A ROBUST NONPERIODIC TILING SYSTEM

Boris Tsirelson

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Abstract. A robust version of Miekisz construction is proposed. It seems to have a good chance to give nonperiodic Gibbs state at a finite temperature.

Introduction

Investigating the problem of low-temperature stability of non-periodic structures, Miekisz[Mi90] constructed a tiling system (a classical lattice gas model with two-body nearest neighbor potential) with nonperiodic ground state such that the period of its Gibbs state tends to infinity when the temperature tends to zero. Or, maybe, the period becomes infinite for some finite temperature; but this possibility seems unlikely, because no care was taken to make the tiling robust. I propose a robust version of the construction; it has a good chance to give nonperiodic Gibbs state at a finite temperature.

Outline of the ground state

The present tiling system is at the same time a two-dimensional substitution system (see Mozes [Mo89, p. 143]) with all the derivation rules having the same height and width N. So, a tiling (a ground state) consists of $N \times N$ squares (blocks of first level), forming $N \times N$ groups (blocks of second level, of size $N^2 \times N^2$), and so on. The structure is essentially the same for all levels. A block contains "centers" (of interaction), connected with "cables" and separated with "insulation" areas, arranged as shown on Fig. 1.

So, the interface between two neighbor blocks contains all the intra-block information in two identical copies.

There are 5 types of centers, each with 4 interfaces (left, right, top, and bottom), and each centre has the property that the information on its left interface determines uniquely the information on the right one, and conversely, and the same for the top/bottom direction.

This architecture is robust in the following sense: any damage of the size small enough is unessential, because it may be by-passed (see Fig. 2) provided that there is no more damage in a neighborhood large enough. The exact formulation follows. It is of such a form: if the fault set is small, then the configuration is a small perturbation of a tiling.

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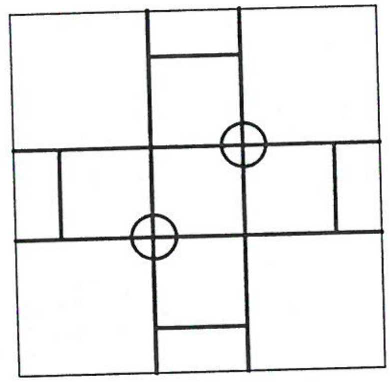


FIGURE 1. Block structure.

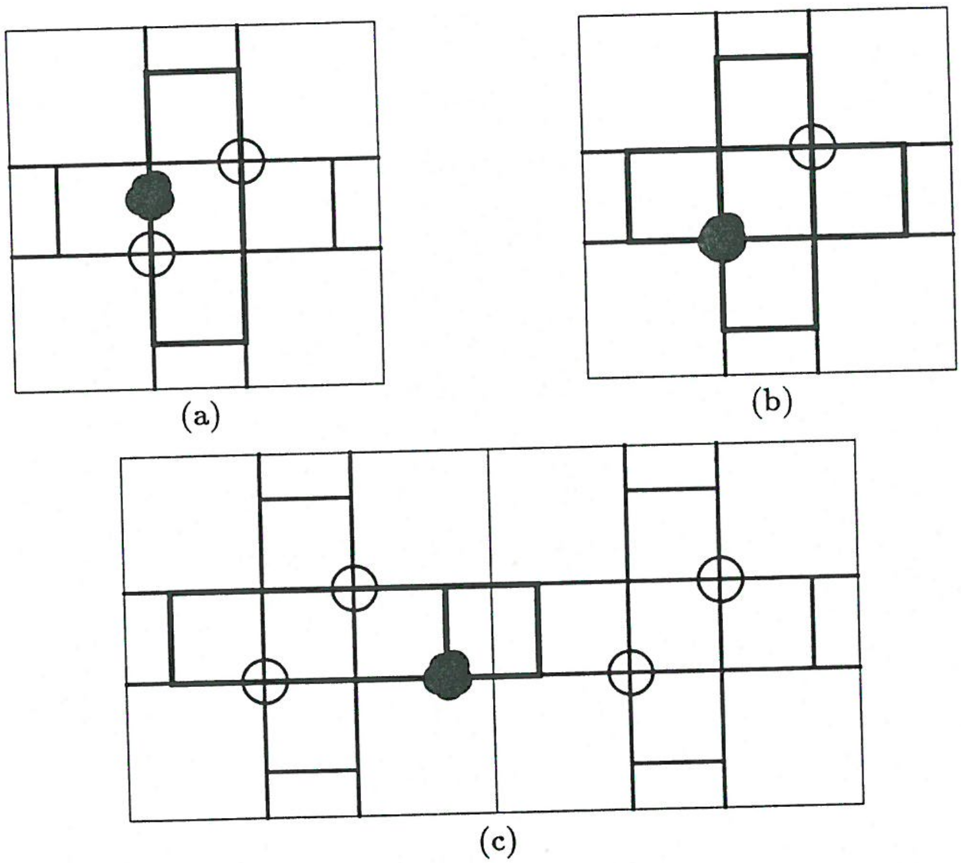


FIGURE 2. By-passing a single damage.

FORMULATION

Definition 1. Fix some $q > 1$ and $0 < r < 1$. For any set X in the two-dimensional lattice $\mathbb{Z}^2$ (or even in any metric space) define $X^{(n)}$ recursively:

$$X^{(0)} = X,$$
$$X^{(n)} = \{x \in X^{(n-1)} : \exists y \in X^{(n-1)} \quad rq^n \le |x-y| \le q^n\}.$$

Definition 2. For any homogeneous random field $\{\xi(x)\}_{x\in\mathbb{Z}^2}$ on the two-dimensional lattice $\mathbb{Z}^2$ define

$$p_n = \mathbb{P}\{x \in X^{(n)}\} \quad \text{(does not depend on } x\text{)},$$

where $X = \{x : \xi(x) \neq 0\}$.

