Gromov's monster group - notes

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Our goal is to present the construction of Gromov's *monster group* - a finitely generated group which does not embed coarsely into any Hilbert space. This is perhaps the most prominent example of how random objects can be useful in geometric group theory. We don't provide all the details and proofs here, since some parts of the construction, involving small cancellation theory and hyperbolicity of random groups, are rather involved. More comprehensive references for the topic include:

- Gromov's original paper [Gro03] (contains lots of ideas, but only sketches of proofs)
- Arzhantseva and Delzant's paper fleshing out Gromov's ideas [AD08] (somewhat hard to read, but contains all ingredients of the construction, including a general approach to graphical small cancellation theory)
- Ollivier's expository paper [Oll03] and references therein (easier to read than the previous two, sketches a different, combinatorial approach to small cancellation in random groups; see also [Oll05])

The main motivation for constructing groups which do not embed coarsely into Hilbert spaces came from looking for counterexamples to the Baum-Connes conjecture, although this is something I know very little about (see references in [AD08] for more context).

In these notes whenever we use the term "with high probability" it means "with probability approaching 1 as relevant parameters (e.g. relation length in a group presentation or size of a graph) go to infinity".

1 Outline of the construction

Theorem 1.1 (main theorem)**.** *There exists a finitely generated group G_∞ which cannot be coarsely embedded into any Hilbert space.*

Recall that a map $f : X \to Y$ between two metric spaces X, Y is a *coarse emebedding* if for all $x, x' \in X$:

$$\rho_-(d(x,x')) \leq d(f(x), f(x')) \leq \rho_+(d(x,x'))$$

for some functions ρ_-, ρ_+ such that $\rho_\pm(t) \xrightarrow{t\to\infty} \infty$. If X is a group with the word metric associated to a finite generating set (or more generally, a graph), then we can actually assume that any coarse embedding is a Lipschitz map, so the only interesting part is the bound on $\rho_-(t)$.

The idea of the construction is to find a group which in an appropriate sense contains an infinite sequence of expander graphs (this is natural since expanders embed badly into Hilbert spaces). More precisely, suppose we have an infinite sequence of expanders $\{\Gamma_n\}_{n=1}^{\infty}$ (satisfying some additional assumptions to be spelled out later). We will start with a group G_0, say, the free group, and inductively construct a sequence of groups $G_1, G_2, G_3, \ldots$ such that:

- G_{n+1} is a quotient of G_n
- Γ_n can be embedded *quasi-isometrically* into G_n and all subsequent quotients

The monster group will then be the direct limit $G_\infty = \varinjlim G_n$.

The groups G_n will be random groups in the following sense. To each graph Γ_n we will associate a random labelling of its edges by generators of G_0. This will define a set of random words $R(\Gamma_n)$ (coming from cycles in Γ_n) and G_{n+1} will be the quotient of G_n by $R(\Gamma_n)$. With high probability the random labelling will be "nice enough", so that the natural map $f_n : \Gamma_n \to G_{n+1} = G_n/\langle R(\Gamma_n)\rangle$ will be a quasi-isometric embedding. Furthermore, the random labelling will satisfy certain small cancellation condition, which is needed to prove that the groups we obtain are infinite, non-elementary and hyperbolic (hyperbolicity will be important for proving the quasi-isometricity of the embedding f_n).

For this construction to work our graphs Γ_n must have sufficiently quickly growing girth (recall that girth(Γ) is the length of the shortest cycle in Γ). Roughly speaking, girth of Γ_n will be equal to the typical relation length in $R(\Gamma_n)$. As the size of Γ_n grows and we take more relations, the relation length has to be large enough, otherwise there will likely be cancellations between the relators and we will not get an infinite group (compare this to the situation in the density model [Oll05] of random groups).

In addition, Γ_n will have to satisfy a certain "thinness" condition, meaning that they cannot have too many distinct paths of given length. How "thin" Γ_n has to be will depend on the spectral radius of G_n. Thus a crucial part of the construction will be obtaining a uniform bound on the spectral radii of the subsequent quotients G_n (otherwise we might have to take thinner and thinner graphs and lose expansion). This is were *property (T)* comes into play - we will show that with high probability a random group in the model above satisfies property (T) and this will imply the bound on the spectral radius.

