A WEAK COMPACTNESS RESULT FOR CRITICAL ELLIPTIC SYSTEMS OF n-HARMONIC TYPE

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Abstract. The paper considers systems of the form

$$-\operatorname{div}\left(|\nabla u|^{n-2}\nabla u\right) = |\nabla u|^{n-2}\Omega\cdot\nabla u$$

on a bounded domain in $\mathbb{R}^n$ with $u \in W^{1,n}$, a matrix $\Omega \in L^n$ (depending on u) and some additional structural assumptions on Ω. We prove that if a sequence of solutions of the above system converges weakly, the limit itself is also a solution. The class of systems considered includes the n-harmonic system and the presented reasoning is a generalization of C. Wang's proof for n-harmonic maps.

1. Introduction

Let $\mathcal{N} \subseteq \mathbb{R}^L$ be a compact, smooth submanifold with no boundary and $U \subseteq \mathbb{R}^n$ a bounded domain. The Sobolev space $W^{1,n}(U,\mathcal{N})$ is defined as

$$W^{1,n}(U,\mathcal{N}) := \left\{u \in W^{1,n}(U,\mathbb{R}^L) : u(x) \in \mathcal{N} \text{ for a.e. } x \in U\right\}.$$

The Dirichlet n-energy functional $E_n\colon W^{1,n}(U,\mathcal{N}) \to \mathbb{R}$ is given by

$$E_n(u) = \int_U |\nabla u|^n \,\mathrm{d}x = \int_U \left(\sum_{\alpha,j}\left|\frac{\partial u^j}{\partial x_\alpha}\right|^2\right)^{n/2} \mathrm{d}x.$$

Here, the sum is taken over all $\alpha = 1,\dots,n$ and $j = 1,\dots,L$.

A map $u \in W^{1,n}(U,\mathcal{N})$ is an n-harmonic map, if it is a critical point of E_n in the space $W^{1,n}(U,\mathcal{N})$ with respect to variations in the range, i.e. it satisfies the Euler-Lagrange system

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{n-2}\nabla u\right) &= |\nabla u|^{n-2}A(u)(\nabla u,\nabla u) \\ &= |\nabla u|^{n-2}\sum_{\alpha=1}^{n} A(u)\left(\frac{\partial u}{\partial x_\alpha},\frac{\partial u}{\partial x_\alpha}\right) \end{aligned} \tag{1.1}$$

called the n-harmonic system. Here div is the divergence operator in $\mathbb{R}^n$ and $A(\cdot)(\cdot,\cdot)$ is the second fundamental form of the submanifold $\mathcal{N} \subseteq \mathbb{R}^L$. We could define the Dirichlet p-energy functional $E_p\colon W^{1,p}(U,\mathcal{N}) \to \mathbb{R}$ and p-harmonic maps for any exponent $1 \le p \le n$ in exactly the same way. However, in this paper we focus on the critical exponent.

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Changyou Wang proved in his 2005 paper [14] that any weak limit of n-harmonic maps is itself n-harmonic. Actually, his result is a bit more general – the sequence of functions u_k is only assumed to satisfy a perturbed n-harmonic system, as long as the perturbation tends to zero in $\left(W^{1,n}(U,\mathcal{N})\right)^*$ with $k \to \infty$. Nonetheless, the following can be considered the main result of Wang's paper [14, Corollary C].

Theorem 1.1 (Wang, 2005). *Assume that $n \geq 2$ and that a sequence $u_k \in W^{1,n}(U,\mathcal{N})$ of n-harmonic maps tends to u weakly in $W^{1,n}(U,\mathcal{N})$, then u is an n-harmonic map.*

Our goal is to generalize Wang's result to a slightly wider class of equations. Namely, the n-harmonic system can be expressed in the form

$$-\delta\left(|du|^{n-2}du\right) = |du|^{n-2}\Omega \cdot du$$

for some matrix Ω of 1-forms. As we explain later, one can choose Ω to be the antisymmetric matrix

$$\Omega_{ij} = \sum_{l=1}^{L}\left(A^i_{jl} - A^j_{il}\right) du^l, \quad i,j = 1,\ldots,L,$$

where A^i_{jl} are the coefficients of the second fundamental form of $\mathcal{N}$ at point u. Note that the matrix Ω depends on the choice of the function u. It is crucial for the proof that the entries of this matrix have particular structure – they are sums of products of the form Ddv, where $D \in W^{1,n}_b := W^{1,n} \cap L^\infty$ and $v \in W^{1,n}$. Therefore, it is natural to assume that the matrix Ω takes the form

$$\Omega_{ij} = \sum_{s=1}^{m} D^s_{ij} dv^s_{ij}, \quad i,j = 1,\ldots,L \tag{1.2}$$

for some m and $D^s_{ij} \in W^{1,n}_b$, $v^s_{ij} \in W^{1,n}$. The antisymmetry of Ω is also important for the proof (see Theorem 2.8).

Our main theorem is formulated in the generality presented above, hence it is a slight generalization of Wang's result.

Theorem 1.2. *Let $U \subseteq \mathbb{R}^n$ be a bounded domain and $n \geq 2$. Assume that each of the functions $u_k \in W^{1,n}(U,\mathbb{R}^L)$ satisfies the system*

$$-\delta\left(|du_k|^{n-2}du_k\right) = |du_k|^{n-2}\Omega_k \cdot du_k, \tag{1.3}$$

where the entries of the antisymmetric matrices $\Omega_k \in L^n(U, \mathrm{so}(L) \otimes \Lambda^1\mathbb{R}^n)$ of 1-forms take the form (1.2) for some m and the coefficients D, v are uniformly bounded:

$$\sup_{i,j,k,s} ||D^{sk}_{ij}||_{W^{1,n}_b},\ ||\nabla v^{sk}_{ij}||_{L^n} < \infty.$$

If in addition $u_k \rightharpoonup u$ in $W^{1,n}$ and $\Omega_k \rightharpoonup \Omega$ in L^n, then u satisfies the limit system

$$-\delta\left(|du|^{n-2}du\right) = |du|^{n-2}\Omega \cdot du. \tag{1.4}$$

The above statement remains valid if we only assume the functions u_k to satisfy the system (1.3) in the perturbed form. Thus the following remark corresponds to the aforementioned theorem of Wang [14, Th. B].

Remark 1.3. *Assume that each of the functions $u_k \in W^{1,n}(U, \mathbb{R}^L)$ satisfies the perturbed system*

$$-\delta\left(|du_k|^{n-2}du_k\right) = |du_k|^{n-2}\Omega_k \cdot du_k + \Phi_k, \tag{1.5}$$

where $\Phi_k \to 0$ w $\left(W^{1,n}(U, \mathbb{R}^L)\right)^*$, and all other assumptions of Theorem 1.2 hold. Then the limit function u satisfies the system (1.4).