Proposition 1. *Let $\xi(x)$ be independent and $r(q-1) > 2$. Then*

$$p_n \le \exp(-C_q(p_0)\cdot 2^n) \quad \textit{for } n \ge 0$$

with $C_q(\epsilon) \to \infty$ for $\epsilon \to 0$.

Proposition 2. *Let $\{\xi(x)\}$ be a pure phase of the Ising model with a temperature $1/\beta$ low enough. And let $r(q-1) > 2$. Then*

$$p_n \le \exp(-C_{q,r}(\beta)\cdot 2^n) \quad \textit{for } n \ge 0$$

with $C_{q,r}(\beta) \to \infty$ for $\beta \to \infty$.

Definition 3. Hierarchical M-neighborhood of a set X is the set

$$X_M = \{y : \exists n\ \exists x \in X^{(n)} \quad |y - x| \le Mq^n\}.$$

(Of course, X_M depends implicitly on q and r, as well as $X^{(n)}$ and p_n do). For the case of Definition 2 we clearly have

$$\mathbb{P}\{x \in X_M\} \le \text{const}\cdot\sum_{n=0}^{\infty}(Mq^n)^2 p_n,$$

the constant being absolute. Hence, for the case of Proposition 1 or 2 we obtain $\mathbb{P}\{x \in X_M\} \to 0$ for $\epsilon \to$ or $\beta \to \infty$, respectively.

Definition 4. A tiling system is robust, if there are q, r, and M such that $r(q-1) > 2$ and any configuration coincides with a tiling outside of X_M, where X is the fault set of the configuration.

Theorem. *There is a robust nonperiodic tiling system.*

The construction

Preliminary.

My system, as well as systems of Mozes, Robinson, and others, is a "substitution-of-tiles" system, that is, both a tiling system and a substitution system. The alphabet of the substitution system coincides with the set of squarelike (proto)tiles of the tiling system, and the symbolic dynamical system defined by the substitution system coincides with that defined by the tiling system. Further, the substitution system has unique derivation. (For definitions see [Mo89]).

At first a rather informal description is given, and then the definition.

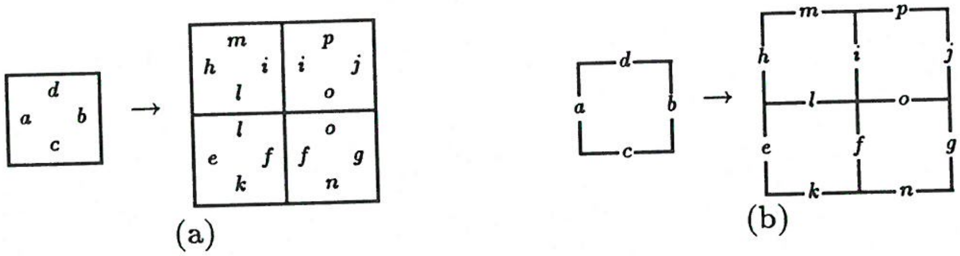


FIGURE 3. A 2×2 derivation rule.

Let us agree on notations suitable for substitution-of-tiles system. A tile is a square with marked edges. The matching (or adjacency) condition demands that abutting edges have identical marks. The right-hand side of a derivation rule is supposed to satisfy the matching condition. A general form of a 2×2 derivation rule is shown on Fig. 3(a). More suitable form is proposed on Fig. 3(b).

The background.

The first component of the marking, which I will call the background, consists in its turn of three components: x, y and p; p runs over $\{0, 1, \dots, p_{\max}\}$, the constant $p_{\max}$ being arbitrary as yet.

Consider the "background component" of a derivation rule, that is, a rule obtaining by keeping the background components and omitting other components within a derivation rule of the system. We restrict ourselves to rules with background component of the form shown on Fig. 4.

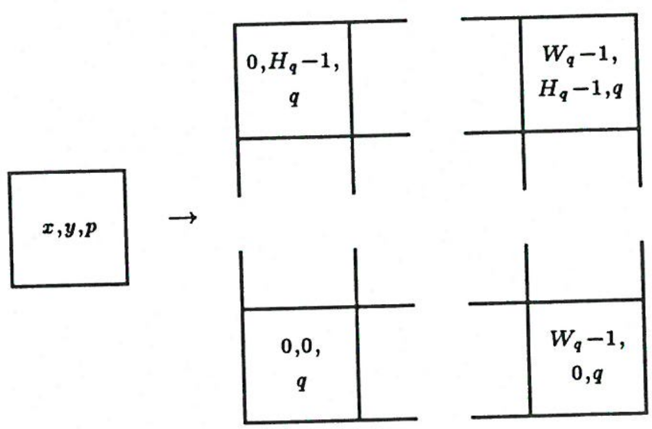
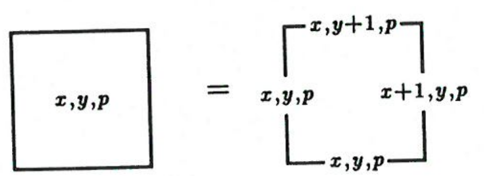
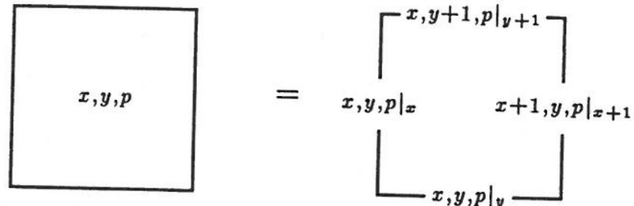


FIGURE 4. Background component of a derivation rule.