There will be many parameters in the construction (like spectral radii and hyperbolicity constants of groups, injectivity radii of quotient maps, girth of graphs) which depend on each other in a complex way - we try to indicate these dependencies in the statements of theorems and lemmas. The issue of how quickly the parameters like girth and hyperbolicity constants grow is related to the notion of a *lacunary hyperbolic group*, see [AD08].

2 Randomly labelled graphs

Let Γ be a (finite or infinite) graph. Consider a finite set of generators $S = \{a_1, \ldots, a_d\}$. From now on all groups will be generated by the fixed set S.

A *labelling* of Γ is assigning to each edge of Γ an element of $S \cup S^{-1}$ and an orientation. When traversing an edge labelled with some generator s, we read off s if we traverse the edge in the direction consistent with the orientation and s^{-1} if we traverse it in the opposite direction.

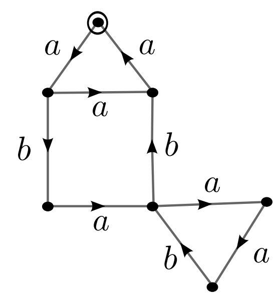

Figure 1: A graph Γ (with a chosen basepoint) labelled with $\{a, b\}$

Each directed path in Γ corresponds to a word over $S \cup S^{-1}$ (not necessarily reduced, i.e. it may have adjacent pairs of the form ss^{-1}). In what follows whenever a graph Γ appears, we implicitly assume that it comes with some labelling, so we will be often confusing paths in the graph and words corresponding to labels read off these paths.

Pick a vertex in Γ and consider the fundamental group of Γ with that vertex as the basepoint. This group is generated by words read along simple cycles in Γ. For example, for the graph from Figure 2 one set of generators is aaa, $ababa^{-1}a^{-1}$ and $a^{-1}b^{-1}aabba$. Denote this set of words by $R(\Gamma)$ (if the graph is not connected, we take the union of these sets over all connected components). Given any group G_0 generated by S, we can now consider a natural quotient group associated to the labelled graph:

$$G(\Gamma) := G_0 / \langle R(\Gamma) \rangle$$

Of course this group does not depend on the basepoint.

In the example above, if G_0 is the free group, the group $G(\Gamma)$ is given by the presentation $\langle a, b | aaa, baba^{-1}, aab \rangle$ (note that we have cyclically reduced the words from $R(\Gamma)$). We can think of the Cayley graph of $G(\Gamma)$ as being formed by “gluing together” copies of Γ.

Note that this notion generalizes the usual presentation of groups in terms of generators and relations - if $G = \langle S | r_1, \ldots, r_k \rangle$ for some words $r_1, \ldots, r_k$, by taking Γ to be a disjoint union of k cycles labelled by $r_1, \ldots, r_k$ and G_0 to be the free group, we get that $G(\Gamma) = G_0 / \langle R(\Gamma) \rangle = G$.

In our construction we will be interested in properties of random labellings. A *random labelling* consists of choosing each edge label and orientation at random, independently and uniformly over all elements of $S \cup S^{-1}$ and orientations.

Now, given a labelled graph Γ and the group $G(\Gamma)$, we have a natural map from the graph (considered as a metric space with the shortest path metric) to the group (with the word metric associated to the generating set S) - we pick a basepoint in Γ and for each vertex we choose a simple path leading to that vertex. The vertex is then mapped to the group element represented by the word read off from the label on the path. Note that this map is well defined, since two different simple paths leading to the same vertex close a cycle, but cycles are killed in G, so the paths represent the same group element.

In general this map doesn't respect the metric structure of Γ. In the example above the basepoint and the rightmost vertex have distance 3 in Γ, but the corresponding words ε (the empty word) and $a^{-1}b^{-1}a$ give us group elements e and a in $G(\Gamma)$, which have distance 1. We can see that this is because there are distinct cycles which have large segments labelled with the same words, which enforces very short relations in G. However, we will see that this cannot happen provided the labelling satisfies certain small cancellation conditions. In particular, we will show that for a randomly chosen labelling the map from Γ to G with high probability will be a quasi-isometry.