With assumptions as above, our proof remains valid with only minor changes. We shall discuss these after we complete the proof of Theorem 1.2. Now we explain how Wang's Theorem 1.1 follows from Theorem 1.2.

Proof of Theorem 1.1. The problem is to show that the n-harmonic system (1.1) is of the form (1.4). More precisely, to express the term $A(u)(\nabla u, \nabla u)$ as $\Omega \cdot du$ for a matrix Ω of the required form. To this end we shall use a trick first introduced by Frederic Hèlein [8, Ch. 4]. The form A can be understood as a symmetric bilinear form on $T\mathbb{R}^L|_{\mathcal{N}}$; it is enough to choose it to be zero on the orthogonal complement of $T\mathcal{N}$. Let $A_{il} = A(u)(e_i, e_l)$ be the matrix of A in the standard basis of $\mathbb{R}^L$. It follows from the orthogonality $A_{il} \perp T\mathcal{N}$ that

$$\sum_{j=1}^{L} A_{il}^j \nabla u^j = 0.$$

Multiplying the above by ∇u^l and summing over $l = 1, 2, \ldots, L$ yields

$$\sum_{j,l=1}^{L} A_{il}^j \nabla u^l \cdot \nabla u^j = 0.$$

Subtracting from $A(\nabla u, \nabla u)$ the expression above, we obtain

$$\begin{aligned} A^i(\nabla u, \nabla u) &= \sum_{j,l=1}^{L} A_{jl}^i \nabla u^l \cdot \nabla u^j \\ &= \sum_{j,l=1}^{L} \left(A_{jl}^i - A_{il}^j\right) \nabla u^l \cdot \nabla u^j \\ &= \sum_{j=1}^{L} \Omega_{ij} \cdot du^j, \end{aligned}$$

where the matrix

$$\Omega_{ij} = \sum_{l=1}^{L} \left(A_{jl}^i - A_{il}^j\right) du^l$$

is antisymmetric, $\Omega \in L^n(U, \mathrm{so}(L) \otimes \Lambda^1\mathbb{R}^n)$. Note also that $A(u)$ as a composition of the functions $u \in W^{1,n}$ and $A \in C_0^\infty$ is in the class $W_b^{1,n}$ and $||A(u)||_{W^{1,n}} \lesssim_{\mathcal{N}} ||u||_{W^{1,n}}$. Hence the matrix Ω is of the required form $\Omega_{ij} = \sum_{s=1}^{m} D_{ij}^s dv_{ij}^s$ (1.2) if we set $m = L$. Moreover, by the Rellich-Kondrashov theorem, the convergence $u_k \rightharpoonup u$ in $W^{1,n}(U, \mathcal{N})$ implies $u_k \to u$

in $L^{n/(n-1)}$. Thus by composing with $A \in C_0^\infty$ we get $A(u_k) \to A(u)$ in $L^{n/(n-1)}$ and finally, $\Omega_k \rightharpoonup \Omega$ in L^n by Lemma 2.1.

We presented the n-harmonic equation in the form (1.4). This allows us to use Theorem 1.2, pass to the limit and conclude that u is an n-harmonic map. □

The proof of Theorem 1.2 presented in this paper follows the proof of Wang. However, we take an approach that allows us to use the same methods in a more general setting. In particular, the equation we consider is not directly connected to the geometry of manifolds, hence it is necessary to use Uhlenbeck decomposition theorem in an abstract formulation due to Tristan Rivière [12].

The key difficulty is that the right-hand side of the equation (1.3) is only bounded in L^1 and therefore we can only assume that it is weakly-* convergent to some measure. The difference of this measure and the right-hand side of the limit equation (1.4) can be proved to satisfy a reverse Hölder-type inequality, then the concentration compactness principle due to Pierre-Louis Lions (Lemma 2.6) together with a capacity argument allows us to prove that the difference is actually zero.

The main idea of the proof is to use the Hodge and Uhlenbeck decompositions in order to obtain expressions with a div-curl structure. Then it is possible to conclude their higher regularity and also convergence through application of the compensated compactness lemma (Proposition 2.5) and the classical div-curl lemma, yielding the desired reverse Hölder's inequality.

2. Preliminaries

2.1. **Notation.** The proof of Theorem 1.2 relies on the particular differential structure of certain terms in our reasoning (see Proposition 2.5). Therefore we find it more convenient to use the language of differential forms. We use the following notation for the differential operators connected to standard differential structure of the space $\mathbb{R}^n$:

- $\Lambda^s\mathbb{R}^n$ – the space of alternating s-forms on $\mathbb{R}^n$ (note that the standard notation would be $\Lambda^s(\mathbb{R}^n)^*$),
- d – the exterior derivative,
- $\alpha \cdot \beta$ – the inner product of two forms $\alpha, \beta \in \Lambda^s\mathbb{R}^n$,
- δ – the codifferential.

Vector and matrix-valued functions are used frequently in this paper and most of the equalities are formulated for vector-valued functions. For the sake of brevity we shall abuse the notation presented above. As an example, for a matrix $\Omega \in M_{L\times L}(\mathbb{R}) \otimes \Lambda^1\mathbb{R}^n$ of 1-forms and a vector $v \in \mathbb{R}^L \otimes \Lambda^1\mathbb{R}^n$ of 1-forms, the product $\Omega \cdot v$ is to be understood as a matrix product with scalar multiplication replaced by the inner product of forms, i.e

$$(\Omega \cdot v)^i = \sum_{j=1}^{L} \Omega_{ij} \cdot v^j.$$

The meaning should be always clear from the context.

We denote the space of bounded Sobolev functions by $W_b^{1,n} = W^{1,n} \cap L^\infty$. Note that this space is closed under multiplication in view of the elementary inequalities

$$||\nabla(fg)||_{L^n} \leq ||f\nabla g||_{L^n} + ||g\nabla f||_{L^n} \leq ||f||_{L^\infty}||\nabla g||_{L^n} + ||g||_{L^\infty}||\nabla f||_{L^n}.$$

The asymptotic inequality symbol $\lesssim$ is used here very often. The statement $A \lesssim B$ for two expressions A, B reads: there is a constant $C > 0$ such that $A \leq CB$. The constant is assumed to depend only on fixed parameters such as the space dimensions n, L. Sometimes, to emphasize the dependence on some of these parameters, we shall write e.g. $\lesssim_n$.