Here and below



when $1 \le x \le W_p-2,\ 1 \le y \le H_p-2$; the full range $0 \le x \le W_p-1,\ 0 \le y \le H_p-1$ is covered as follows:



where $x+1$ is treated in $\mathbb{Z}_{W_p}$, that is, $(W_p-1)+1=0$, and similarly $y \in \mathbb{Z}_{H_p}$; and

$$p|_z = \begin{cases} p, & \text{for } z \neq 0 \\ 0, & \text{for } z = 0. \end{cases}$$

Hence, Fig. 4 is equivalent to Fig. 5.

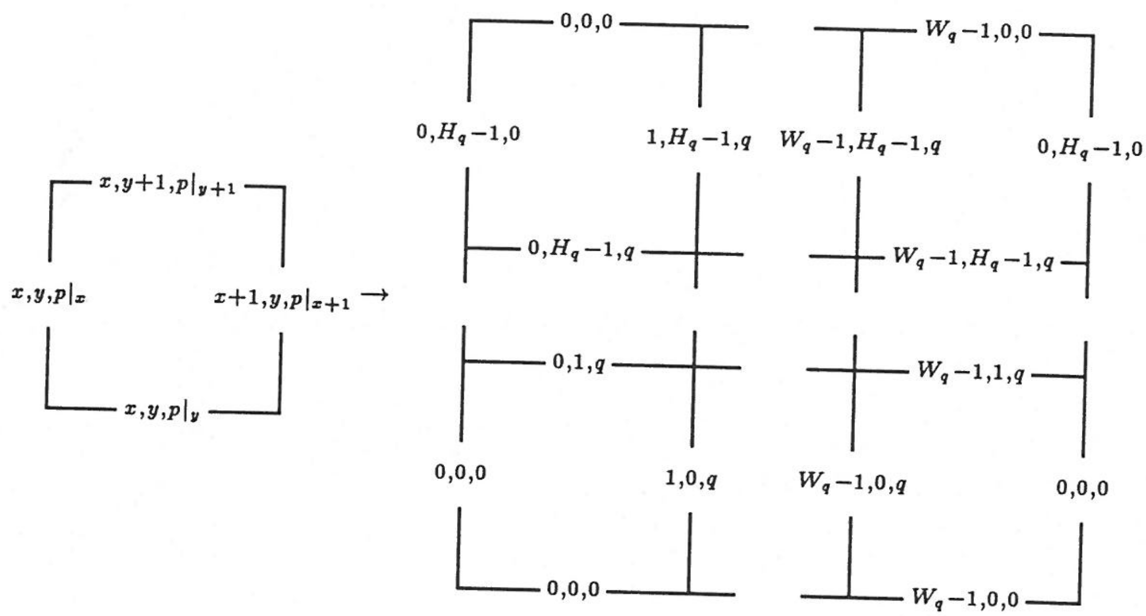


FIGURE 5. Background component of a derivation rule (another form)

Note that p-component is a constant (denoted here by q) within a block, and the size of a block, $W_q \times H_q$, is a function of it. A triple (x, y, p) is called admissible, if $p \in \{0, 1, \ldots, p_{\max}\}$, $x \in \{0, \ldots, W_p - 1\}$, and $y \in \{0, \ldots, H_p - 1\}$.

The foreground for the background.

One component of the marking, called background, was considered above. Now we start considering other component of the marking, called foreground. It consists of three components X, Y, P, the triple (X, Y, P) being supposed admissible, as well as (x, y, p) triple. We have to define their values on the whole right-hand side of each derivation rule. In this subsection, however, they are described on a subset, because now we consider only right-hand side foreground that corresponds to the left-hand side background (x, y, p).

From now on we demand $W_p \geq 4$, $H_p \geq 4$ for all p (instead of weaker demand of Mozes $W_p \geq 2$, $H_p \geq 2$).

The relevant marking is shown on Fig. 6.

The connection between the foreground and the background.