3 Small cancellation conditions

Since we won't be providing any details regarding small cancellation conditions which hold for random groups (see [Oll07] and other papers by Ollivier for a comprehensive treatment), we only describe the idea briefly. We start with the classical notion of small cancellation. Suppose we have a set of words R over $S \cup S^{-1}$. A *piece* is a subword which appears in two distinct words from R. For example if $R = \{aaab, aacc, abcd\}$, then aa and ab are pieces of length 2. *Small cancellation conditions* say that for a given set of relations R pieces cannot be too long. The two most common conditions are:

- $C'(\lambda)$: R satisfies $C'(\lambda)$ condition if for every piece w appearing in a word $r \in R$ we have $|w| < \lambda|r|$, where $\lambda > 0$ (note the strict inequality)
- $C(p)$: R satisfies $C(p)$ condition if every word in R is a concatenation of at least p pieces, where p is a natural number

For groups we usually assume that the set of relations R is closed under cyclic permutations and taking inverses.

For example, the presentation $\mathbb{Z}^2 = \langle a, b | aba^{-1}b^{-1}, ba^{-1}b^{-1}a, a^{-1}b^{-1}ab, b^{-1}aba^{-1} \rangle$ satisfies $C(4)$ condition (since all pieces are single letters), but not $C'(1/4)$ (since the inequality in the definition of $C'(\lambda)$ is strict). Of course $C'(\lambda)$ condition implies $C(\lfloor 1/\lambda \rfloor + 1)$ condition.

It is a classical result that if the group presentation satisfies $C'(1/6)$ condition, then the group is hyperbolic. We will need a similar result in the context of presenation coming from labelled graphs - this is called *graphical small cancellation theory*.

Given a labelled graph Γ, a graphical piece is a word which appears as a label on two distinct paths embedded in Γ. In the example from Figure 2, aa is a graphical piece, since it appears on two distincts paths (in the upper and the lower right triangle).

The graphical small cancellation conditions are:

- $Gr'(\lambda)$: a labelled graph Γ satisfies $Gr'(\lambda)$ condition if for every piece w appearing on a simple cycle c we have $|w| < \lambda|c|$
- $Gr(p)$: a labelled graph Γ satisfies $Gr(p)$ condition if every simple cycle in Γ is a concatenation of at least p pieces

There are also other variants of the definitions, see [Oll03]. In the case where Γ is a disjoint union of cycles, graphical small cancellation conditions reduce to the classical small cancellation conditions. The fact we will need is that for some $\lambda < 1/6$ a random group satisfies $Gr'(\lambda)$ condition - this will imply that the group is hyperbolic, infinite, non-elementary and torsion-free.

4 Assumptions about expanders

In the construction we will need a family of bounded-degree expanders $\{\Gamma_n\}_{n=1}^{\infty}$ satisfying the following conditions:

- $|\Gamma_n| \to \infty$ as $n \to \infty$ (where $|\Gamma_n|$ is the number of vertices in Γ_n)
- $\mathrm{diam}(\Gamma_n) \leq C\mathrm{girth}(\Gamma_n)$ for all n and some universal constant $C > 0$ (in particular $\mathrm{girth}(\Gamma_n) \to \infty$ as $n \to \infty$)
- graphs Γ_n are b-thin for some $b > 0$, meaning that:

$$\#\{\text{simple paths in } \Gamma_n \text{ of length at most } \frac{1}{2}\mathrm{girth}(\Gamma_n)\} \leq K\,(2d)^{b\,\mathrm{girth}(\Gamma_n)}$$

  for some universal constant $K > 0$ (recall that d is the number of generators in the group presentations we are considering).

Note that since obviously $\mathrm{diam}(\Gamma_n) \geq \frac{1}{2}\mathrm{girth}(\Gamma_n)$ we must have $C > 1/2$. From now on we will often denote $\mathrm{girth}(\Gamma_n)$ by g_n. The high girth assumption will be crucial in the construction, in particular we will need a subsequence of $\{\Gamma_n\}_{n=1}^{\infty}$ such that the girths grow exponentially quickly (in groups defined by labelled graphs the girth will correspond roughly to the shortest relation length).

In what follows we will need the thinness parameter b to be sufficiently small. A simple way to obtain arbitrarily small thinness while preserving the expander property of Γ_n is to to perform a subdivision of edges. Fix $j \geq 2$ and denote by Γ_n^j a graph obtained from Γ_n by dividing each edge of Γ_n into j new edges (this introduces a number of new vertices of degree 2). Obviously both diameter and girth increase j times after this operation, so the assumptions on girth are still satisfied. However, the number of paths in Γ_n^j of length at most $\frac{1}{2}\mathrm{girth}(\Gamma_n^j)$ is at most j^2 times the number of paths in Γ_n of length at most $\frac{1}{2}\mathrm{girth}\Gamma_n$ (subdividing the edges doesn't introduce any new paths apart from the possibility of choosing the starting and end vertex in j^2 ways). Thus:

$$\#\{\text{simple paths in } \Gamma_n^j \text{ of length at most } \frac{1}{2}\text{girth}(\Gamma_n^j)\} \leq j^2 K\, 2^{b\,\text{girth}(\Gamma_n)} = j^2 K\, 2^{\frac{b}{j}\text{girth}(\Gamma_n^j)}$$

and the new thinness is $\frac{b}{j}$. The subdivision decreases the spectral gap of Γ_n roughly by a factor of j^2, so as long as j is independent of n we stil obtain an expander family.