2.2. **Technical lemmata.** First we note two elementary observations, which we shall use frequently. The utility of the second one comes from the fact that the Sobolev and Rellich-Kondrashov theorems imply a continuous embedding $W^{1,p} \hookrightarrow L^q$ for $q \leq p^* = \frac{np}{n-p}$, which is compact only for $q < p^*$. Therefore we may assume a weakly convergent sequence in $W^{1,p}$ to be strongly convergent (up to a subsequence) in L^q for $q < p^*$, but this is not necessarily true for $q = p^*$.

Lemma 2.1 (on weak convergence)**.**

- *Let $p, q, r \geq 1$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $u_k \to u$ in L^p and $v_k \rightharpoonup v$ in L^q, then $u_k v_k \rightharpoonup uv$ in L^r.*
- *Let the space U have finite measure, $1 < p < \infty$, $v_k \in L^p(U)$ be a bounded sequence and $v \in L^p(U)$. If $v_k \rightharpoonup v$ in $L^1(U)$, then also $v_k \rightharpoonup v$ in $L^p(U)$.*

The next lemma gives us particular test functions, which will be used to remove the singularities from the equation in Section 3.3. The construction relies on the existence of unbounded functions in the space $W^{1,n}(\mathbb{R}^n)$. The proof follows the paper of Goldstein et al. [6, Lemma 3.2].

Lemma 2.2. *There is a sequence of functions $\varphi_k \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \varphi_k \leq 1$, $\varphi_k \equiv 1$ on the ball $B(0, r_k)$, $\varphi_k \equiv 0$ outside the ball $B(0, R_k)$, where $0 < r_k < R_k \to 0$ and $||\varphi_k||_{W^{1,n}} \to 0$.*

Proof. Let us define the function

$$\eta(r) = \log\log(e - \log r) \quad \text{for } r \in (0,1).$$

For $r \in (0,1)$ we have $e - \log r > e$ and hence $\log(e - \log r) > 1$; this implies that $\eta(r) > 0$. The function η has an infinite limit at zero and the asymptotics of its derivative can be estimated as follows:

$$|\eta'(r)| = \frac{1}{r(e-\log r)\log(e-\log r)} \lesssim \frac{1}{r(e-\log r)} \sim -\frac{1}{r\log r},$$

Cauchy's condensation test then concludes that the function $r^{n-1}|\eta'(r)|^n$ has a finite integral on the interval $(0,1)$, if only $n > 1$.

Now we introduce the radial function $f(x) = \eta(|x|)$. We will show that $f \in W^{1,n}(\mathbb{B}^n)$. It is easy to check that $f \in L^n(\mathbb{B}^n)$. Rotational invariance yields $|\nabla f(x)| = |\eta'(|x|)|$, so we can compute

$$\int_{\mathbb{B}^n} |\nabla f(x)|^n \,\mathrm{d}x = \int_0^1 \int_{S_r} |\eta'(r)|^n \,\mathrm{d}y \,\mathrm{d}r = \int_0^1 |S_r| \, |\eta'(r)|^n \,\mathrm{d}r < \infty,$$

where the last integral is finite by our previous considerations.

The idea of the construction of the sequence φ_k is the following: divide the graph of f into horizontal 'slices', i.e. choose φ_k so that they satisfy $\sum_{k=1}^{m} \varphi_k = \min(f, m)$ for each $m = 1, 2, \ldots$. We formalize it in a slightly different manner. Choose a function $\zeta \in C^\infty(\mathbb{R})$ satisfying $0 \le \zeta \le 1$, $\zeta \equiv 0$ on $(-\infty, 0]$, $\zeta \equiv 1$ on $[1, \infty)$. Define $\varphi_k(x) = \zeta(f(x) - k)$ for $x \in \mathbb{B}^n$ and $k = 1, 2, \ldots$. Evidently $0 \le \varphi_k \le 1$. Moreover,

$$\begin{aligned} f(x) &\le k &&\Rightarrow \varphi_k(x) = 0, \\ f(x) &\ge k+1 &&\Rightarrow \varphi_k(x) = 1, \end{aligned}$$

which implies that φ_k has the required behaviour. Since the function f is smooth except at the origin, the function φ_k is smooth in $\mathbb{B}^n$ and can be extended by zero to a function $\varphi_k \in C_0^\infty(\mathbb{R}^n)$. Finally, $|\nabla \varphi_k| \lesssim |\nabla f| \cdot \chi_{\{f > k\}}$ and $f \in W^{1,n}$ imply that $||\varphi_k||_{W^{1,n}} \to 0$. □

The following compactness theorem for maps with p-Laplacian bounded in L^1 is due to Hardt, Lin and Mou and independently to Courilleau [2, Th. 1.1]. Here we state it with $p = n$.

Lemma 2.3. *Let $U \subseteq \mathbb{R}^n$ be a bounded domain and u_k be a bounded sequence in $W^{1,n}(U, \mathbb{R}^L)$ such that the sequence*

$$\operatorname{div}\left(|\nabla u_k|^{n-2} \nabla u_k\right)$$

is bounded in $L^1(U, \mathbb{R}^L)$. Assume that $u_k \rightharpoonup u$ in $W^{1,n}(U, \mathbb{R}^L)$. Then there is a subsequence for which we have $\nabla u_k \to \nabla u$ in $L^q(U, \mathbb{R}^L)$ for any $1 \le q < n$.

The proof of Remark 1.3 requires a more generalized version of the above lemma.

Lemma 2.4. *Let $U \subseteq \mathbb{R}^n$ be a bounded domain and u_k be a bounded sequence in $W^{1,n}(U, \mathbb{R}^L)$ such that*

$$\operatorname{div}\left(|\nabla u_k|^{n-2} \nabla u_k\right) = f_k + \Phi_k,$$

where the sequence f_k is bounded in $L^1(U, \mathbb{R}^L)$ and $\Phi_k \to 0$ in $\left(W^{1,n}(U, \mathbb{R}^L)\right)^*$. Assume that $u_k \rightharpoonup u$ in $W^{1,n}(U, \mathbb{R}^L)$. Then there is a subsequence for which we have $\nabla u_k \to \nabla u$ in $L^q(U, \mathbb{R}^L)$ for any $1 \le q < n$.

The proof of Courilleau [2, Th. 1.1] applies here as well. The only difference lies in the estimation of the integral

$$\int_U \operatorname{div}\left(|\nabla u_k|^{n-2} \nabla u_k\right) \vec{\beta}_\varepsilon(u_k - u) \xi \, \mathrm{d}x$$

appearing in [2], which in our case is decomposed into a sum of two terms. Since $\Phi_k \to 0$ in $\left(W^{1,n}\right)^*$ and the sequence $\vec{\beta}_\varepsilon(u_k - u)\xi$ is bounded in $W^{1,n}$, the additional term $\left\langle \Phi_k, \vec{\beta}_\varepsilon(u_k - u)\xi \right\rangle$ tends to zero.

