching process with offspring distribution $Y \sim \text{Poiss}(\lambda)$ for $\lambda = np$. Let $\mathbb{P}_{n,p}(T = k)$ denote the distribution of the total progeny T of the first process and let $\mathbb{P}_\lambda(T^* = k)$ denote the distribution of the total progeny T^* of the second process.

Prove that for any $k \geq 1$ we have

$$\mathbb{P}_{n,p}(T \geq k) = \mathbb{P}_\lambda(T^* \geq k) + e_n(k),$$

where $|e_n(k)| \leq \frac{k\lambda^2}{n}$.

Hint: if $W \sim \text{Bin}(n, p)$ and $V \sim \text{Poiss}(np)$, how can we couple W and V so that $\mathbb{P}(W \neq V)$ is as small as possible?