5 Main lemmas: quasi-isometries and hyperbolicity

The essence of the inductive construction described in the outline is the following lemma:

Lemma 5.1. *Let G_0 be hyperbolic, infinite with spectral radius $\rho < 1$. Let Γ be a graph satisfying the last two assumptions from Section 4. Fix $\lambda < 1/6$ and pick a random labelling of Γ. Then, provided that $g = \text{girth}(\Gamma)$ is large enough and thinness of Γ is small enough, with high probability we have the following:*

(a) For any word w labelling a (not necessairly simple) path in Γ we have:

$$\|\overline{w}\|_{G_0} \geq A(|w| - \varepsilon g)$$

for some constant $A > 0$, depending only on ρ, and $\varepsilon > 0$, depending on ρ and λ (here $\overline{w}$ denotes the element of G_0 represented by the word w and $\|\cdot\|_{G_0}$ is its distance from the identity in G_0)

(b) $G = G_0(\Gamma)$ satisfies $Gr'(\lambda)$ small cancellation condition

The lemma says that all words appearing on simple paths in Γ can be embedded as quasi-geodesics in G_0 (remember that ϕ which maps w to $\overline{w}$ is a map from the set of paths in Γ, or the universal cover of Γ, and not from Γ). Note that G_0 doesn't "know" anything about the metric structure of Γ. The idea of the proof is that for short lengths a randomly labelled path in Γ will emulate the simple random walk in G_0 and for long words we will employ hyperbolicity to promote local quasi-geodesics to global quasi-geodesics.

Proof. Let W_n denote the simple random walk in G_0. Since G_0 has spectral radius $\rho < 1$, we have the bound on return probabilities:

$$\mathbb{P}(W_n = e) \lesssim \rho^n = (2d)^{-\kappa n}$$

for $\kappa = -\log_{2d}\rho$ and sufficiently large n ("$\lesssim$" means an inequality up to a universal constant). We want to apply this inequality for all n, so we can e.g. introduce a factor of 1/2 in the exponent - we get that for all n:

$$\mathbb{P}(W_n = e) \lesssim (2d)^{-\frac{1}{2}\kappa n}$$

Let w be a word appearing in Γ on a fixed path of length between εg and $\frac{g}{2}$, where $\varepsilon > 0$ is some small universal constant to be determined later. Since this has to be a simple path, its labels are chosen independently and with the same distribution under the random labelling; thus w has the same distribution as W_n. It's easy to show [Oll04, Proposition 17] that:

$$\mathbb{P}\left(\|\overline{w}\|_{G_0} \leq \frac{\kappa}{2(1-\kappa)} |w| \right) \leq (2d)^{-\frac{\kappa}{4}|w|} \tag{1}$$

for sufficiently large $|w|$ (i.e. the simple random walk on G_0 has linear speed). Take g large enough so that this estimate holds for all w of length between εg and $\frac{g}{2}$. Let $A = \frac{\kappa}{2(1-\kappa)}$. We call a word w *bad* if $\|\overline{w}\|_{G_0} \leq A|w|$. The estimate (1) says that:

$$\mathbb{P}\left(\text{a fixed path of length between } \varepsilon g \text{ and } \frac{g}{2} \text{ gives a bad word} \right) \leq (2d)^{-\frac{\kappa}{4}|w|}$$

Now we do the union bound over all possible paths of length at most $\frac{g}{2}$ in Γ. If Γ is b-thin, then:

$$\#\{\text{simple paths in } \Gamma \text{ of length at most } \frac{g}{2}\} \leq K\, (2d)^{bg}$$

for some universal constant $K > 0$. Thus:

$$\begin{aligned} \mathbb{P}\left(\exists \text{ a path of length between } \varepsilon g \text{ and } \frac{g}{2} \text{ which gives a bad word} \right) &\leq K\, (2d)^{bg} (2d)^{-\frac{\kappa}{4}|w|} \\ &\leq K\, (2d)^{(b-\frac{\kappa}{4}\varepsilon)g} \end{aligned}$$

as $|w| \geq \varepsilon g$. We want this probability to go to 0 as $g \to \infty$, so we need:

$$b < \frac{\kappa}{4}\varepsilon$$

By discussion in Section 4 we can assume that b satisfies this condition by performing a subdivision of edges if necessary, replacing Γ with Γ^j. Crucially, what j we need to take depends only on ρ and not on G_0 or any other parameters.