2.3. **Spaces $\mathcal{H}^1$ and** BMO**.** Now we recall the basic properties of the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ and the space $\mathrm{BMO}(\mathbb{R}^n)$.

A function $f \in L^1(\mathbb{R}^n)$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$, if

$$f_* := \sup_{\varepsilon > 0} |\phi_\varepsilon * f| \in L^1(\mathbb{R}^n),$$

where $\phi_\varepsilon(x) := \varepsilon^{-n}\phi\left(\frac{x}{\varepsilon}\right)$ and $\phi \in C_0^\infty(\mathbb{R}^n)$ is a fixed nonnegative function satisfying $\int_{\mathbb{R}^n} \phi \,\mathrm{d}y = 1$. The space $\mathcal{H}^1(\mathbb{R}^n)$ equipped with the norm

$$||f||_{\mathcal{H}^1(\mathbb{R}^n)} := ||f||_{L^1(\mathbb{R}^n)} + ||f_*||_{L^1(\mathbb{R}^n)}.$$

is a Banach space.

A function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to the space $\mathrm{BMO}(\mathbb{R}^n)$ (bounded mean oscillation, see [10]), if

$$||f||_{\mathrm{BMO}(\mathbb{R}^n)} := \sup\left\{\frac{1}{|B|}\int_B |f - f_B| \,\mathrm{d}y : B - \text{ball in } \mathbb{R}^n\right\} < \infty,$$

where $f_B = \frac{1}{|B|}\int_B f \,\mathrm{d}y$ is the mean of f over the ball B.

A function $f \in \mathrm{BMO}(\mathbb{R}^n)$ belongs to the subspace $\mathrm{VMO}(\mathbb{R}^n)$ (vanishing mean oscillation), if in addition we have the convergence

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |f - f_B| \,\mathrm{d}y \overset{r\to 0}{\longrightarrow} 0$$

uniformly in $x \in \mathbb{R}^n$. This is a closed subspace of $\mathrm{BMO}(\mathbb{R}^n)$ and it can be alternatively characterised as the closure of the subspace $C_0^\infty(\mathbb{R}^n)$ in the norm $\mathrm{BMO}(\mathbb{R}^n)$. Note that for a function $f \in W^{1,n}(\mathbb{R}^n)$ we have

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |f - f_B| \,\mathrm{d}y \lesssim_n \left(\int_{B(x,r)} |\nabla f|^n \,\mathrm{d}y\right)^{1/n}$$

thanks to the Poincaré inequality, hence $W^{1,n}(\mathbb{R}^n) \subseteq \mathrm{VMO}(\mathbb{R}^n)$ and

$$||f||_{\mathrm{BMO}(\mathbb{R}^n)} \lesssim ||\nabla f||_{L^n(\mathbb{R}^n)}.$$

Fefferman and Stein [5] proved that the dual space to $\mathcal{H}^1(\mathbb{R}^n)$ is $\mathrm{BMO}(\mathbb{R}^n)$. Moreover,

$$\left|\int_{\mathbb{R}^n} fg \,\mathrm{d}y\right| \lesssim ||f||_{\mathcal{H}^1(\mathbb{R}^n)}||g||_{\mathrm{BMO}(\mathbb{R}^n)},$$

for any $f \in \mathcal{H}^1(\mathbb{R}^n)$ and $g \in L^\infty \cap \mathrm{BMO}(\mathbb{R}^n)$. Additionally, the dual space to $\mathrm{VMO}(\mathbb{R}^n)$ is $\mathcal{H}^1(\mathbb{R}^n)$.

We recall an important theorem of Coifman, Lions, Meyer and Semmes [1] (see also [4]) on compensated compactness.

Proposition 2.5. *Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Assume that we have $f \in W^{1,p}(\mathbb{R}^n)$ and $g \in W^{1,q}(\mathbb{R}^n, \Lambda^2\mathbb{R}^n)$. Then $df \cdot \delta g \in \mathcal{H}^1(\mathbb{R}^n)$, moreover*

$$||df \cdot \delta g||_{\mathcal{H}^1(\mathbb{R}^n)} \lesssim ||\nabla f||_{L^p(\mathbb{R}^n)}||\nabla g||_{L^q(\mathbb{R}^n)}.$$

In particular, by the duality of $\mathcal{H}^1(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$ we obtain the inequality

$$\left|\int_{\mathbb{R}^n} h df \cdot \delta g \,\mathrm{d}y\right| \lesssim ||\nabla f||_{L^p(\mathbb{R}^n)}||\nabla g||_{L^q(\mathbb{R}^n)}||\nabla h||_{L^n(\mathbb{R}^n)}$$

for any $h \in W^{1,n}_b(\mathbb{R}^n)$.

2.4. **Concentration-compactness.** We shall use the concentration-compactness principle of P.-L. Lions.

Lemma 2.6. *Let μ, ν be finite positive measures on the ball $\mathbb{B} \subseteq \mathbb{R}^n$. Assume that*

$$\left(\int_B |\phi|^q \, \mathrm{d}\nu \right)^{1/q} \lesssim \left(\int_B |\phi|^p \, \mathrm{d}\mu \right)^{1/p} \quad \text{for all } \phi \in C_0^\infty(B),$$

where $1 \leq p < q \leq \infty$. Then there exists an at most countable set J, a sequence of distinct points $(x_j)_{j\in J}$ in $\mathbb{B}$ and a positive sequence $(a_j)_{j\in J}$ such that

$$\nu = \sum_{j\in J} a_j \delta_{x_j}.$$

In particular, $\sum_{j\in J} a_j < \infty$.

In the statement above we have the ball B in place of the space $\mathbb{R}^n$, see [11, Remark 1.5]. We shall apply the lemma also to signed measures, as explained below.

Remark 2.7. *In Lemma 2.6 we can drop the assumption that ν is positive. Then we need to assume the inequality*

$$\left| \int_B |\phi|^q \, \mathrm{d}\nu \right|^{1/q} \lesssim \left(\int_B |\phi|^p \, \mathrm{d}\mu \right)^{1/p} \quad \text{for all } \phi \in C_0^\infty(B)$$

and as a result the measure ν has the form

$$\nu = \sum_{j\in J} a_j \delta_{x_j},$$

where $a_j \in \mathbb{R}$, $\sum_{j\in J} |a_j| < \infty$.