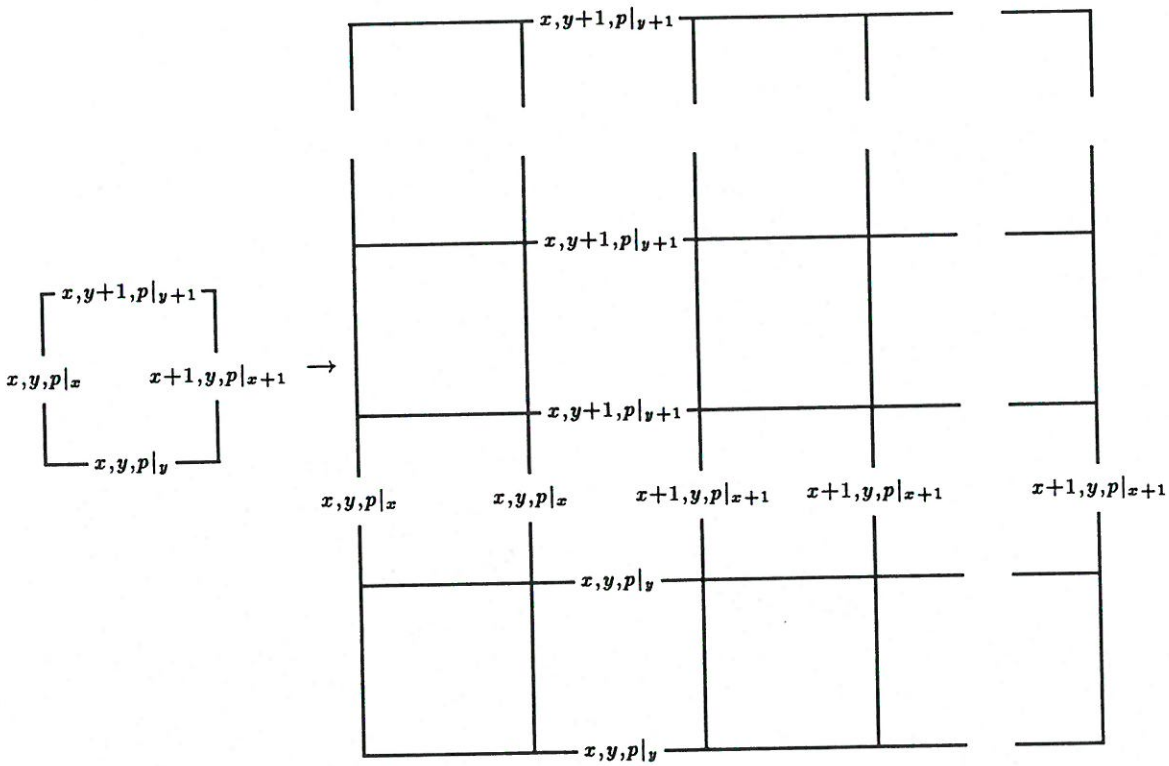


FIGURE 6. The part of right-hand side foreground corresponding to the left-hand side background.

We postulate a connection between background (Figs. 4,5) and foreground (Fig. 6) components of right-hand side marking:

$$q \in G(x, y, p);$$

the function G, that maps triples (x, y, p) into subsets of $\{0, 1, \ldots, p_{\max}\}$, will be specified later.

The foreground for the foreground: connectors.

We use 20 types of foreground marking; 17 of them are described in this subsection. They are shown on Fig. 7 with corresponding graphic symbols. Fig. 7(a) means: no foreground marking at all. Fig. 7(b) represents two types: vertical connector (shown) and horizontal connector (similar). Also Fig. 7(c) represents four types. The two directions, "up" and "right", are always interchangeable. However, the symmetry of "left" and "right" is broken on Figs. 7(f,g), representing (in common) four types. Similarly, Figs. 7(h,i) represent four types.

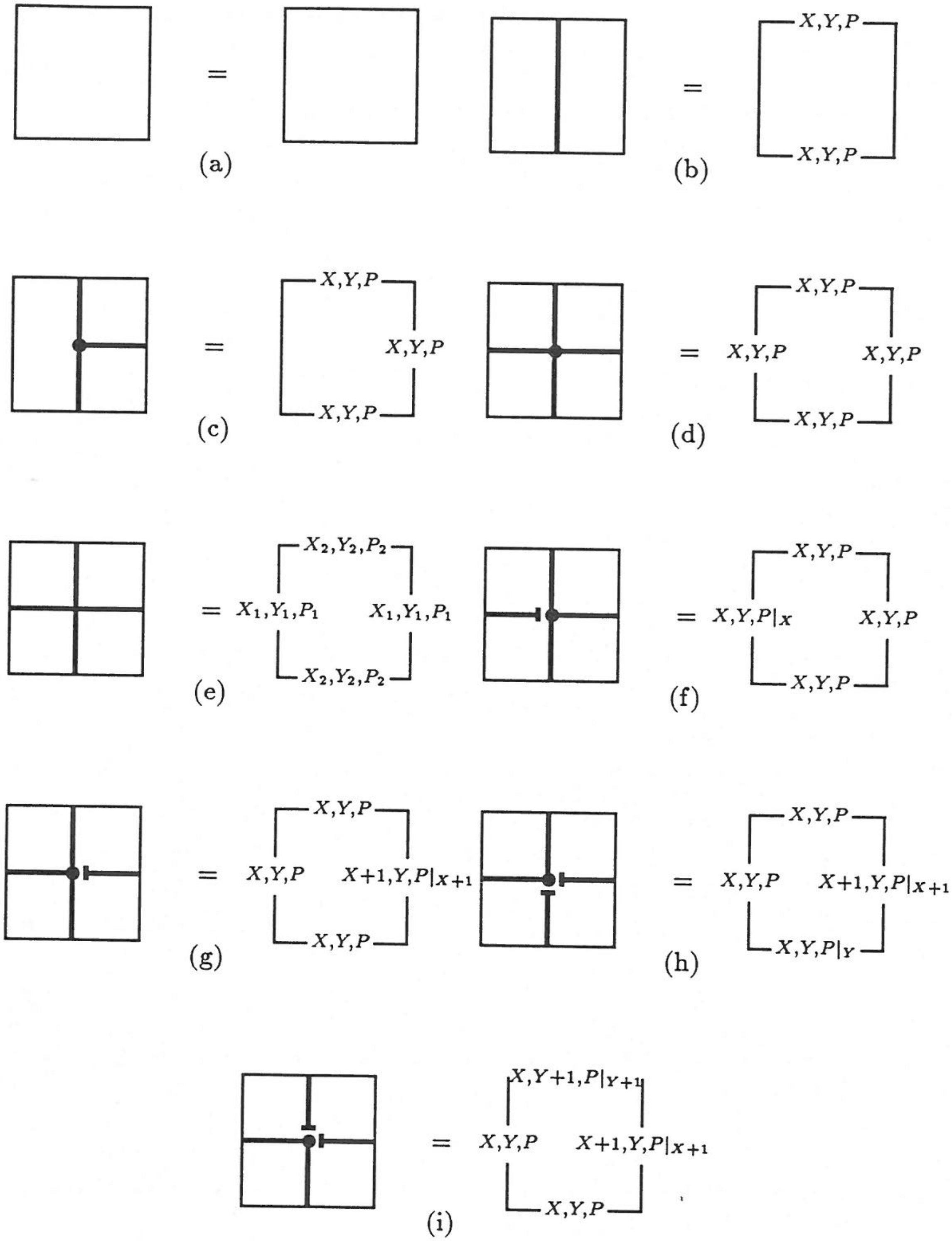


FIGURE 7. Foreground marking types: connectors.