Thus if b is small enough (depending on ρ), then with high probability every word w appearing in Γ on a path of length between εg and $\frac{g}{2}$ satisfies:

$$\|\overline{w}\|_{G_0} \geq A|w|$$

Recall that a map $f : X \to Y$ is an (α, β, γ)-*local quasi-isometric embedding* if for every x, x' such that $d(x, x') \leq \gamma$ we have:

$$\alpha d(x, x') - \beta \leq d(f(x), f(x')) \leq \frac{1}{\alpha} d(x, y) + \beta$$

Thus the map $w \mapsto \overline{w}$ is an $(A, A\varepsilon g, \frac{g}{2})$-local quasi-isometric embedding. We want to have a *global* quasi-isometric embedding, i.e. have the same inequality (with possibly worse parameters) for w of all lengths.

To get this, we use the fact that in hyperbolic groups local quasi-geodesics are in fact global quasi-geodesics:

Proposition 5.2. *[AD08, Theorem 3.7] If X is a δ-hyperbolic space, then for sufficiently large γ (depending on α, β and δ) every (α, β, γ)-local quasi-isometric embedding in X is an $(\frac{\alpha}{2}, \beta)$-quasi-isometric embedding. Furthermore, one can take:*

$$\gamma(\alpha, \beta, \delta) = \eta(\alpha, \delta) + 8\beta$$

where η is a function independent of β.

We assumed that G_0 is hyperbolic and in our case $\alpha = A$, $\beta = A\varepsilon g$ and $\gamma = \frac{g}{2}$. So we need:

$$\frac{g}{2} \geq \eta(A, \delta) + 8A\varepsilon g$$

where δ is the hyperbolicity constant of G_0. It is enough if we take ε such that, say, $8A\varepsilon < 1/4$ (depending only on ρ) and g large enough (depending only on on ρ and δ).

Thus the map $w \to \overline{w}$ is an $\left(\frac{A}{2}, A\varepsilon g\right)$-quasi-isometric embedding and for every word w we have:

$$\|\overline{w}\|_{G_0} \geq A(|w| - \varepsilon g)$$

as desired.

We skip the proof of the small cancellation condition, as it is rather involved. □

We know how to quasi-isometrically embed words appearing on paths in Γ into G_0. The next step is to obtain a quasi-isometric embedding of the graph Γ into the quotient group $G_0(\Gamma)$. Let $f : \Gamma \to G_0(\Gamma)$ denote the natural map from the labelled graph to the quotient (this comes of course from the map $w \mapsto \overline{w}$ on words).

Theorem 5.3. *Let Γ be a labelled graph satisfying $Gr'(\lambda)$ condition for some $\lambda < 1/6$ and such that that the map ϕ from Lemma 5.1, mapping w to $\overline{w}$, is an $\left(\frac{A}{2}, A\varepsilon g\right)$-quasi-isometric embedding. Then:*

(a) The group $G_0(\Gamma)$ is infinite hyperbolic torsion free

(b) The injectivity radius of the quotient map $\pi : G_0 \to G_0(\Gamma)$ is at least $\frac{Ag}{4}$

(c) The map $f : \Gamma \to G_0(\Gamma)$ is an $\left(\frac{A}{4C}, A\varepsilon g\right)$-quasi-isometric embedding

Recall that C is the constant such that $\mathrm{diam}(\Gamma) \leq C\mathrm{girth}(\Gamma)$ and *injectivity radius* of π is the largest r such that the quotient map $\pi : G_0 \to G_0(\Gamma)$ is injective on the ball of radius r around identity in G_0.

Proof. We skip the proofs of parts (a) and (b), since they are more involved (see [Oll03]). We only show how to put the pieces together to obtain the quasi-isometric embedding.