Proof. Apply Hahn's decomposition to obtain $\nu = \nu^+ - \nu^-$, where ν^+, ν^- are mutually singular positive measures. As in the proof of Lemma 2.6 [11], we can show that the inequality above holds for any bounded function ϕ. Hence it also holds for measures ν^+ and ν^- in place of ν. The application of Lemma 2.6 ends the proof. $\square$

2.5. **The Uhlenbeck decomposition.** The last key tool of the proof is a decomposition first introduced by Karen Uhlenbeck [13]. It was used by Wang [14] to choose an orthogonal basis of the tangent bundle, for which the connection matrix of 1-forms is coclosed. Here we use an abstract statement of this theorem due to Tristan Rivière [12] – it does not refer to the geometry of manifolds and thus is better adjusted to the proof of Theorem 1.2.

Theorem 2.8. *There is $\varepsilon > 0$ such that for any antisymmetric matrix $\Omega \in L^n(\mathbb{B}^n, \mathrm{so}(L) \otimes \Lambda^1\mathbb{R}^n)$ of 1-forms on $\mathbb{B}^n$ satisfying $||\Omega||_{L^n} < \varepsilon$ there exists a function $P \in W^{1,n}(\mathbb{B}^n, \mathrm{SO}(L))$ taking values in orthogonal matrices such that the matrix*

$$\widetilde{\Omega} = P^{-1}dP + P^{-1}\Omega P \tag{2.1}$$

is coclosed, i.e. $\delta\widetilde{\Omega} = 0$, moreover

$$||\widetilde{\Omega}||_{L^n} + ||dP||_{L^n} \lesssim_{n,L} ||\Omega||_{L^n}. \tag{2.2}$$

Remark 2.9. *It is easily checked using a substitution of the form $x \mapsto x_0 + rx$ that in fact Theorem 2.8 holds for every ball $B(x_0, r) \subseteq \mathbb{R}^n$ with the same value of $\varepsilon > 0$ and the same constant in the inequality* (2.2).

3. Proof of the main theorem

3.1. **Application of the Uhlenbeck decomposition.** Suppose that the assumptions of Theorem 2.8 are satisfied for every k, and so we can apply the Uhlenbeck decomposition to each of the matrices Ω_k and choose P_k, $\widetilde{\Omega}$ as in (2.1), (2.2). Without loss of generality $P_k \rightharpoonup P$ in $W^{1,n}$ and $\widetilde{\Omega}_k \rightharpoonup \widetilde{\Omega}$ in L^n. Then it is easily checked that P, $\widetilde{\Omega}$ satisfy the conditions (2.1), (2.2) for the matrix Ω.

Recall that our system is the following:

$$-\delta\left(|du|^{n-2}du\right) = |du|^{n-2}\Omega \cdot du. \tag{1.4}$$

Now we transform it using the above decomposition. We calculate:

$$-\delta(|du|^{n-2}P^{-1}du) = -dP^{-1}|du|^{n-2}du - P^{-1}\delta(|du|^{n-2}du).$$

Taking into account the equality $dP^{-1} = -P^{-1}dPP^{-1}$ for the first summand and the equation (1.4) for the second, we obtain

$$\begin{aligned}
-\delta(|du|^{n-2}P^{-1}du) &= P^{-1}dPP^{-1}|du|^{n-2}du + P^{-1}|du|^{n-2}\Omega \cdot du \\
&= |du|^{n-2}\left(P^{-1}dP + P^{-1}\Omega P\right) \cdot P^{-1}du \\
&= |du|^{n-2}\widetilde{\Omega} \cdot P^{-1}du.
\end{aligned}$$

Thus we obtained a coclosed matrix $\widetilde{\Omega}$ in place of Ω. It is also important that $\widetilde{\Omega}$ has the form required by Theorem 1.2. Recall that

$$\Omega_{ij} = \sum_{s=1}^{m} D^s_{ij} dv^s_{ij}, \quad i, j = 1, \ldots, L, \tag{1.2}$$

for some m and $D^s_{ij} \in W^{1,n}_b$, $v^s_{ij} \in L^n$. We shall check that this is the case for the matrix $\widetilde{\Omega} = P^{-1}dP + P^{-1}\Omega P$. Indeed, the first term is

$$\left(P^{-1}dP\right)_{ij} = \sum_{s=1}^{L} P_{si} dP_{sj},$$

and so it has the form (1.2) with L summands. The second term is

$$\left(P^{-1}\Omega P\right)_{ij} = \sum_{s,t=1}^{L} P_{si}\Omega_{st}P_{tj}.$$

Thanks to the form of Ω and the fact that the space $W^{1,n}_b$ is closed under multiplication, the entries of the second term are also of the form (1.2). Therefore $\widetilde{\Omega}$ has the form (1.2) with $L + mL^2$ summands instead of m.

3.2. **Local convergence.** Our aim is to prove that each side of the system (1.3) converges to the corresponding term of the system (1.4) in the sense of distributions. Since Theorem 2.8 on the Uhlenbeck decomposition applies only to matrices Ω of small L^n norm, we shall prove the convergence locally, on sufficiently small balls. The following local lemma is the key point of the proof of Theorem 1.2. It proves our claim under an additional assumption that the norm $||\Omega||_{L^n}$ is small enough, up to possible point singularities.

Lemma 3.1. *Let the functions $u_k \in W^{1,n}(B, \mathbb{R}^L)$, $\Omega_k \in L^n(B, \mathrm{so}(L) \otimes \Lambda^1\mathbb{R}^n)$ satisfy the conditions of Theorem 1.2 on a ball $B \subseteq U$, $u_k \rightharpoonup u$ in $W^{1,n}$, $\Omega_k \rightharpoonup \Omega$ in L^n. Suppose that $||\Omega_k||_{L^n(B)} < \varepsilon$ for each k with $\varepsilon = \varepsilon(n, L)$ from Theorem 2.8. Then $u \in W^{1,n}(B, \mathbb{R}^L)$ satisfies the system (1.4) with possible singularities, i.e.*

$$-\delta(|du|^{n-2}P^{-1}du) = |du|^{n-2}\widetilde{\Omega} \cdot P^{-1}du + \sum_{j \in J} a_j \delta_{x_j} \text{ in } B, \tag{3.1}$$

where the set J is at most countable, $x_j \in B$, $a_j \in \mathbb{R}^L$, $\sum_{j \in J} |a_j| < \infty$.