When a tile on left-hand side of a derivation rule is the vertical connector (Fig. 7(b)), the corresponding right-hand side is designed as shown on Figs. 8, 9. Of course, it have to be combined with Fig. 6. For other types see Figs. 10–12 (and next subsection). For the empty type (Fig. 7(a)) we use Fig. 6 only.

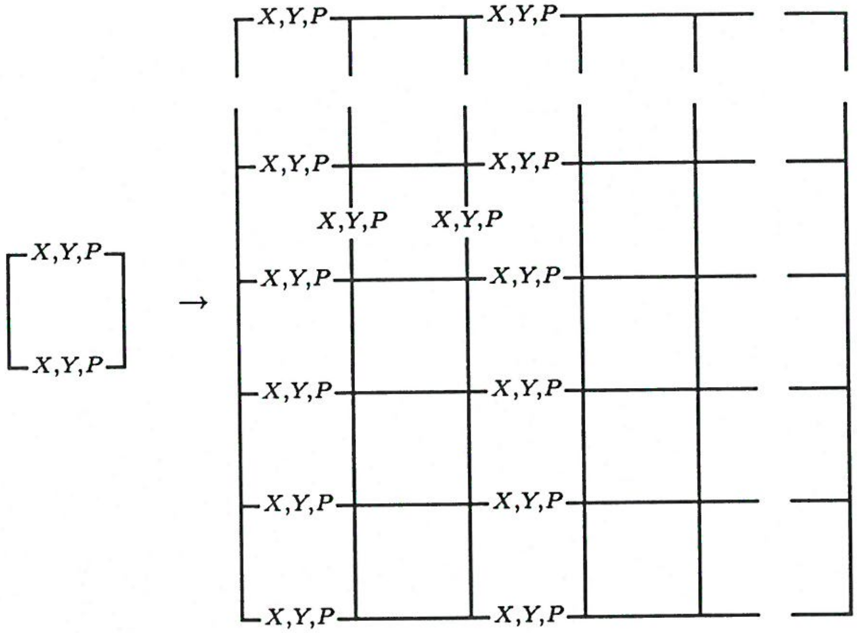


FIGURE 8. Implementation of the vertical connector.

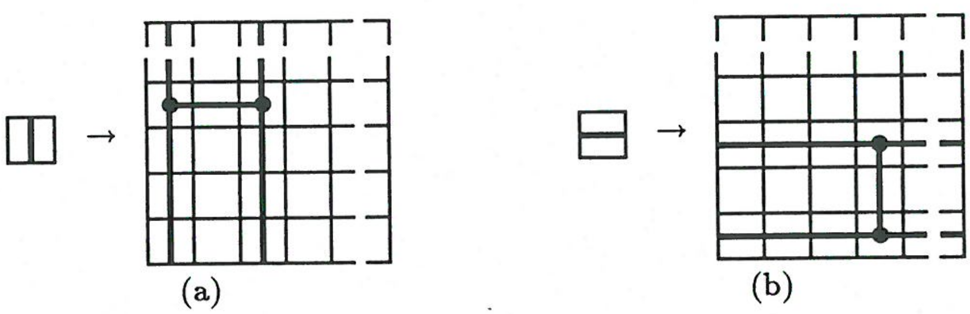


FIGURE 9. Implementation of the vertical (a) and horizontal (b) connectors.

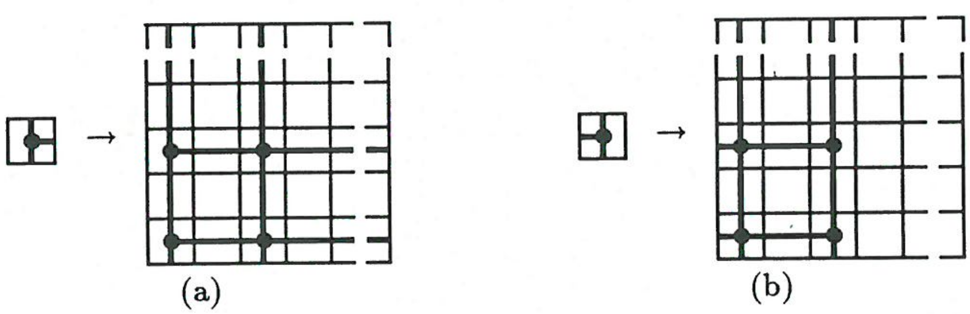


FIGURE 10. Implementation of three-sided junctions (two more cases are similar).