Let r be the injectivity radius of the quotient map. Take any $x, y \in \Gamma$. We have two possible cases:

(1) $d(f(x), f(y)) \le r$

Because the quotient map is injective on a ball of radius r, we can actually find points from G_0 which are mapped onto $f(x)$ and $f(y)$, respectively, such that:

$$d(f(x), f(y)) = d(\phi(x), \phi(y)) \ge \frac{A}{2} d(x, y) - A\varepsilon g$$

by the quasi-isometry assumption about ϕ (here we are confusing x and y with the paths representing them in Γ with respect to a given basepoint).

(2) $d(f(x), f(y)) > r$

Since the distance between x and y is at most $\mathrm{diam}(\Gamma)$, we have in particular:

$$d(f(x), f(y)) > r \ge r \frac{d(x, y)}{\mathrm{diam}(\Gamma)} \ge \frac{Ag}{4} \frac{1}{Cg} d(x, y) = \frac{A}{4C} d(x, y)$$

by part (b) and the assumption $\mathrm{diam}(\Gamma) \le C\mathrm{girth}(\Gamma)$.

From $C \ge 1/2$ we have $\frac{A}{4C} \le \frac{A}{2}$, so by combining the two bounds above we get the desired quasi-isometry.

□

6 Gromov monster group: construction

Now we proceed to construct the Gromov monster group.

Start from any group G_0 which is non-elementary hyperbolic, infinite and torsion-free (for example the free group F_d). Let $\{\Gamma_n\}_{n=1}^{\infty}$ be the expander sequence satisfying assumptions from Section 4 and let $g_n = \mathrm{girth}(\Gamma_n)$ (we identify Γ_n with the subdivision Γ_n^j to avoid notational complications).

We put a random labelling on Γ_1 and consider the random group $G_1 := G_0(\Gamma_1)$. We can assume that $\mathrm{girth}(\Gamma_1)$ is high enough and that Γ_1 is thin enough. By Lemma 5.1 with high probability the assumptions of Theorem 5.3 will be satisifed and G_1 will be an infinite, torsion-free hyperbolic group. Furthermore, the natural map $f_1 : \Gamma_1 \to G_1$ is an $\left(\frac{A}{4C}, A\varepsilon g_1\right)$-quasi-isometric embedding, where A and ε depend only on the spectral radius of G_0.

Now we can iterate this construction. Suppose that at the n-th step we have a group G_n which is hyperbolic, infinite and torsion-free. By passing to a subsequence, we can always assume that Γ_{n+1} has sufficiently high girth and is thin enough, so that when we put a random labelling on Γ_{n+1}, with high probability the assumptions of Theorem blaa will be satisifed. Thus we get the next quotient group $G_{n+1} = G_n(\Gamma_{n+1})$, which is hyperbolic, infinite and torsion-free, and the map $f_{n+1} : \Gamma_{n+1} \to G_{n+1}$ is an $\left(\frac{A}{4C}, A\varepsilon g_n\right)$-quasi-isometric embedding.

Note that at each step the parameters A, ε and the parameter j of the subdivision, needed to obtain thinness, depend on the spectral radius of the group G_n of the previous step (and not otherwise on the group). It could happen that this spectral radius is not bounded away from 1, so the quasi-isometry constants could get worse as n grows and Γ_n could cease being an expander sequence. In what follows we assume that for all groups G_n their spectral radii are uniformly bounded from above by ρ, the spectral radius of G_1. This is the case if G_1 has property (T) ([AD08]), which is true for random groups (see [Sil03]).

In this way we obtain a sequence of successive quotients:

$$G_0 \twoheadrightarrow G_1 \twoheadrightarrow \ldots \twoheadrightarrow G_n \twoheadrightarrow \ldots$$

and maps $f_n : \Gamma_n \to G_n$ such that for any $x, y \in \Gamma_n$ we have:

$$\frac{A}{4C} d(x, y) - A\varepsilon g_n \leq d(f_n(x), f_n(y)) \leq d(x, y)$$

Let G_∞ be the direct limit of the sequence G_n. It is a finitely generated (possibly infinitely presented) group which "coarsely contains" a sequence of expanders in the following sense:

Theorem 6.1. *There exist maps $f_n : \Gamma_n \to G_\infty$ such that for any $x, y \in \Gamma_n$ we have:*

$$\frac{A}{4C} d(x, y) - A\varepsilon g_n \leq d(f_n(x), f_n(y)) \leq d(x, y)$$

Proof. As each for each group G_n we have a quotient map from G_n to G_∞, this also defines the maps $f_n : \Gamma_n \to G_\infty$ in a natural way (compose $f_n : \Gamma_n \to G_n$ with the quotient to G_∞).