Proof. The assumption $||\Omega_k||_{L^n(B)} < \varepsilon$ allows us to apply the Uhlenbeck decomposition inside the ball B. Let us adopt the notation introduced in Section 3.1. The system (1.4) takes an equivalent form

$$-\delta(|du|^{n-2}P^{-1}du) = |du|^{n-2}\widetilde{\Omega} \cdot P^{-1}du.$$

It follows from Lemma 2.3 that we can assume the convergence $du_k \to du$ in $L^q(B)$ for $1 \le q < n$. Using this and the convergence $P_k^{-1} \to P^{-1}$ in L^p for every $p < \infty$ we obtain, by Lemma 2.1,

$$|du_k|^{n-2}P_k^{-1}du_k \rightharpoonup |du|^{n-2}P^{-1}du \text{ in } L^1(B).$$

Hence

$$\delta\left(|du_k|^{n-2}P_k^{-1}du_k\right) \to \delta\left(|du|^{n-2}P^{-1}du\right) \text{ in } \mathcal{D}'(B).$$

We shall now investigate the convergence of the term $|du_k|^{n-2}\widetilde{\Omega}_k \cdot P_k^{-1}du_k$. Consider extensions of u_k, P_k^{-1} from the ball B to the whole of $\mathbb{R}^n$, vanishing outside the ball $2B$ and with corresponding norm estimates. Namely, we require the norms

$$||\nabla u_k||_{L^n(\mathbb{R}^n)},\ ||\nabla P_k^{-1}||_{L^n(\mathbb{R}^n)},\ ||P_k^{-1}||_{L^\infty(\mathbb{R}^n)}$$

not to exceed the corresponding norms on the ball B times a constant. This is possible for the norm L^∞, as the matrices P_k are orthogonal. We have

$$|||du_k|^{n-2}P_k^{-1}du_k||_{L^{n/(n-1)}(\mathbb{R}^n)} \le ||P_k^{-1}||_{L^\infty(\mathbb{R}^n)}||\nabla u_k||^{n-1}_{L^n(\mathbb{R}^n)} \lesssim ||\nabla u_k||^{n-1}_{L^n(B)}.$$

Thus we can apply the Hodge decomposition theorem (cf. Iwaniec, Martin [9]) for $|du_k|^{n-2}P_k^{-1}du_k \in L^{n/(n-1)}(\mathbb{R}^n, \mathbb{R}^L \otimes \Lambda^1\mathbb{R}^n)$ and choose

$$f_k \in W^{1,n/(n-1)}\left(\mathbb{R}^n, \mathbb{R}^L\right),\ g_k \in W^{1,n/(n-1)}\left(\mathbb{R}^n, \mathbb{R}^L \otimes \Lambda^2\mathbb{R}^n\right)$$

satisfying

$$|du_k|^{n-2}P_k^{-1}du_k = df_k + \delta g_k, \tag{3.2}$$

$$||\nabla f_k||_{L^{n/(n-1)}(\mathbb{R}^n)} + ||\nabla g_k||_{L^{n/(n-1)}(\mathbb{R}^n)} \lesssim ||\nabla u_k||^{n-1}_{L^n(B)}.$$

By these estimates, we can assume without loss of generality that $f_k \rightharpoonup f$ and $g_k \rightharpoonup g$ in $W^{1,n/(n-1)}(B)$. Passing to the limit in the equation (3.2) and using Lemma 2.3 again, we obtain

$$|du|^{n-2}P^{-1}du = df + \delta g \text{ in } B.$$

The right-hand side term of our equation can then be rewritten as

$$|du_k|^{n-2}\widetilde{\Omega}_k \cdot P_k^{-1}du_k = \widetilde{\Omega}_k \cdot df_k + \widetilde{\Omega}_k \cdot \delta g_k.$$

Since $df_k \rightharpoonup df$ in $L^{n/(n-1)}(B)$, $\widetilde{\Omega}_k \rightharpoonup \widetilde{\Omega}$ in $L^n(B)$ and $\delta\widetilde{\Omega}_k = 0$ in B, the classical div-curl lemma gives

$$\widetilde{\Omega}_k \cdot df_k \to \widetilde{\Omega} \cdot df \text{ in } \mathcal{D}'(B).$$

However, we can show this directly using integration by parts. Recall that $f_k \to f$ in $L^{n/(n-1)}(B)$ and $\widetilde{\Omega}_k \rightharpoonup \widetilde{\Omega}$ in L^n, so $\widetilde{\Omega}_k \cdot f_k \rightharpoonup \widetilde{\Omega} \cdot f$ in L^1. For any $\phi \in C_0^\infty(B)$,

$$\begin{aligned}
\int_{\mathbb{R}^n} \widetilde{\Omega}_k \cdot df_k\phi \,\mathrm{d}x &= -\int_{\mathbb{R}^n} \widetilde{\Omega}_k \cdot f_k d\phi \,\mathrm{d}x \\
&\to -\int_{\mathbb{R}^n} \widetilde{\Omega} \cdot f d\phi \,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \widetilde{\Omega} \cdot df\phi \,\mathrm{d}x.
\end{aligned}$$

Recall that by our assumptions each entry of the matrix $\widetilde{\Omega}_k$ is a sum of $L + mL^2$ 1-forms, each of the form Ddv, where $D \in W_b^{1,n}$ and $v \in W^{1,n}$; the norms of D, v are uniformly bounded. For the sake of simplicity we shall consider only one summand at a time and assume $\widetilde{\Omega}_k = D_k dv_k$ for further calculations; here D_k takes matrix values and v_k vector values. At the end, one just needs to sum up the estimates obtained in this way. Without loss of generality we assume that $D_k \rightharpoonup D$ and $v_k \rightharpoonup v$ in their corresponding spaces. Let us decompose the difference $\widetilde{\Omega}_k \cdot \delta g_k - \widetilde{\Omega} \cdot \delta g$ in the following manner:

$$\begin{aligned}
\widetilde{\Omega}_k \cdot \delta g_k - \widetilde{\Omega} \cdot \delta g &= D_k dv_k \cdot \delta g_k - Ddv \cdot \delta g \\
&= (D_k - D)d(v_k - v) \cdot \delta(g_k - g) \\
&\quad + Dd(v_k - v) \cdot \delta(g_k - g) \\
&\quad + D_k d(v_k - v) \cdot \delta g + (D_k dv \cdot \delta g_k - Ddv \cdot \delta g) \\
&= I + II + IIIa + IIIb.
\end{aligned}$$

Convergence of the summands III to 0. The convergence of $IIIa$, $IIIb$ to 0 in $\mathcal{D}'(B)$ is a straightforward consequence of the Sobolev embedding theorem and technical Lemma 2.1. Indeed, we know that $D_k \rightharpoonup D$, $v_k \rightharpoonup v$ in $W^{1,n}$ (and thus strongly in L^p for $p < \infty$), $g_k \rightharpoonup g$ in $W^{1,n/(n-1)}$ and the sequence D_k is bounded in L^∞. This implies $D_k d(v_k - v) \rightharpoonup 0$ in L^1, so also in L^n, and finally $IIIa \rightharpoonup 0$ in L^1. Similarly, $D_k^T \delta g_k \rightharpoonup D^T \delta g$ in L^1, so also in $L^{n/(n-1)}$, and by Lemma 2.1 again we have $dv \cdot D_k^T \delta g_k \rightharpoonup dv \cdot D^T \delta g$ in L^1. This, together with the linear algebra identity $Ddv \cdot \delta g = dv \cdot D^T \delta g$, yields $IIIb \rightharpoonup 0$ in L^1.