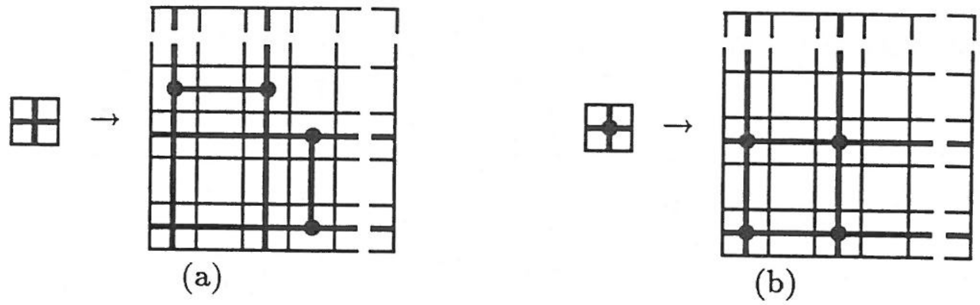


FIGURE 11. Implementation of cross (a) and four-sided junction (b).

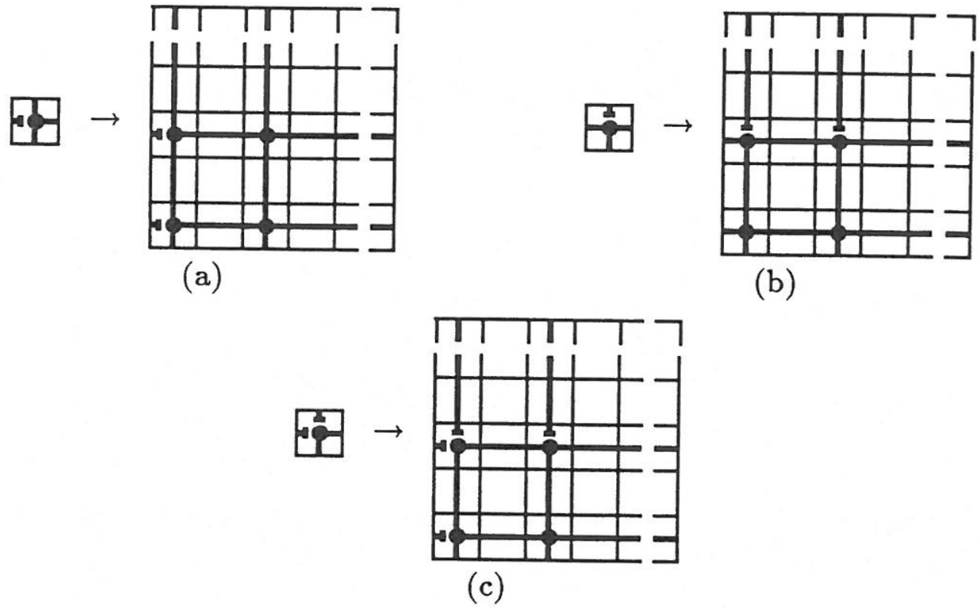
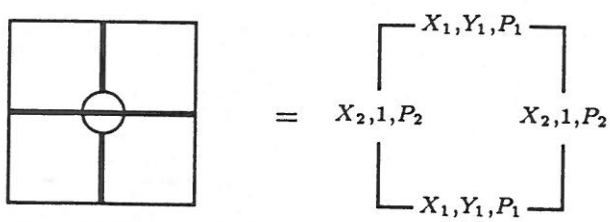


FIGURE 12. Implementation of boundary junctions (two more cases are similar to (a,b), and three more cases are similar to (c).

The foreground for the foreground: interaction centers.

We use 20 types of foreground marking of a tile. Most of the types (17) were considered. Others are: function generator (Fig. 13; 2 types), and level adapter (Fig. 14; 1 type). The function G was mentioned in subsection "The connection between the foreground and the background" and was not specified yet.



with
$X_2 = 1, 2,$
$P_2 \in G(X_1, Y_1, P_1).$

FIGURE 13. Function generator.

The last type, level adapter, is especially important: it is the only type establishing a correspondence between background and foreground. Thus, both components of marking are shown on Fig. 14 (foreground marking above, background - below).

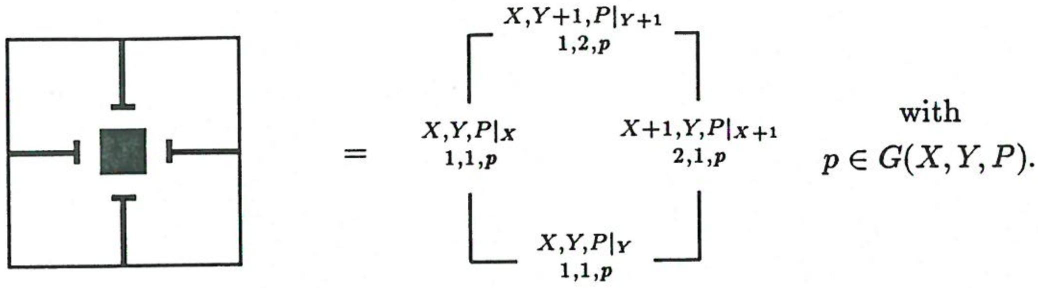


FIGURE 14. Level adapter.

By comparing Fig. 14 with Fig. 5 we see that the level adapter tile may appear within right-hand side of a derivation rule at position $(1, 1)$ only. In fact, the $(1, 1)$ position is always occupied with such a tile; now Fig. 6 may be pictured as Fig. 15.

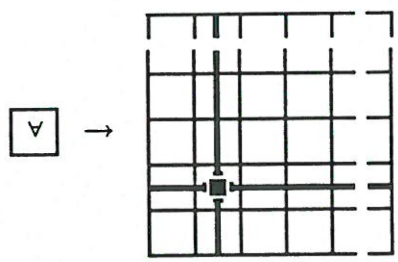


FIGURE 15. The part of right-hand side foreground corresponding to the left-hand side background.