We already know that the quasi-isometric embedings $f_n : \Gamma_n \to G_n$ satisfy the inequality above. So the expander Γ_n embeds nicely into G_n, but it could get squashed when passing through subsequent quotient maps. We can ensure that this does not happen in the following way. The map $f_n : \Gamma_n \to G_n$ is 1-Lipschitz, so if the injectivity radius of π_n is greater than the diameter of Γ_n, the image $f_n(\Gamma_n)$ will map injectively into all subsequent groups $G_{n+1}, G_{n+2}, \ldots$ and thus into G_∞. This will give us the inequality in the statement of the theorem.

To ensure that the injectivity radius is greater than the diameter, note that the injectivity radius of the map $\pi_n : G_n \to G_{n+1}$ is at least $\frac{Ag_{n+1}}{4}$ by Theorem 5.3 and we have $\operatorname{diam}(\Gamma_n) \leq Cg_n$. So it is enough to have $\frac{Ag_{n+1}}{4} \geq Cg_n$, which can be ensured by passing to a subsequence of $\{\Gamma_n\}_{n=1}^{\infty}$. □

Technically speaking this does not give us a coarse embedding of the disjoint union $\sqcup \Gamma_n$ into G_∞, since we can have sequences of pairs (x_n, y_n) such that $d(x_n, y_n) \to \infty$, but $\frac{A}{4C} d(x_n, y_n) - A\varepsilon g_n$ stays bounded, for example when $d(x_n, y_n) = o(g_n)$. Nevertheless this is enough to prove our main result.

Note that the condition $g_{n+1} \geq \frac{C}{4A} g_n$ from the proof implies the exponential growth of the girth sequence g_n. This is related to the notion of lacunarity mentioned in the introduction.

From this we almost immediately get the main result:

Theorem 6.2. *The group G_∞ cannnot be coarsely embedded into any Hilbert space.*

Proof. It is well known that if $\{\Gamma_n\}_{n=1}^\infty$ is a family of expanders, then one can find sequences of vertices $x_n, y_n \in \Gamma_n$ such that $d(x_n, y_n) = \Omega(\log|\Gamma_n|)$, but under any Lipschitz embedding of the family Γ_n into a Hilbert space, the images of x_n, y_n stay within a bounded distance from each other (in other words, an expander on n vertices has ℓ^2 distortion $\Omega(\log n)$).

Suppose there exists $F : G_\infty \to \mathcal{H}$ - a Lipschitz coarse embedding into a Hilbert space. By composing f_n with F we obtain Lipschitz embeddings $F_n : \Gamma_n \to \mathcal{H}$.

Let $x_n, y_n \in \Gamma_n$ be such that $d(x_n, y_n) \geq K \log|\Gamma_n|$ for some $K > 0$, but $F_n(x_n)$, $F_n(y_n)$ are within distance bounded from each other. As F is a coarse embedding, $d(f_n(x_n), f_n(y_n))$ is bounded if and only if $d(F_n(x_n), F_n(y_n))$ is bounded.

However, by Theorem 5.3:

$$d(f_n(x_n), f_n(y_n)) \geq \frac{A}{4C} d(x_n, y_n) - A\varepsilon g_n \geq \frac{AK}{4C} \log|\Gamma_n| - A\varepsilon g_n$$

Assumptions about high girth of our expanders imply that $\log|\Gamma_n| \geq L g_n$ for some $L > 0$ (both g_n and $\operatorname{diam}(\Gamma_n)$ are of the order of $\sim \log|\Gamma_n|$), so:

$$d(f_n(x_n), f_n(y_n)) \geq \frac{AKL}{4C} g_n - A\varepsilon g_n$$

If we take ε to be small enough in our construction, the right hand side goes to infinity as $g_n \to \infty$. Thus $d(f_n(x_n), f_n(y_n))$ goes to infinity while $d(F_n(x_n), F_n(y_n))$ stays bounded - a contradiction. □

The only remaining point to be addressed is whether the expander family $\{\Gamma_n\}_{n=1}^\infty$ satisfying the assumptions from Section 4 actually exists. It turns out that one can use Phillips-Lubotzky-Sarnak expanders for this purpose (see discussion in Section 7 from [AD08]).

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