Convergence of the summand II **to** 0**.** Consider extensions of the functions v_k, v, D from the ball B to the space $\mathbb{R}^n$ with corresponding norm estimates; we can assume the convergence $v_k \rightharpoonup v$ in $W^{1,n}$, $g_k \rightharpoonup g$ in $W^{1,n/(n-1)}$ is preserved. It follows that $d(v_k - v) \cdot \delta(g_k - g) \to 0$ in $\mathcal{D}'(\mathbb{R}^n)$; the argument is similar to that for the term $\widetilde{\Omega}_k \cdot df_k$. By Proposition 2.5, the sequence $d(v_k - v) \cdot \delta(g_k - g)$ is bounded in $\mathcal{H}^1(\mathbb{R}^n)$, so without loss of generality it is weakly-* convergent to some function $s \in \mathcal{H}^1(\mathbb{R}^n)$. Then we know that

$$\int \phi d(v_k - v) \cdot \delta(g_k - g) \, \mathrm{d}x \to 0 \text{ for } \phi \in C_0^\infty(\mathbb{R}^n),$$

$$\int \phi d(v_k - v) \cdot \delta(g_k - g) \, \mathrm{d}x \to \int \phi s \, \mathrm{d}x \text{ for } \phi \in \mathrm{VMO}(\mathbb{R}^n).$$

which shows that $s = 0$. Choose any test function $\phi \in C_0^\infty(B)$; since $\phi D \in W^{1,n}(\mathbb{R}^n)$, we have $\phi D \in \mathrm{VMO}(\mathbb{R}^n)$, and so

$$\int \phi D d(v_k - v) \cdot \delta(g_k - g) \, \mathrm{d}x \to 0,$$

which proves $II \to 0$ in $\mathcal{D}'(B)$.

Convergence of the summand I**.** For $k = 1, 2, \ldots$ we define the auxiliary measures

$$\begin{aligned} \mathrm{d}\nu_k &= ((D_k - D) d(v_k - v) \cdot \delta(g_k - g)) \, \mathrm{d}x, \\ \mathrm{d}\mu_k &= \left(|\nabla(D_k - D)|^n + |\nabla(v_k - v)|^n + |\nabla(g_k - g)|^{n/(n-1)} \right) \mathrm{d}x. \end{aligned}$$

Up to this point, we have proved that for any $\phi \in C_0^\infty(B)$,

$$\begin{aligned} &\int \phi \left(|du_k|^{n-2} \widetilde{\Omega}_k \cdot P_k^{-1} du_k - |du|^{n-2} \widetilde{\Omega} \cdot P^{-1} du \right) \mathrm{d}x \\ &= \int \phi (D_k - D) d(v_k - v) \cdot \delta(g_k - g) \, \mathrm{d}x + o(1) \\ &= \int \phi \, \mathrm{d}\nu_k + o(1). \end{aligned}$$

Fix any $\phi \in C_0^\infty(B)$. Now we derive the following estimate:

$$\left| \int \phi^{n+1} \, \mathrm{d}\nu_k \right| \lesssim \left(\int |\phi|^n \, \mathrm{d}\mu_k \right)^{(n+1)/n} + o(1).$$

First, observe that

$$\begin{aligned} \int \phi^{n+1} \, \mathrm{d}\nu_k &= \int \phi^{n+1} (D_k - D) d(v_k - v) \cdot \delta(g_k - g) \, \mathrm{d}x \\ &= \int \phi (D_k - D) d(\phi(v_k - v)) \cdot \delta(\phi^{n-1}(g_k - g)) \, \mathrm{d}x + o(1). \end{aligned}$$

Indeed, the difference of the integrands is a sum of products, each of them having at least one of the factors $v_k - v$, $g_k - g$ not differentiated. Recall that by the Sobolev and Rellich-Kondrashov theorems,

$$\begin{array}{llll} D_k - D & \to 0 & \text{weakly in } W^{1,n}, & \text{strongly in } L^p,\ p < \infty, \\ v_k - v & \to 0 & \text{weakly in } W^{1,n}, & \text{strongly in } L^p,\ p < \infty, \\ g_k - g & \to 0 & \text{weakly in } W^{1,n/(n-1)}, & \text{strongly in } L^p,\ p < \frac{n}{n-2}. \end{array}$$

Hence the difference converges weakly to zero in L^1 by Hölder's inequality and Lemma 2.1. We apply Proposition 2.5 to the obtained integral and get

$$\left|\int \phi(D_k - D) d(\phi(v_k - v)) \cdot \delta(\phi^{n-1}(g_k - g)) \, \mathrm{d}x\right|$$
$$\leq ||\phi(D_k - D)||_{\mathrm{BMO}} ||\nabla(\phi(v_k - v))||_{L^n} ||\nabla(\phi^{n-1}(g_k - g))||_{L^{n/(n-1)}}.$$

Each of the factors is estimated in the same manner:

$$\begin{aligned} ||\phi(D_k - D)||_{\mathrm{BMO}} &\lesssim ||\nabla(\phi(D_k - D))||_{L^n} \\ &\leq ||\phi\nabla(D_k - D)||_{L^n} + ||\nabla\phi||_{L^\infty} ||D_k - D||_{L^n} \\ &\leq \left(\int |\phi|^n \, \mathrm{d}\mu_k\right)^{1/n} + o(1), \end{aligned}$$

$$\begin{aligned} ||\nabla(\phi(v_k - v))||_{L^n} &\leq ||\phi\nabla(v_k - v)||_{L^n} + ||\nabla\phi||_{L^\infty} ||v_k - v||_{L^n} \\ &\leq \left(\int |\phi|^n \, \mathrm{d}\mu_k\right)^{1/n} + o(1), \end{aligned}$$

$$\begin{aligned} ||\nabla(\phi^{n-1}(g_k - g))||_{L^{\frac{n}{n-1}}} &\leq ||\phi^{n-1}\nabla(g_k - g)||_{L^{\frac{n}{n-1}}} + ||\nabla\phi^{n-1}||_{L^\infty} ||g_k - g||_{L^{\frac{n}{n-1}}} \\ &\leq \left(\int |\phi|^n \, \mathrm{d}\mu_k\right)^{(n-1)/n} + o(1). \end{aligned}$$