The right-hand side foreground marking for corresponding derivation rules is shown on Figs. 16, 17. The common part shown on Fig. 15 is pictured explicitly on Fig. 17(a), but is meant on Figs. 9–12, 16.

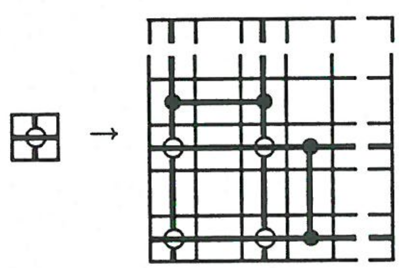
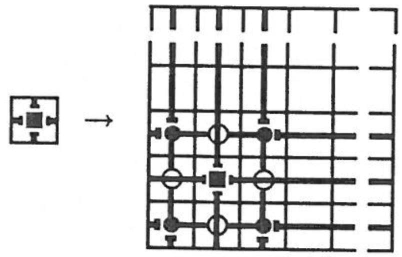


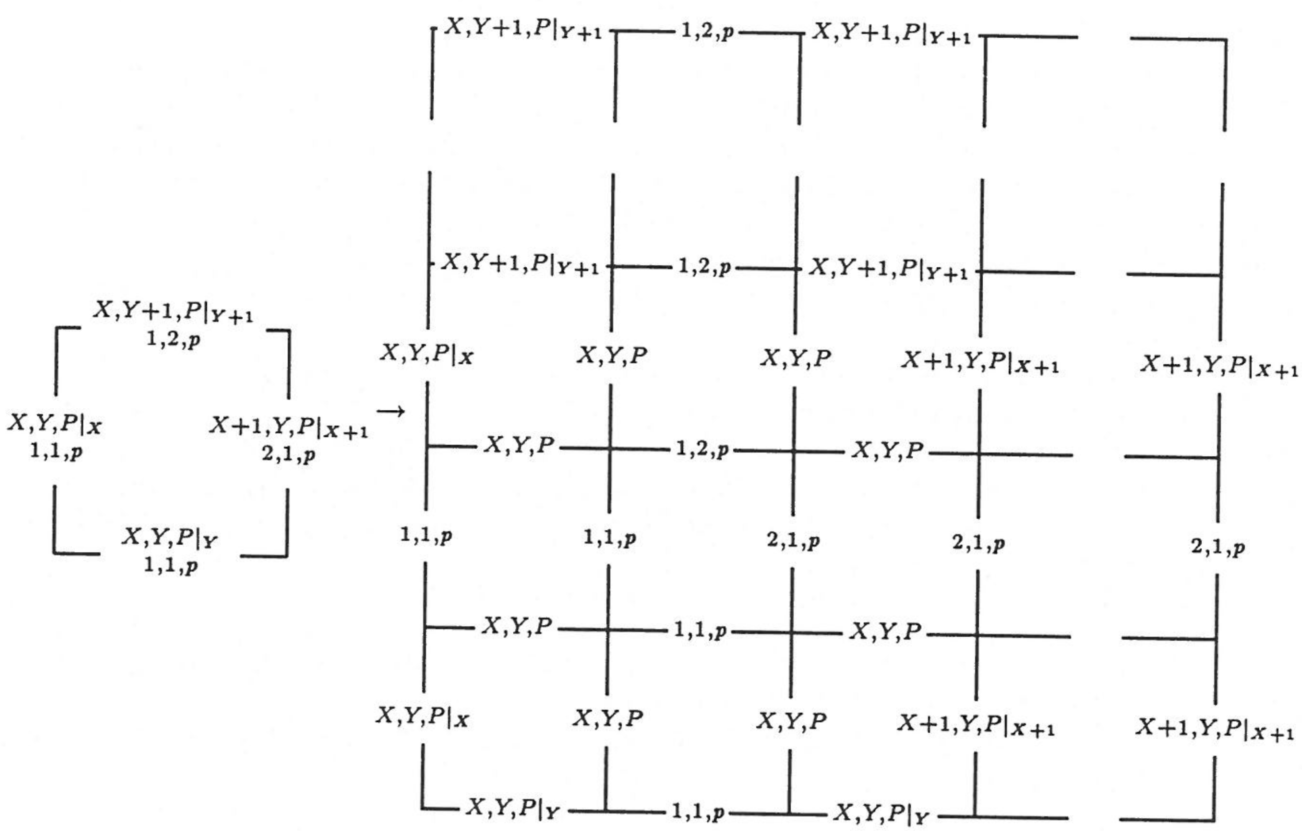
FIGURE 16. Implementation of the function generator.

The definition.

Now we summarize the construction and give the definition. We introduced 20 types of foreground marking with 20 corresponding "parametric derivation rules." Each parametric derivation rule contains parameters x, y, p running over admissible triples (though x, y are fixed at 1 for the level adapter). Admissible triples was defined by relations $0 \le p \le p_{\max}$, $0 \le x < W_p$, $0 \le y < H_p$; the numbers W_p, H_p are not specified yet, but $W_p \ge 4$, $H_p \ge 4$. Except of the empty type



(a)



(b)

FIGURE 17. Implementation of level adapter. Both foreground and background are shown on left-hand side, whereas foreground only is shown on right-hand side.

(Fig. 7(a)), each derivation rule contains more parameters X, Y, P or X_1, Y_1, P_1 and X_2, Y_2, P_2, each triple running over admissible triples (though X_2, Y_2 are restricted for the function generator). Special restrictions were imposed on parameters for the function generator ($P_2 \in G(X_1, Y_1, P_1)$) and level adapter ($p \in G(X, Y, P)$) in terms of a multi-valued function G not specified yet. The function G was used also in the restriction $q \in G(x, y, p)$ imposed on all derivation rules; q was introduced on Fig. 5.