Multiplying these inequalities gives

$$\left|\int \phi^{n+1} \, \mathrm{d}\nu_k\right| \lesssim \left(\int |\phi|^n \, \mathrm{d}\mu_k\right)^{(n+1)/n} + o(1),$$

as claimed. Now it follows from Hölder's inequality that the densities of measures ν_k and μ_k are uniformly bounded in $L^1(B)$, so we can assume without loss of generality that

$$\nu_k \longrightarrow \nu, \ \mu_k \longrightarrow \mu \ \text{ weakly-* in } M(B).$$

Moreover, since the measures μ_k are positive, μ is positive as well. The obtained estimate gives us

$$\left|\int \phi^{n+1} \, \mathrm{d}\nu\right| \lesssim \left(\int |\phi|^n \, \mathrm{d}\mu\right)^{(n+1)/n}.$$

If we replace ϕ by ϕ^2, where $\phi \in C_0^\infty(B)$, we obtain

$$\left|\int |\phi|^{2(n+1)} \, \mathrm{d}\nu\right|^{\frac{1}{2(n+1)}} \lesssim \left(\int |\phi|^{2n} \, \mathrm{d}\mu\right)^{\frac{1}{2n}}.$$

Then Lemma 2.6 (concentration compactness) and Remark 2.7 imply that ν is a finite measure of the form

$$\nu = \sum_{j \in J} a_j \delta_{x_j},$$

where the set J is at most countable, $x_j \in B$, $a_j \in \mathbb{R}^L$, $\sum_{j\in J} |a_j| < \infty$. Thus we proved that passing to the limit in the system (1.3) yields

$$-\delta(|du|^{n-2} P^{-1} du) = |du|^{n-2} \widetilde{\Omega} \cdot P^{-1} du + \sum_{j \in J} a_j \delta_{x_j} \text{ in } B. \tag{3.1}$$

□

3.3. Removing the singularities.

Proof of Theorem 1.2. We begin by showing that in the obtained system (3.1) the singular term $\sum_{j\in J} a_j \delta_{x_j}$ is in fact necessarily zero. To this end we use technical Lemma 2.2. Let us choose the functions φ_k as constructed in the lemma.

Choose any $j_0 \in J$ and for simplicity assume that $x_{j_0} = 0 \in B$. The equation (3.1) can be tested with the function $\psi_k = (\pm\varphi_k, \ldots, \pm\varphi_k)$, where the signs are chosen to match the signs of the coordinates of the vector a_{j_0}. Thus we obtain the following:

$$\underset{\displaystyle 0}{\underset{\displaystyle\downarrow}{\left\langle -\delta(|du|^{n-2}P^{-1}du), \psi_k \right\rangle}} = \underset{\displaystyle 0}{\underset{\displaystyle\downarrow}{\left\langle |du|^{n-2}\widetilde{\Omega}P^{-1}\cdot du, \psi_k \right\rangle}} + |a_{j_0}| + \underset{\displaystyle 0}{\underset{\displaystyle\downarrow}{\textstyle\sum_{j\neq j_0} a_j \cdot \psi_k(x_j)}}$$

Indeed, the condition $||\varphi_k||_{W^{1,n}} \to 0$ implies the convergence of the first term. Since $0 \le \varphi_k \le 1$ and $\operatorname{supp}\varphi_k \subseteq B(0, R_k)$, $R_k \to 0$, the convergence of the second term follows from the Lebesgue dominated convergence theorem. The third term is similar: $a \in l^1$ and for each $j \neq 0$ and k large enough we have $\psi_k(x_j) = 0$. Therefore we proved that $a_{j_0} = 0$; in consequence the whole singular part is zero.

Now we can proceed from local to global considerations. Since the sequence $|\Omega_k|^n$ is bounded in $L^1(U)$, without loss of generality it is weakly-* convergent to some finite positive measure η on U. Let $\varepsilon > 0$ be as in Lemma 3.1 and let us define the set

$$S = \{x \in U : \eta(\{x\}) \ge \varepsilon^n\}.$$

The set S is finite, as $|S|\varepsilon^n \le \eta(S) \le \eta(U) < \infty$. For any point $x \in U \setminus S$ there is a radius $r > 0$ for which $\eta(B(x, 2r)) < \varepsilon^n$. We have

$$\limsup_{k\to\infty} \int_{B(x,r)} |\Omega_k|^n \, \mathrm{d}y \le \eta(B(x, 2r)),$$

hence $\int_{B(x,r)} |\Omega_k|^n \, \mathrm{d}y < \varepsilon^n$ for k large enough. Lemma 3.1 and our previous remarks on the singularities allow us to conclude that u satisfies the system (1.4) on the ball $B(x, r)$. Hence u satisfies the system (1.4) on the set $U \setminus S$.

In order to finish the proof, we exploit Lemma 2.2 again. Assume that the set S consists of points $x_1, \ldots, x_s$. Choose any test function $\psi \in C_0^\infty(U, \mathbb{R}^L)$ and decompose it into two parts as follows:

$$\psi(x) = \psi(x) \sum_{j=1}^{s} \varphi_k(x - x_j) + \psi(x) \left(1 - \sum_{j=1}^{s} \varphi_k(x - x_j)\right) =: \psi_{0,k}(x) + \psi_{1,k}(x).$$

Note that $\operatorname{supp}\psi_{1,k} \subseteq U \setminus S$ and so we can test the system (1.4) with the function $\psi_{1,k}$. On the other hand, $\psi_{0,k} \to 0$ in $W^{1,n}(U)$ and in a dominated way, so by passing to the limit $k \to \infty$ we obtain the system (1.4) tested with the funtion ψ. This ends the proof. □

Proof of Remark 1.3. Assume that the functions u_k satisfy the perturbed system (1.5). We only need to show that in this case the conclusion of Lemma 3.1 holds as well. Indeed, the remaining part of the proof of Theorem 1.2 is still valid. Observe that in our case the Uhlenbeck decomposition gives the following:

$$-\delta(|du_k|^{n-2}P_k^{-1}du_k) = |du_k|^{n-2}\widetilde{\Omega}_k \cdot P_k^{-1}\nabla u_k + P_k^{-1}\Phi_k.$$

For any $\psi \in C_0^\infty(B, \mathbb{R}^L)$ we have

$$\left|\langle P_k^{-1}\Phi_k, \psi\rangle\right| = |\langle \Phi_k, P_k\psi\rangle| \leq ||\Phi_k||_{(W^{1,n}(U,\mathbb{R}^L))^*}||P_k\psi||_{W^{1,n}(U)} \to 0,$$

as the sequence P_k is bounded in $W^{1,n}$. Hence the additional term tends to zero in the sense of distributions. The proof of convergence for other terms remains the same, although we have to replace Lemma 2.3 with a slightly generalized version, i.e. Lemma 2.4. □

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