Denote by T the set of all the 20 types of foreground marking. Our parametric derivation rules are in one-to-one correspondence with elements of T. We introduce now a principal function F that summarize figures 9–12, 15–17(a). It is a function of three arguments x, y, t, where $t \in T$ and $x, y \in \{0, 1, \dots\}$, and $F(x, y, t) \in T$. This function is defined as follows: for given x, y, t consider the parametric derivation rule corresponding to t, and consider the tile with coordinates (x, y) in the right-hand side of the rule; it is pictured as having a definite type of foreground marking; this type is just $F(x, y, t)$.

We do not claim that the foreground marking of a single tile determines uniquely the corresponding type of foreground marking. But it is not an impediment to the definition of F. Indeed, mentioned figures prescribe a definite type to each element of right-hand side block. Note also that the right-hand side block is pictured as finite but arbitrary large; correspondingly, $F(x, y, t)$ is defined for all x, y, and in fact it does not depend on x when $x \geq 5$, nor on y when $y \geq 5$.

Note that $F(1, 1, t)$ does not depend on t; it is always the level adapter (see Fig. 15).

By using the function F we might easily construct a single "substitution-of-tiles" system having desired properties. However, we prefer to leave in our construction enough freedom to adapt a wide class of substitution systems, as Mozes [Mo89] did. This is why we do not demand $\{0, 1, \dots, p_{\max}\} = T$ and $q \in G(x, y, p) \iff q = F(x, y, p)$. Instead we introduce a function $g : \{0, 1, \dots, p_{\max}\} \to T$, and demand

$$q \in G(x, y, p) \Longrightarrow g(q) = F(x, y, g(p)).$$

Any collection $(p_{\max}, g, G)$ satisfying this condition determines a definite "substitution-of-tiles" system of the considered class.

Now we finish summarizing the construction and start defining the system for given $p_{\max}, g, G$.

To define the tiling system we have to define its set of tiles, that is, the set of admissible combinations of marks on edges of a tile. General form of background marking was specified after Fig. 4; it is parametrized by admissible triples (x, y, p). We ascribe each tile (according to its background marking) to the corresponding type

$$t = F(x, y, g(p))$$

and impose the corresponding demand on the marking. For example, if a tile is ascribed to the "vertical connector" type, then its foreground marking is of the form shown on the right-hand side of Fig. 7(b) with arbitrary admissible triple (X, Y, P). So, "vertical connector" tiles are parametrized by x, y, p, X, Y, P under condition $F(x, y, g(p)) =$ "vertical connector". Similarly, "level adapter" tiles are parametrized by x, y, p, X, Y, P under conditions $x = 1$, $y = 1$, $F(x, y, g(p)) =$

"level adapter" (the latter being always true for these x, y), and $p \in G(X, Y, P)$. And so on. Thereby the tiling system is defined.

The substitution system may be defined now as follows. Its alphabet coincides with the set of tiles defined above. Its derivation rules result from the 20 above "parametric derivation rules" by substituting admissible values for the parameters. These values must satisfy conditions assosiated with corresponding parametric derivation rules, and besides the following condition: the tile in left-hand side of the derivation rule belongs to the alphabet, and its type coincides with the type of the parametric derivation rule. (The type of a tile means $F(x, y, g(p))$, as before). Under these conditions all tiles in the right-hand side belong to the alphabet, too. Thereby the substitution system is defined.

It is easy to see that these two systems, the tiling system and the substitution system, are equivalent in the sense that both generate one and the same closed shift-invariant set ("subshift") in the space of configurations. And the substitution system has unique derivation.

We do not claim that the subshift is non-empty; it depends on the function G.

The simplest non-trivial case follows: $\{0, 1, \dots, p_{\max}\} = T$, $W_p = H_p = 4$ for all p, and $q \in G(x, y, p) \iff q = F(x, y, p)$. In this case the substitution system is deterministic; in fact, for every letter of its alphabet there is exactly one derivation rule belonging to it.

UNIVERSALITY

We defined a class of tiling systems. Now we claim an universality property of the class, close to the universality property of Mozes' class. Our system is able to simulate any substitution system with unique derivation provided that each derivation rule is of hight and weight ≥ 4. That is, for any such system there are $p_{\max}, g, G$ giving (via our construction) a tiling system that is equivalent to the given substitution system in measure-theoretic sense: there is a map of one subshift to other which is one-to-one almost everywhere with respect to any invariant measure.

Two following lemmas are used in the proof.

Lemma1. *There is no shift-invariant probability measure on the set of all non-empty zero-density subsets of the lattice $\mathbb{Z}^2$.*

Lemma 2. *Let $\Omega = \Omega_{p_{\max}, g, G}$ be the subshift corresponding to a tiling system of our class, and let two tilings $\omega_1, \omega_2 \in \Omega$ have the same x, y, X, Y components of marking (on the whole lattice), that is, $x_{\omega_1}(e) = x_{\omega_2}(e)$ for each edge e of the lattice, and the same for y, X, Y. Then $g(p_{\omega_1}(e)) = g(p_{\omega_2}(e))$ and $g(P_{\omega_1}(e)) = g(P_{\omega_2}(e))$ almost everywhere, that is, for all e except a zero-density set.*

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SCHOOL OF MATHEMATICAL SCIENCES, TEL AVIV UNIVERSITY, TEL AVIV 69978, ISRAEL
E-mail address: Tsirel@math.TAU.Ac.IL